Many-particle entanglement

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Abstract

The quantum state of a system of k particles can be viewed as a tensor of order k. Local stochastic operations on a quantum state correspond to the application of linear maps to the indices of the corresponding tensor and are studied in entanglement theory and implemented in current quantum information science experiments.

Interestingly, the notion of tensor transformation is at the heart of the study of the computational complexity of algebraic problems such as the multiplication of matrices. Strassen's breakthrough algorithm for the multiplication of d-by-d matrices that runs faster than your standard d^3 high-school algorithm spurred a whole development of tensor theory.

In these notes (based on lectures given at the Boulder summer school on condensed matter physics 2018), I will explain these connections in the case $k = 3$ and show how they elucidate our understanding of both entanglement and matrix multiplication. For more recent developments on the subject, please see my presentations on tensors at https://prezi.com/user/christandl/.

I will review these connections and present recent results on quantum information-inspired functionals that can serve as obstructions for asymptotic tensor transformations.

Contents

1. Introduction

I will give an introduction to the world of tensors from the point of view of a resource theory: the resource tensor will be transformed by operations on its indices (local operations) to another tensor. Inspired by information-theory, this will be done on two levels: the single-copy level, where one tensor is to be transformed and the multi-copy level, where many tensor products of tensors can be transformed (for the multicopy case, I refer to the URL given in the abstract). Both setups were considered by Strassen in the context of fast algorithms for matrix multiplication (matrix multiplication is a bilinear operation that can be viewed as a tensor of order three), but are so natural to quantum information theory (where k-particle quantum states can be viewed as tensors of order k) that the these two worlds naturally meet - for the benefit of both.

1.1. Tensors

We denote the d-dimensional complex vector space by $\mathbb{C}^d$. We call

$$t \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$$

a *tensor* of order three. This definition as well as most results presented extend naturally to tensors of order k. For simplicity of the exposition, however, we will restrict to $k = 3$, the first nontrivial case. Most results mentioned, however, will hold true for arbitrary k.

Choosing e_i as the vector with a one in the i'th position and zeros otherwise, we can write t in coordinates in the *computational basis*

$$t = \sum_{i,j,k=1}^{a,b,c} t_{ijk} e_i^A \otimes e_j^B \otimes e_k^C,$$

where we drop the superscript when clear from the context. It is sometimes nice to think of a tensor as the list of c matrices $\{T_k\}_{k=1}^c$, where $(T_k)_{ij} = t_{ijk}$, obtained by fixing the third index. These matrices are called slices. Note that one can also slice the tensor by fixing the first index or by fixing the second index.

When we talk about tensors, we mostly have not chosen an inner product yet. The length of the tensor is then also not well-defined. Note that by endowing $\mathbb{C}^d$ with the standard inner product $\langle v, v' \rangle = \sum_i \bar{v}_i v'_i$ and extending it to three tensor factors via

$$\langle \alpha \otimes \beta \otimes \gamma, \alpha' \otimes \beta' \otimes \gamma' \rangle := \langle \alpha, \alpha' \rangle \langle \beta, \beta' \rangle \langle \gamma, \gamma' \rangle$$

and by demanding linearity, we might regard tensors as (unnormalised) three-particle quantum states.

Example 1.1 (Matrix)**.** Let $c = 1$, then t may be regarded as a matrix, since

$$t = \sum_{i,j}^{a,b} t_{ij1} e_i^A \otimes e_j^B \otimes e_1^C = \sum_{i,j}^{a,b} t'_{ij} e_i^A \otimes {e'}_j^B,$$

where we defined ${e'}_j^B := e_j^B \otimes e_1^C$ and $t'_{ij} := t_{ij1}$. Two important examples (where $a = b$) are the unit matrix (or the Einstein-Podolsky-Rosen (EPR) state of a levels)

$$\mathrm{EPR}_a = \mathbf{1}_a = \sum_{i=1}^{a} e_i^A \otimes e_i^B$$

and the rank one matrices

$$v \otimes w = \sum_{i,j=1}^{a,b} v_i w_j e_i^A \otimes e_j^B,$$

where $v = \sum_{i=1}^{a} v_i e_i$ and $w = \sum_{j=1}^{b} w_j e_j$. A nice special case is $v = w$ being the all-ones vector, leading to the all ones matrix

$$\sum_{i,j=1}^{a} e_i^A \otimes e_j^B.$$

Example 1.2 (Three qubits)**.** When $a = b = c = 2$, two important tensors are

$$\mathrm{GHZ} = \sum_{i=1}^{2} e_i^A \otimes e_i^B \otimes e_i^C = e_1^A \otimes e_1^B \otimes e_1^C + e_2^A \otimes e_2^B \otimes e_2^C$$

and

$$W = e_1^A \otimes e_1^B \otimes e_2^C + e_1^A \otimes e_2^B \otimes e_1^C + e_2^A \otimes e_1^B \otimes e_1^C$$

known as the Greenberger-Horne-Zeilinger state (GHZ) and the W-state, respectively. Of course, we also have product states, and matrices, embedded in the three possible ways.

Example 1.3 (Matrix Multiplication)**.** Let A and B be two d-by-d matrices, i.e. elements of $\mathbb{M}_d$. Denote by MaMu_d'' the map that multiplies the two.

$$\begin{aligned} \mathrm{MaMu}_d'' &: \mathbb{M}_d \times \mathbb{M}_d \to \mathbb{M}_d \\ \mathrm{MaMu}_d'' &: (A, B) \mapsto A \cdot B \end{aligned}$$

This map is bilinear and can be turned into a trilinear map by dualising the target space

$$\begin{aligned} \mathrm{MaMu}_d' &: \mathbb{M}_d \times \mathbb{M}_d \times \mathbb{M}_d^* \to \mathbb{C} \\ \mathrm{MaMu}_d' &: (A, B, C) \mapsto C(A \cdot B) \end{aligned}$$

Writing the matrices in coordinates, noting that $\mathbb{M}_d = \mathbb{C}^d \otimes (\mathbb{C}^d)^*$,

$$\begin{aligned} A &= \sum_{ij} a_{ij} e_i \otimes e_j^* \\ B &= \sum_{ij} b_{j'k} e_{j'} \otimes e_k^* \end{aligned}$$

we find

$$A \cdot B = \sum_{ik} \sum_{j} a_{ij} b_{jk} e_i \otimes e_k^*$$

and with

$$C = \sum_{ij} c_{k'i'} e_{i'}^* \otimes e_{k'}$$

we find

$$C(A \cdot B) = \sum_{ijk} a_{ij} b_{jk} c_{ki}.$$

Freely identifying vector spaces and their duals we see that MaMu_d' can be regarded as the tensor

$$\mathrm{MaMu}_d = \sum_{i,j,k=1}^{d} e_{ij} \otimes e_{jk} \otimes e_{ki},$$

where $e_{ij} = e_i \otimes e_j$. Surprisingly, this is exactly the tensor obtained by arranging three EPR_d states pairwise between the vector spaces. This observation connects quantum information theory to that of the study of the algebraic operation matrix multiplication.

Exercise 1.4. Show that t is product iff both matrices obtained by flattening (grouping two of the tensor spaces) $t_{A:BC}$ and $t_{AB:C}$ are rank one.

1.2. Tensor product and direct sum

An important operation on tensors is the (Kronecker) tensor product, which turns the tensors

$$t \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$$
$$t' \in \mathbb{C}^{a'} \otimes \mathbb{C}^{b'} \otimes \mathbb{C}^{c'}$$

into the tensor

$$t \otimes t' \in (\mathbb{C}^a \otimes \mathbb{C}^{a'}) \otimes (\mathbb{C}^b \otimes \mathbb{C}^{b'}) \otimes (\mathbb{C}^c \otimes \mathbb{C}^{c'}).$$

Observe that the resulting tensor is again a tensor of order three, which we indicated by the brackets. In some circumstances, it might be useful to regard it as a tensor of order six, but we will indicate this in case we wish to do so.

Whereas the tensor product is very natural from a quantum physical point of view, the direct sum

$$t \oplus t' \in (\mathbb{C}^a \oplus \mathbb{C}^{a'}) \otimes (\mathbb{C}^b \oplus \mathbb{C}^{b'}) \otimes (\mathbb{C}^c \oplus \mathbb{C}^{c'})$$

is less so, although its mathematical operation is more transparent: the resulting tensor has a $2 \times 2 \times 2$ block structure with t and t' on the main diagonal. The tensor product in contrast has natural $a \times b \times c$ block structure with the tensors $t_{ijk}t'$ in the block ijk.

Example 1.5 (Unit tensor)**.** Denote by GHZ_1 the product tensor $e_1 \otimes e_1 \otimes e_1$ in $\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}$. Note that

$$\mathrm{GHZ}_r := \underbrace{\mathrm{GHZ}_1 \oplus \cdots \oplus \mathrm{GHZ}_1}_{r} = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$$

is the tensor with ones down the diagonal. In analogy to the unit matrix it is known as the unit tensor and in algebraic complexity usually denoted by $\langle r \rangle$. Note that this notation has nothing to do with the bra-ket notation from quantum mechanics. Note that GHZ_2 is the GHZ state we introduced before. We call GHZ_r the GHZ state with r levels.

Example 1.6 (Matrix multiplication)**.** Let

$$t = \sum_i e_i \otimes e_i \otimes e_1 \in \mathbb{C}^r \otimes \mathbb{C}^r \otimes \mathbb{C}$$
$$t' = \sum_i e_i \otimes e_1 \otimes e_i \in \mathbb{C}^r \otimes \mathbb{C} \otimes \mathbb{C}^r$$
$$t'' = \sum_i e_1 \otimes e_i \otimes e_i \in \mathbb{C} \otimes \mathbb{C}^r \otimes \mathbb{C}^r$$

be three embedded EPR pairs of r levels. Then it is easily seen that

$$\mathrm{MaMu}_d = t \otimes t' \otimes t''.$$

Example 1.7 (Recursivity)**.** It is a simple but important observation that

$$\mathrm{EPR}_a \otimes \mathrm{EPR}_b = \mathrm{EPR}_{ab}$$
$$\mathrm{GHZ}_a \otimes \mathrm{GHZ}_b = \mathrm{GHZ}_{ab}$$
$$\mathrm{MaMu}_a \otimes \mathrm{MaMu}_b = \mathrm{MaMu}_{ab}$$

and thus

$$\underbrace{\mathrm{EPR}_2 \otimes \cdots \otimes \mathrm{EPR}_2}_{n} = \mathrm{EPR}_{2^n}$$
$$\underbrace{\mathrm{GHZ}_2 \otimes \cdots \otimes \mathrm{GHZ}_2}_{n} = \mathrm{GHZ}_{2^n}$$
$$\underbrace{\mathrm{MaMu}_2 \otimes \cdots \otimes \mathrm{MaMu}_2}_{n} = \mathrm{MaMu}_{2^n}$$

Exercise 1.8. Show that $t \oplus t = \mathrm{GHZ}_2 \otimes t$.

2. Single-Copy

We will now discuss transformations between tensors.

2.1. Restrictions

Let

$$t \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$$
$$t' \in \mathbb{C}^{a'} \otimes \mathbb{C}^{b'} \otimes \mathbb{C}^{c'}$$

we say that t *restricts to* t' and write $t \geq t'$ if there exist linear maps

$$A : \mathbb{C}^a \to \mathbb{C}^{a'}$$
$$B : \mathbb{C}^b \to \mathbb{C}^{b'}$$
$$C : \mathbb{C}^c \to \mathbb{C}^{c'}$$

s.th. $(A \otimes B \otimes C)t = t'$. We say that t and t' are *equivalent* and write $t \cong t'$ if $t \geq t'$ and $t' \geq t$. Note that $\geq$ is a preorder on the set of tensors (i.e. transitive and reflexive). If the equivalence is implemented via an invertible transformation, we say that t and t' are isomorphic, and write $t \sim t'$. Note that isomorphic tensors necessarily have $a = a'$, $b = b'$ and $c = c'$, but that this is not necessary for equivalence.

Note that in quantum information language t restricts to t' corresponds to saying that t can be transformed into t' by stochastic local operations (and classical communication) or SLOCC for short. The maps A, B and C then corresponds to the (unnormalised) Kraus operators implementing the transformation. This can be seen as follows. Any LOCC operation Λ acting on a density operator ρ is also a separable operation, i.e. of the form

$$\Lambda(\rho) = \sum_i (A_i \otimes B_i \otimes C_i)\rho(A_i \otimes B_i \otimes C_i)^\dagger$$

for some linear maps A_i, B_i, C_i. Post-selection on an outcome i corresponds to considering the map

$$\Lambda'(\rho) = (A_i \otimes B_i \otimes C_i)\rho(A_i \otimes B_i \otimes C_i)^\dagger$$

restricting in addition to pure states $\rho = tt^\dagger$ and dropping the subscript we find

$$\Lambda'(tt^\dagger) = \big((A \otimes B \otimes C)t\big)\big((A \otimes B \otimes C)t\big)^\dagger$$

which shows that LOCC with post-selection (called SLOCC) when applied on pure states corresponds to the notion of restriction. Note that SLOCC operations can be implemented without classical communication.

Example 2.1 (Matrix)**.** Let $c = 1$, then t is equivalent to EPR_r, where r is the Schmidt rank of t. The equivalence can be implemented as follows. First bring t into its Schmidt form

$$\sum_{i=1}^{r} \sigma_i e_i \otimes e_i$$

by applying local unitaries. Then apply the matrix $\text{diag}(\sigma_1^{-1}, \ldots, \sigma_r^{-1}, \underbrace{0, \cdots, 0}_{a-r})$ to the first space. We thus see that t is isomorphic to EPR_r. It is now easy to show that $\text{EPR}_r \geq \text{EPR}_{r'}$ iff $r \geq r'$. Combining the two statements we find that $t \geq t'$ iff $R(t) \geq R(t')$, where $R(t)$ is the Schmidt rank of t.

Example 2.2 (Three qubits)**.** There are six equivalence (even under isometry) classes of tensors (excluding the nul tensor) given by GHZ, W, the EPR state between AB, BC or AC and the product state. The relationship among those *entanglement classes* is as follows (plus permutations of EPR pairs): The relation

$$e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 \geq e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2)$$

is implemented by the map $A : e_{1/2} \mapsto e_1$. The other relations are

$$e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \geq e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2)$$

and

$$e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) \geq e_1 \otimes e_1 \otimes e_1.$$

Which maps implement these relations? Can you show that there are not other relations?

Example 2.3 (Matrix Multiplication)**.** In 1969, Strassen showed that

$$\text{GHZ}_7 \geq \text{MaMu}_2.$$

Note that this statement is equivalent to writing down MaMu_2 as a sum of seven product states

$$\text{MaMu}_2 = \sum_{i=1}^{7} \alpha_i \otimes \beta_i \otimes \gamma_i,$$

since the linear maps

$$\begin{aligned} A &: e_i \mapsto \alpha_i \\ B &: e_i \mapsto \beta_i \\ C &: e_i \mapsto \gamma_i \end{aligned}$$

then implement the restriction. Here is the decomposition, which can be verified through direct computation

$$\begin{aligned} \text{MaMu}_2 = &(e_- \otimes e_2) \otimes (e_2 \otimes e_+) \otimes (e_1 \otimes e_1) \\ &+ (e_1 \otimes e_1) \otimes (e_- \otimes e_2) \otimes (e_2 \otimes e_+) \\ &+ (e_2 \otimes e_+) \otimes (e_1 \otimes e_1) \otimes (e_- \otimes e_2) \\ &- (e_- \otimes e_1) \otimes (e_1 \otimes e_+) \otimes (e_2 \otimes e_2) \\ &- (e_2 \otimes e_2) \otimes (e_- \otimes e_1) \otimes (e_1 \otimes e_+) \\ &- (e_1 \otimes e_+) \otimes (e_2 \otimes e_2) \otimes (e_- \otimes e_1) \\ &+ \text{EPR} \otimes \text{EPR} \otimes \text{EPR} \end{aligned}$$

where we introduced the states $e_\pm := e_1 \pm e_2$. The decomposition is easy to remember, since once one has the first vector, the first six are obtained by cyclic shift and simultaneous bit flip — i.e. applying the symmetries of MaMu_2.

Exercise 2.4. Verify Strassen's decomposition of MaMu_2 into 7 terms.

It is natural to ask for the *cost* of creating a tensor t by restriction from GHZ states,

$$R(t) := \min\{r : \mathrm{GHZ}_r \geq t\}$$

and for the *value*

$$Q(t) := \max\{r : t \geq \mathrm{GHZ}_r\}.$$

They are commonly known as the rank and subrank of t, as they both reduce to the matrix rank in the case $c = 1$. In quantum information theory, it is natural to call $\log R(t)$ the (single copy) SLOCC entanglement cost and $\log Q(t)$ the (single copy) SLOCC distillable entanglement.

It is a very difficult problem (NP hard) to compute the tensor rank of a tensor and thus more generally to decide whether one tensor restricts to another one. Even slow systematic methods are hard to come by.

Exercise 2.5. Show that tensor rank is not multiplicative under the tensor product.

Question 2.6 (Rank)**.** Two seemingly simple research questions on the rank of small tensors:

- The rank of MaMu_2 is less than seven (indeed it is equal to seven). Actually, any odd cycle of EPR pairs has rank at most $2^k - 1$, where k is the number of EPR pairs (`https://arxiv.org/abs/1606.04085`). Can you improve on this? Even a rank 30 decomposition for the five-cycle would be nice to have (or a new rank 31 decomposition).
- The rank of $W \otimes W$ as a six-tensor is strictly smaller than 9 (`https://arxiv.org/abs/1705.09379`). What is rank of $\mathrm{MaMu}_2 \otimes \mathrm{MaMu}_2$ when regarded as a six tensor?

2.2. Degenerations

It turns out that allowing approximate restrictions will allow us to use algebro-geometric methods. We say that t *degenerates* to t', and write $t \trianglerighteq t'$ if there are tensors $\{t_i\}_{i \in \mathbb{N}}$ s.th. $t_i \geq t'$ for all i and $\lim_{i \to \infty} t_i = t$.

Obviously $t \geq t'$ implies $t \trianglerighteq t'$ (take all $t_i = t$), but there are cases, where $t \not\geq t'$ but $t \trianglerighteq t'$. The easiest (and quite canonical one) is given in the following example.

Example 2.7 (3 qubits)**.** Let $\epsilon > 0$. Note that

$$\begin{aligned} e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 &\geq \frac{1}{\epsilon} \left((e_1 + \epsilon e_2)^{\otimes 3} - e_1 \otimes e_1 \otimes e_1 \right) \\ &= e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 + O(\epsilon). \end{aligned}$$

Since ϵ was arbitrary, we find

$$\mathrm{GHZ}_2 \trianglerighteq W.$$

Question 2.8 (Border rank)**.** MaMu_2 corresponds to the triangle graph of EPR_2 pairs. What is the rank of the diamond graph? My best upper bound is 28, my best border rank upper bound is 27, my best numerical border rank upper bound is 26. Can you beat me (or just make my numerics rigorous)? Border rank $\underline{R}(t)$ is defined by $\min\{r : \mathrm{GHZ}_r \trianglerighteq t\}$.

Let $G = GL(a) \times GL(b) \times GL(c)$, where GL denotes the general linear group. Define the orbit of t as

$$G.t := \{t' : t \sim t'\}$$

Thinking of $G.t$ as a subset of $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$, we can take its Euclidean closure, which we denote by $\overline{G.t}$. It turns out that this is equivalent to considering the Zariski closure (the topology, where closed sets are cut out by zero-sets of polynomials). Note that by definition of degeneration we have

$$\overline{G.t} = \{t' : t \trianglerighteq t'\}.$$

$\overline{G.t}$ is an algebraic variety and is thus given by the zero-set of a set of polynomials.

Thus in order to show that $t \not\trianglerighteq t'$, it is sufficient to provide a polynomial p on the set of tensors, s.th. $p(\overline{G.t}) = 0$, but $p(t') \neq 0$.

Example 2.9 (3 qubits)**.** The Cayley hyperdeterminant

$$\begin{aligned} Det(t) := {} & t_{111}^2 t_{222}^2 + t_{112}^2 t_{221}^2 + t_{121}^2 t_{212}^2 + t_{211}^2 t_{122}^2 \\ & - 2t_{111}t_{112}t_{122}t_{222} - 2t_{111}t_{121}t_{212}t_{222} - 2t_{111}t_{211}t_{122}t_{222} \\ & - 2t_{112}t_{221}t_{212}t_{121} - 2t_{112}t_{122}t_{221}t_{211} - 2t_{121}t_{122}t_{212}t_{211} \\ & + 4t_{111}t_{122}t_{212}t_{221} + 4t_{112}t_{121}t_{211}t_{222} \end{aligned}$$

satisfies

$$Det(A \otimes B \otimes Ct) = det(A)det(B)det(C)Det(t)$$

as we will see a little later. In addition is easily seen to give $Det(\mathrm{GHZ}) = 2$, but $Det(W) = 0$. Thus $W \not\trianglerighteq \mathrm{GHZ}$. This completes our understanding of the relations between the different three qubit entanglement classes.

Since $\overline{G.t}$ is invariant under the action of G (by definition), polynomials that are covariant (i.e. transform like a vector in a representation) with respect to G will do, and those can be constructed with help of representation theory. We have seen this in the case of the Cayley hyperdeterminant, but want now to understand the general construction. For this, we need to use some representation theory of $\mathrm{GL}(d)$. This is not too difficult, since it is a generalisation of the familiar $\mathrm{GL}(2)$ (which is just $\mathbb{C}\backslash\{0\} \times \mathrm{SL}(2)$).

It turns out that the polynomial irreducible representations of $\mathrm{GL}(d)$ can be described by the action of the diagonal matrices (known as the maximal torus), $t = \mathrm{diag}(t_1, \ldots, t_d)$. Then for every Young diagram $\lambda = (\lambda_1, \ldots, \lambda_d)$ with non-increasing integers λ_i, i.e. $\lambda_i \geq \lambda_{i+1}$, there is a unique (up to isomorphism) irreducible representation V_λ specified by a unique (up to scale) highest weight vector v_λ, s.th.

$$t.v_\lambda = t^\lambda v_\lambda,$$

where $t^\lambda = t_1^{\lambda_1} \cdots t_d^{\lambda_d}$. $n = \sum_i \lambda_i$ is the degree of the representation. The name *highest weight* refers to the fact that all other vectors v with

$$t.v = t^w v,$$

for a vector of integers w will satisfy $\lambda > w$ in lexicographical order.

Example 2.10. If $d = 2$ and we restrict to $\mathrm{SL}(2)$, then $\lambda_1 - \lambda_2 = 2j$ is the spin of the representation.

There is a nice way of realising these representations due to Weyl. Consider the space $(\mathbb{C}^d)^{\otimes n}$ which has a natural action of $\mathrm{GL}(d)$ by simultaneous action on all factors. Since any polynomial finite dimensional representation of $\mathrm{GL}(d)$ breaks up into irreducible representations (just as for a finite group), we obtain a direct sum decomposition

$$(\mathbb{C}^d)^{\otimes n} = \bigoplus_\lambda V_\lambda \otimes [\lambda],$$

where λ ranges over all such diagrams with degree n. $[\lambda]$ is the multiplicity space and is actually irreducible as a representation of the symmetric group S_n which acts on $(\mathbb{C}^d)^{\otimes n}$ and whose action commutes with that of $\mathrm{GL}(d)$. This statement is known as Schur-Weyl duality. We will denote the projector onto $V_\lambda \otimes [\lambda]$ by P_λ.

Considering now

$$(\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c)^{\otimes n} = (\mathbb{C}^a)^{\otimes n} \otimes (\mathbb{C}^b)^{\otimes n} \otimes (\mathbb{C}^c)^{\otimes n},$$

and applying three times the Schur-Weyl duality, we find

$$\bigoplus_{\lambda_A, \lambda_B, \lambda_C} V_{\lambda_A} \otimes V_{\lambda_B} \otimes V_{\lambda_C} \otimes [\lambda_A] \otimes [\lambda_B] \otimes [\lambda_C]$$

Suitable polynomials are now obtained by a clever choice of a highest weight vector v in one of the components $\lambda = (\lambda_A, \lambda_B, \lambda_C)$. The polynomial and v depend in the following way on each other:

$$p_v(t) := v^* t^{\otimes n}.$$

Example 2.11 (Cayley's hyperdeterminant)**.** Choose $\lambda_A = \lambda_B = \lambda_C = (2, 2)$ and the highest weight vector

$$v = (\mathrm{Sing}_{12} \otimes \mathrm{Sing}_{34}) \otimes (\mathrm{Sing}_{12} \otimes \mathrm{Sing}_{34}) \otimes (\mathrm{Sing}_{13} \otimes \mathrm{Sing}_{24}),$$

where $\mathrm{Sing}_{12} := e_1 \otimes e_2 - e_2 \otimes e_1$ is the singlet between tensor factor 1 and 2. Using

$$(A \otimes A)\mathrm{Sing} = \det(A)\mathrm{Sing}$$

it is now easy to show that the Cayley hyperdeterminant has the previously claimed covariance property with respect to the triple $\mathrm{GL}(2)$ action.

Exercise 2.12. We have seen that the W-state does not degenerate to GHZ. Show that $W^{\otimes 2}$ restricts to GHZ.

This ends the exposition of pure state entanglement transformation under SLOCC and its basic connection to matrix multiplication, algebraic geometry and representation theory. Interestingly, restriction and degeneration seem to simplify when we consider the asymptotic restriction relation, denoted by $t \gtrsim t'$, which is defined as

$$t^{\otimes n + o(n)} \geq t'^{\otimes n}.$$

(Asymptotic restriction stays the same when we consider degeneration instead.) For more information and recent developments, see the the exposition `https://prezi.com/user/christandl/` of the two papers on entanglement polytopes (`https://arxiv.org/abs/1208.0365`) and the quantum functionals (`https://arxiv.org/abs/1709.07851`).