

Field Theory Approach to Diffusion-Limited Reactions:

1. Models and Mappings

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Field Theory Approach to Diffusion-Limited Reactions

1. Models and Mappings

How to turn stochastic particle models into a field theory, with no phenomenology.

2. Single-Species Annihilation

Field theoretic renormalization group calculation for $A + A \rightarrow 0$ reaction in gory detail.

3. Applications

Higher order reactions, disorder, Lévy flights, two-species reactions, coupled reactions.

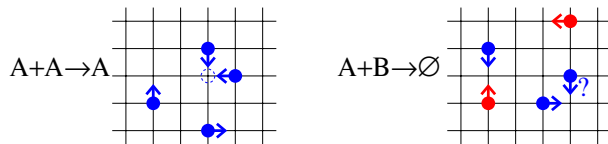
4. Active to Absorbing State Transitions

Directed percolation, branching and annihilating random walks, and all that.

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Diffusion-Limited Reactions

One or more species of particles undergoing random walks on a lattice, with a reactions occurring for particles on the same lattice site



Example: for $A + A \rightarrow 0$ the density of particles decays as

$$\rho(t) = \begin{cases} Ct^{-1} & \text{for } d > 2 \\ \tilde{A} \ln t / Dt & \text{for } d = 2 \\ A_d (Dt)^{-d/2} & \text{for } d < 2 \end{cases}$$

where A_d and \tilde{A} are **universal numbers!**

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Field Theory Approach to Diffusion-Limited Reactions: 1. Models and Mappings

Master Equation for Lattice Models

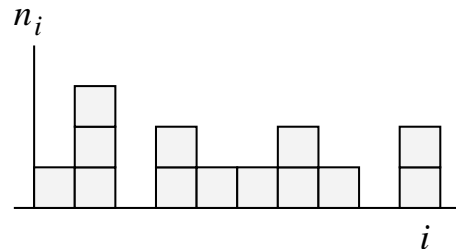
Doi Representation

Mapping to Field Theory

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Stochastic Classical Particles on a Lattice

Consider a set of lattice sites labeled $i = 1, 2, 3, \dots$, and each site is occupied by n_1, n_2, n_3, \dots particles.



Define

- ▶ $\alpha =$ a particular state, i.e., $\alpha = \{n_1, n_2, n_3, \dots\}$
- ▶ $P(\alpha, t) =$ the probability of obtaining state α at time t .

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Probability Master Equation

Dynamical processes (hops, reactions, decays) will cause a change of state from α to β .

$$w_{\alpha \rightarrow \beta} = \text{rate of transition from } \alpha \text{ to } \beta, \text{ defines dynamics}$$

Master Equation

$$\frac{d}{dt} P(\alpha, t) = \sum_{\beta} \left[\underbrace{w_{\beta \rightarrow \alpha} P(\beta, t)}_{\text{flow into } \alpha} - \underbrace{w_{\alpha \rightarrow \beta} P(\alpha, t)}_{\text{flow out of } \alpha} \right]$$

- ▶ $\sum_{\alpha} P(\alpha, t) = 1$ preserved by the master equation
- ▶ Initial conditions $P(\alpha, 0)$ need to be specified

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Master Equation for $A \rightarrow 0$ Decay

Consider a single lattice site that contains some number of identical particles. These particles decay at rate λ .

The rate for a transition from n to m particles is

$$w_{n \rightarrow m} = \begin{cases} 0 & \text{for } m \neq n - 1 \\ n\lambda & \text{for } m = n - 1 \end{cases}$$

and the master equation is

$$\frac{d}{dt} P(n, t) = \lambda \left[(n+1)P(n+1, t) - nP(n, t) \right]$$

Wait! That doesn't look like exponential decay ...

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Master Equation to Differential Equation

Let $\rho(t) = \langle n \rangle = \sum_n n P(n, t)$ be the average number of particles at time t . Then

$$\begin{aligned} \dot{\rho} &= \sum_n n \dot{P}(n, t) = \sum_n n \left[\lambda(n+1)P(n+1, t) - \lambda n P(n, t) \right] \\ &= \lambda \sum_n n(n+1)P(n+1, t) - \lambda \sum_n n^2 P(n, t) \\ &= \lambda \sum_m (m-1)m P(m, t) - \lambda \sum_n n^2 P(n, t) \\ &= -\lambda \sum_m m P(m, t) \\ &= -\lambda \rho \end{aligned}$$

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A + A → 0 Reaction

Again, consider a single lattice site, with the rule that a pair of particles may annihilate each other. The rates are

$$w_{n \rightarrow m} = \begin{cases} 0 & \text{for } m \neq n - 2 \\ n(n-1)\lambda & \text{for } m = n - 2 \end{cases}$$

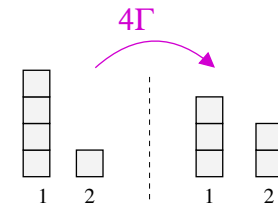
and the master equation is

$$\frac{d}{dt}P(n, t) = \lambda \left[(n+2)(n+1)P(n+2, t) - n(n-1)P(n, t) \right]$$

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Hop

Now consider two sites, $i = 1$ and 2, with a rate Γ of hopping from site 1 to site 2.



$$w_{(n_1, n_2) \rightarrow (m_1, m_2)} = \begin{cases} 0 & \text{for } m_1 \neq n_1 - 1 \text{ or } m_2 \neq n_2 + 1 \\ n_1 \Gamma & \text{for } m_1 = n_1 - 1 \text{ and } m_2 = n_2 + 1 \end{cases}$$

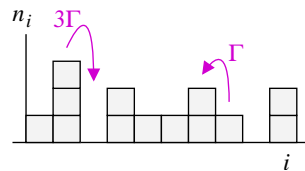
and the master equation is

$$\frac{d}{dt}P(n_1, n_2, t) = \Gamma \left[(n_1 + 1)P(n_1 + 1, n_2 - 1, t) - n_1 P(n_1, n_2, t) \right]$$

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Diffusion

Consider a one-dimensional chain of lattice sites $i = 1, 2, \dots$ and let all particles hop left or right with rate Γ . The master equation is



$$\frac{d}{dt}P(\alpha, t) = \Gamma \sum_{\langle ij \rangle} \left[(n_i + 1)P(n_i + 1, n_j - 1, \dots, t) - n_i P(\alpha, t) \right. \\ \left. (n_j + 1)P(n_i - 1, n_j + 1, \dots, t) - n_j P(\alpha, t) \right]$$

Define $\rho(x, t) = \sum_{\alpha} n_i P(\alpha, t)$ where $x = i\Delta x$.

For small Δx this becomes the **diffusion equation**:

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2} \quad D = \Gamma \Delta x^2 = \text{diffusion constant.}$$

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Master Equation for A + A → 0 Diffusion-Limited Reaction

$$\frac{d}{dt}P(\{n\}, t) = \frac{D}{\Delta x^2} \sum_{\langle ij \rangle} \left[(n_i + 1)P(\dots, n_i + 1, n_j - 1, \dots, t) - n_i P(\{n\}, t) \right. \\ \left. + (n_j + 1)P(\dots, n_i - 1, n_j + 1, \dots, t) - n_j P(\{n\}, t) \right] \\ + \lambda \sum_i \left[(n_i + 2)(n_i + 1)P(\dots, n_i + 2, \dots, t) - n_i(n_i - 1)P(\{n\}, t) \right]$$

with $P(\{n\}, 0) = \prod_i \frac{n_0^{n_i} e^{-n_0}}{n_i!}$ for random initial conditions.

Yuck!

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Field Theory Approach to Diffusion-Limited Reactions: 1. Models and Mappings

Master Equation for Lattice Models

Doi Representation

Mapping to Field Theory

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Doi Occupation Number Representation (Fock Space)

For a single lattice site:

- ▶ Introduce a creation operator \hat{a}^\dagger and annihilation operator \hat{a} , with commutator $[\hat{a}, \hat{a}^\dagger] = 1$.
- ▶ Represent the state of zero particles by $|0\rangle$, defined via $\hat{a}|0\rangle = 0$.
- ▶ Represent a state of n particles by $|n\rangle = \hat{a}^{\dagger n}|0\rangle$. (Note: normalization differs from usual QM.)
- ▶ For this state

$$\hat{a}^\dagger |n\rangle = |n+1\rangle \quad \hat{a} |n\rangle = n |n-1\rangle \quad \hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle$$

For multiple lattice sites: introduce a pair $\hat{a}_i, \hat{a}_i^\dagger$ at each site

$$\{n\} = (n_1, n_2, \dots) \Leftrightarrow |\{n\}\rangle = \prod_i \hat{a}_i^{\dagger n_i} |0\rangle$$

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Doi Representation, part II

We can pack the probability P into a Fock space state:

$$|\phi(t)\rangle = \sum_{\{n\}} P(\{n\}, t) |\{n\}\rangle$$

and re-write the master equation in Schrödinger-like form:

$$\frac{d}{dt} |\phi(t)\rangle = -\hat{H} |\phi(t)\rangle$$

Why do this? Because it is a simpler description of the dynamics.
For $A + A \rightarrow 0$ diffusion-limited reaction we get

$$\hat{H} = \Gamma \sum_{\langle ij \rangle} (a_i^\dagger - a_j^\dagger)(a_i - a_j) - \lambda \sum_i (1 - a_i^{\dagger 2}) a_i^2$$

and formal solution $|\phi(t)\rangle = \exp(-\hat{H}t) |\phi(0)\rangle$.

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$A + A \rightarrow 0$ on a Single Site

Master equation:

$$\frac{d}{dt} P(n, t) = \lambda [(n+2)(n+1)P(n+2, t) - n(n-1)P(n, t)]$$

Multiply by $|n\rangle$ and sum over n :

$$\begin{aligned} \frac{d}{dt} |\phi(t)\rangle &= \lambda \sum_n P(n+2, t) (n+2)(n+1) |n\rangle - \lambda \sum_n P(n, t) n(n-1) |n\rangle \\ &= \lambda \sum_n P(n+2, t) \hat{a}^2 |n+2\rangle - \lambda \sum_n P(n, t) \hat{a}^{\dagger 2} \hat{a}^2 |n\rangle \\ &= \lambda (\hat{a}^2 - \hat{a}^{\dagger 2} \hat{a}^2) \sum_n P(n, t) |n\rangle \\ &= \lambda (1 - \hat{a}^{\dagger 2}) \hat{a}^2 |\phi(t)\rangle = -\hat{H} |\phi(t)\rangle \end{aligned}$$

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Hop from Site 1 to Site 2

Master Equation

$$\frac{d}{dt}P(n_1, n_2, t) = \Gamma \left[(n_1+1)P(n_1+1, n_2-1, t) - n_1P(n_1, n_2, t) \right]$$

Multiply by $|n_1, n_2\rangle$ and sum over n_1 and n_2 :

$$\begin{aligned} \frac{d}{dt}|\phi(t)\rangle &= \Gamma \sum_{n_1, n_2} P(n_1+1, n_2-1, t) (n_1+1)|n_1, n_2\rangle \\ &\quad - \Gamma \sum_{n_1, n_2} P(n_1, n_2, t) n_1|n_1, n_2\rangle \\ &= \Gamma \sum_{n_1, n_2} P(n_1+1, n_2-1, t) \hat{a}_2^\dagger \hat{a}_1 |n_1+1, n_2-1\rangle \\ &\quad - \Gamma \sum_{n_1, n_2} P(n_1, n_2, t) \hat{a}_1^\dagger \hat{a}_1 |n_1, n_2\rangle \\ &= \Gamma(\hat{a}_2^\dagger - \hat{a}_1^\dagger) \hat{a}_1 |\phi(t)\rangle \end{aligned}$$

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Diffusion

► Hop from site 1 to site 2:

$$\hat{H}_{1 \rightarrow 2} = \Gamma(\hat{a}_1^\dagger - \hat{a}_2^\dagger) \hat{a}_1$$

► Allow for the reverse hop with the same rate:

$$\hat{H}_{1 \leftrightarrow 2} = \Gamma(\hat{a}_1^\dagger - \hat{a}_2^\dagger)(\hat{a}_1 - \hat{a}_2)$$

► For hops between all neighboring lattice sites:

$$\hat{H}_D = \frac{D}{(\Delta x)^2} \sum_{\langle ij \rangle} (\hat{a}_i^\dagger - \hat{a}_j^\dagger)(\hat{a}_i - \hat{a}_j)$$

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Multiple Species

With two species reactions, such as $A + B \rightarrow 0$ reaction, the master equation is defined in terms of A -particle and B -particle occupation numbers

$$P(\{m\}, \{n\}, t)$$

For the Doi representation, we introduce distinct creation and annihilation operators for each species: \hat{a}_i , \hat{a}_i^\dagger , \hat{b}_i , and \hat{b}_i^\dagger , and define state

$$|\phi(t)\rangle = \sum_{\{m\}, \{n\}} P(\{m\}, \{n\}, t) \prod_i \hat{a}_i^{\dagger m_i} \hat{b}_i^{\dagger n_i} |0\rangle$$

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Doi Hamiltonians

Each process contributes two terms to \hat{H} , of the form

$$(\text{rate}) \left[(\text{reactants}) - (\text{reaction}) \right]$$

(reactants) = creation and annihilation operator for each reactant, normal ordered

(reaction) = annihilation operator for each reactant, creation operator for each product, normal ordered

Examples:

$$A + A \rightarrow 0 \quad \lambda[\hat{a}^{\dagger 2} \hat{a}^2 - \hat{a}^2] \quad A + A \rightarrow A \quad \lambda[\hat{a}^{\dagger 2} \hat{a}^2 - \hat{a}^\dagger \hat{a}^2]$$

$$A \rightarrow A + A \quad \lambda[\hat{a}^\dagger \hat{a} - \hat{a}^{\dagger 2} \hat{a}] \quad A + B \rightarrow C \quad \lambda[\hat{a}^\dagger \hat{b}^\dagger \hat{a} \hat{b} - \hat{c}^\dagger \hat{a} \hat{b}]$$

$$\text{Hop } 1 \rightarrow 2 \quad \Gamma[\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_1]$$

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Observables

For our classical particle system, averages are given by

$$\langle A(t) \rangle = \sum_{\{n\}} A(\{n\}) P(\{n\}, t)$$

To map this to the Doi representation, we need a projection state $\langle \mathcal{P} | = \langle 0 | e^{\sum_i \hat{a}_i}$, which has properties

$$\langle \mathcal{P} | 0 \rangle = 1 \quad \langle \mathcal{P} | \hat{a}_i^\dagger = \langle \mathcal{P} | \Rightarrow \langle \mathcal{P} | n \rangle = 1$$

Then, for the operator $\hat{A} = A(\{n_i \rightarrow \hat{a}_i^\dagger \hat{a}_i\})$ we get

$$\langle A(t) \rangle = \langle \mathcal{P} | \hat{A} | \phi(t) \rangle = \langle \mathcal{P} | \hat{A} e^{-\hat{H}t} | \phi(0) \rangle$$

Note: for Poisson initial conditions, $|\phi(0)\rangle = \prod_i e^{-n_0 + n_0 \hat{a}_i^\dagger} |0\rangle$

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Probability Conservation

Check Normalization

For some initial state $P(\{n\}, 0)$ we average the identity operator:

$$\langle \mathcal{P} | \hat{1} | \phi(0) \rangle = \langle \mathcal{P} | \sum_{\{n\}} P(\{n\}, 0) | \{n\} \rangle = \sum_{\{n\}} P(\{n\}, 0) = 1 \checkmark$$

Check Probability Conservation

What is the condition on \hat{H} ? We need

$$1 = \langle \mathcal{P} | \hat{1} e^{-\hat{H}t} | \phi(0) \rangle = \langle \mathcal{P} | (1 - \hat{H}t + \frac{1}{2}t^2 \hat{H}^2 - \dots) | \phi(0) \rangle$$

for all t , so we require $\langle \mathcal{P} | \hat{H} = 0$. Equivalently, $\hat{H} \rightarrow 0$ as $\hat{a}_i^\dagger \rightarrow 1$.

Note: \hat{H} need not be hermitian!

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Coherent States

The coherent state $|\phi\rangle = \exp(-\frac{1}{2}|\phi|^2 + \phi a^\dagger) |0\rangle$ with complex ϕ has properties

$$\hat{a} |\phi\rangle = \phi |\phi\rangle, \quad \langle \phi | \hat{a}^\dagger = \langle \phi | \phi^*$$

and the overlap relation

$$\langle \phi_1 | \phi_2 \rangle = \exp(-\frac{1}{2}|\phi_1|^2 - \frac{1}{2}|\phi_2|^2 + \phi_1^* \phi_2)$$

Can construct a resolution of the identity operator

$$\hat{1} = \sum_n \frac{1}{n!} |n\rangle \langle n| = \sum_{m,n} \frac{1}{n!} |n\rangle \langle m| \delta_{mn} = \int \frac{d^2\phi}{\pi} |\phi\rangle \langle \phi|$$

by use of

$$\delta_{mn} = \frac{1}{m! \pi} \int \phi^{*m} \phi^n e^{-|\phi|^2} d^2\phi$$

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Coherent State Representation

Given $\hat{H}(\hat{a}_i^\dagger, \hat{a}_i)$, take formal solution $|\phi(t)\rangle = e^{-\hat{H}t}|\phi(0)\rangle$ and divide time t into a number of small increments Δt via

$$e^{-\hat{H}t} = \exp(-\hat{H}\Delta t)^{t/\Delta t}$$

Insert $\hat{1} = \int \frac{d^2\phi}{\pi} |\phi\rangle\langle\phi|$ between each successive time step:

$$\int \dots |\phi_{t+\Delta t}\rangle\langle\phi_{t+\Delta t}| e^{-\hat{H}\Delta t} |\phi_t\rangle\langle\phi_t| e^{-\hat{H}\Delta t} |\phi_{t-\Delta t}\rangle\langle\phi_{t-\Delta t}| \dots$$

Now focus on a single matrix element:

$$\begin{aligned} \langle\phi_t| e^{-\hat{H}\Delta t} |\phi_{t-\Delta t}\rangle &= e^{-H(\phi_t^*, \phi_{t-\Delta t})\Delta t} \langle\phi_t|\phi_{t-\Delta t}\rangle \\ &\simeq e^{-H(\phi_t^*, \phi_t)\Delta t} e^{\phi_t^* \phi_{t-\Delta t} - \frac{1}{2}|\phi_t|^2 - \frac{1}{2}|\phi_{t-\Delta t}|^2} \\ &\simeq e^{-H(\phi_t^*, \phi_t)\Delta t} e^{-\phi_t^* \partial_t \phi_t \Delta t} \end{aligned}$$

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Path Integral

String together many time slices in the limit $\Delta t \rightarrow 0$ and we get

$$e^{-\hat{H}t} \rightarrow \int \mathcal{D}\phi^* \mathcal{D}\phi \exp\left(-\int_0^t dt' [\phi^* \partial_{t'} \phi + H(\phi^*, \phi)]\right)$$

where $\prod_j \left(\frac{d^2\phi_j}{\pi}\right) \rightarrow \mathcal{D}\phi^* \mathcal{D}\phi$

Generalize to multiple lattice sites:

$$e^{-\hat{H}t} \rightarrow \int \prod_j (\mathcal{D}\phi_j^* \mathcal{D}\phi_j) e^{-S[\{\phi_j^*\}, \{\phi_j\}]}$$

with

$$S = \int_0^t \sum_j dt' [\phi_j^* \partial_{t'} \phi_j + H(\{\phi_j^*\}, \{\phi_j\})]$$

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Path Integral Observables

$$\langle A(t) \rangle = \langle \mathcal{P} | \hat{A} e^{-\hat{H}t} | \phi(0) \rangle = \mathcal{N}^{-1} \int \prod_j (\mathcal{D}\phi_j^* \mathcal{D}\phi_j) A(\phi(t)) e^{-S[\phi^*, \phi]}$$

with action S given by

$$S = \sum_i \left\{ -\phi_i(t) + \int_0^t dt' [\phi_i^* \partial_{t'} \phi_i + H(\{\phi_i^*\}, \{\phi_i\})] - n_0 \phi_i^*(0) \right\}$$

Can eliminate projection state term by field shift $\phi^* \rightarrow 1 + \tilde{\phi}$:

$$\int_0^t (1 + \tilde{\phi}) \partial_{t'} \phi dt' = \phi(t) - \phi(0) + \int_0^t \tilde{\phi} \partial_t \phi dt'$$

which takes $H \rightarrow H(\{1 + \tilde{\phi}_j\}, \{\phi_j\})$

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Diffusion

$$\begin{aligned} S_D &= \int dt \left[\sum_i \tilde{\phi}_i \partial_t \phi_i + \frac{D}{\Delta x^2} \sum_{\langle ij \rangle} (\tilde{\phi}_i - \tilde{\phi}_j) (\phi_i - \phi_j) \right] - \sum_i n_0 \tilde{\phi}_i(0) \\ &= \int dt d^d x \left[\tilde{\phi} \partial_t \phi + D \nabla \tilde{\phi} \cdot \nabla \phi - n_0 \tilde{\phi} \delta(t) \right] \\ &= \int dt d^d x \left[\tilde{\phi} (\partial_t - D \nabla^2) \phi - n_0 \tilde{\phi} \delta(t) \right] \end{aligned}$$

Action is linear in $\tilde{\phi}$. Extremum:

$$\frac{\delta S_D}{\delta \tilde{\phi}} = \partial_t \phi - D \nabla^2 \phi - n_0 \delta(t) = 0$$

is the plain old diffusion equation: $\partial_t \phi = D \nabla^2 \phi + n_0 \delta(t)$

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Diffusion-Limited $A + A \rightarrow 0$ Reaction

The reaction part of the hamiltonian is

$$H_{\text{reac}} = -\lambda \sum_i (1 - \phi_i^{*2}) \phi_i^2 \rightarrow \int d^d x (2\lambda_0 \tilde{\phi} \phi^2 + \lambda_0 \tilde{\phi}^2 \phi^2)$$

with the field shift $\phi^* \rightarrow 1 + \tilde{\phi}$ and $\lambda_0 = \lambda/\Delta x^d$.

Thus the complete $A + A \rightarrow 0$ action is

$$S = \int d^d x dt \left[\tilde{\phi}(\partial_t - D\nabla^2)\phi + 2\lambda_0 \tilde{\phi} \phi^2 + \lambda_0 \tilde{\phi}^2 \phi^2 - n_0 \tilde{\phi} \delta(t) \right]$$

Now we're ready to do some calculations!

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The $A + A \rightarrow 0$ Stochastic PDE

Can make the action linear in $\tilde{\phi}$ via an auxiliary field η :

$$e^{-\lambda_0 \tilde{\phi}^2 \phi^2} \propto \int d\eta \exp \left\{ -\frac{1}{2} \eta^2 + i\eta \sqrt{2\lambda_0} \tilde{\phi} \phi \right\}$$

resulting in averages

$$\int \mathcal{D}\eta e^{-\eta^2/2} \int \mathcal{D}\phi \int \mathcal{D}\tilde{\phi} e^{-\int \tilde{\phi}(\partial_t - D\nabla^2)\phi + 2\lambda_0 \tilde{\phi} \phi^2 + i\eta \sqrt{2\lambda_0} \tilde{\phi} \phi}$$

The $\tilde{\phi}$ integration creates a δ -function that enforces

$$\partial_t \phi = D\nabla^2 \phi - 2\lambda_0 \phi^2 + i\sqrt{2\lambda_0} \phi \eta$$

A stochastic reaction-diffusion equation with multiplicative noise
that is complex!?

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Summary and Observations

- ▶ Mapping to Doi representation simplifies the master equation by getting rid of pesky factors involving n .
- ▶ This Fock space description natural for identical particles acting independently, not restricted to quantum mechanics
- ▶ Fock space dynamics can be converted to a field theory without resorting to Langevin-type phenomenology
- ▶ Technique can easily include multiple species, long-range hops, birth/death processes, convected fields
- ▶ Mechanical forces not so easily included. \hat{H} is **not** an energy but rather rates.
- ▶ SPDE's are fraught with peril!

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Exercises

1. Diffusion equation:

- (a) Show that the diffusion master equation is equivalent to

$$\frac{d}{dt} \langle n_i \rangle = \frac{D}{\Delta x^2} \sum_j (\langle n_j \rangle - \langle n_i \rangle)$$

where the sum on j runs over nearest neighbors of site i .

- (b) Show that $\rho(x, t) = \langle n_i \rangle$ with $x = i \Delta x$ satisfies $\partial_t \rho = D \partial_x^2 \rho$ as $\Delta x \rightarrow 0$.
- (c) Generalize the result to a d -dimensional hypercubic lattice.

2. Consider the decay $A \rightarrow 0$ on a single lattice site. Map the problem from the master equation to the Doi hamiltonian to the shifted field theory. Show that the $\mathcal{D}_{\tilde{\phi}}$ integration yields the expected result

$$\partial_t \phi = -\lambda \phi + n_0 \delta(t)$$

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Exercises

3. Write down the Doi hamiltonian for the reversible reaction $\ell A + m B \rightleftharpoons n C$, with rates λ for the forward reaction and μ for the reverse reaction. Here ℓA , for example, means ℓ A particles are required for the reaction.

4. Coherent states:

- (a) Determine the coefficients C_n in the expansion of the coherent state $|\phi\rangle = \sum_n C_n |n\rangle$
- (b) Confirm the identity

$$\delta_{mn} = \frac{1}{m! \pi} \int \phi^{*m} \phi^n e^{-|\phi|^2} d^2 \phi$$

- (c) Use the results from (a) and (b) confirm $\hat{1} = \int \frac{d^2 \phi}{\pi} |\phi\rangle \langle \phi|$.
Note that $d^2 \phi = d(\text{Re} \phi) d(\text{Im} \phi)$.

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