I. The Magnetization Field.

In theoretical physics, one objective is to explain what has been seen in past experiments; the other is to predict what will be seen in future experiments. In either case, the theory must be expressed in terms of quantities conveniently accessible to experimental measurement. In condensed matter physics, these measurements, with a few notable exceptions, involve properties averaged over regions of space large compared with atomic dimensions, but small compared with typical spatial variations discernible by the apparatus. In particular, phenomena in practical applications of magnetically ordered materials are not conveniently discussed in terms of individual electronic spins $\vec{S}$. Instead, one uses the concept of a magnetization vector field $\vec{M}(\vec{x},t)$ per unit volume, defined as a suitable average over the spin vectors in a neighborhood of the position vector $\vec{x}$, evaluated at time t. In a magnetic insulator, with the spins localized at lattice positions, that average is

$$\frac{1}{\Omega_r}\sum_{\vec{r}_i\in\Omega_r}\mu_i\vec{S}_i$$

where μ_i is the Bohr magneton of the i th spin localized within the small volume Ω_r surrounding $\vec{r}$. Since the $\vec{S}_i$ are quantum mechanical objects, $\vec{M}$, in principle, also obeys quantum mechanics, but since the number of spins in this definition is large, one expects to be able to treat $\vec{M}$ as a classical vector. Up to the time of this writing, there seems to be no convincing, widely accepted experimental demonstations of residual quantum aspects, although they must, of course, exist. In the case of magnetic metals, there are no well defined lattice sites that carry the usual half- integral or integral spins. One then must define a local spin density operator $\vec{s}_{\alpha,\beta} = \psi^\dagger_\alpha(\vec{r})\vec{\sigma}_{\alpha\beta}\psi_\beta(\vec{r})$ in terms of the Pauli matrix vector $\vec{\sigma}$ of the individual electrons and the two-component electron fields ψ. This is averaged over a small volume to give the $\vec{M}$ field as in the previous case, and once again $\vec{M}$ behaves almost like a classical vector field[1]. It will be treated as such in most of this book. To avoid involvements in still unanswered basic questions in the magnetism of metals, we shall use the simpler picture of magnetic insulators wherever it is necessary to establish a correspondence between the field- and atomic viewpoints.

An equation is needed to describe the evolution of $\vec{M}(\vec{r},t)$. Even in the classical approximation to this field, this equation must reflect the realities of the spin-dependent forces in the magnetic sample, and of the motion of the constituent spins. Two approaches are possible: a relatively easy one that uses only symmetry principles, such as crystal symmetries, time-reversal symmetries and purely phenomenological considerations, and a considerably harder one that starts with the dynamics of the individual quantum spins coupled to their environment and to one another through exchange and dipolar forces. A drawback of the purely phenomenological approach is that , in contrast to the second, 'atomistic'approac h, it cannot determine the constants that enter the equations for $\vec{M}$.

Actually, a complete theory would establish equations of motion, not only of $\vec{M}$, but also of all the degrees of freedom of the environment in which it is embedded and to which $\vec{M}$ is firmly coupled. Perhaps the most important degrees of freedom are the orbital ones, seen by the spins through spin-orbit coupling. Also, in metals the dynamics of the conduction electrons is accompanied by magnetic field variations, which are coupled to the magnetization. Evidently, a researcher studying the motion of $\vec{M}(\vec{r},t)$ from the point of view of applications does not wish to carry around the 'baggage' of these extra degrees of freedom. To get rid of these, one can imagine solving for the equations of motion of the extra variables, with $\vec{M}(\vec{r},t)$ regarded as given, and the solution will be some functional of that given field. Then one would substitute the solution into the coupling terms in the equation for $\vec{M}(x,t)$, leaving one with an equation for $\vec{M}(\vec{r},t)$ alone. As will be shown later on, this program can actually be carried out in some cases. However, there is a price to be paid for this: the resulting equation of motion cannot be local in the time variable. The reason is well known from mechanics: The evolution of a 'complete' set of dynamical variables has the property that, given the values of the variables at *just one* sin *gle time* in the past completely determines the values of the variables at the present time. If the equations for some of the variables are solved in terms of the remaining ones, the time evolution of the latter will depend on their entire past. However, it is usually the case that, if one is interested only in sufficiently slow time variations, the resulting equations become local in time, at least asymptotically. In most of the problems discussed here, the equations for $\vec{M}$ may be deemed local in time at rates of variation slower than several gigahertz.

To begin with, we describe the phenomenological approach. First, suppose that the temperature is sufficiently far below the Curie point so that the magnetization is very nearly saturated. A field $\vec{H}$ applied to it will result in a torque $\vec{M} \times \vec{H}$. The newtonian equation of motion is then $d\vec{A}/dt = \vec{M} \times \vec{H}$, where $\vec{A}$ is the angular momentum. Associated with the spin $\vec{S}$ of an electron is an angular momentum $g\mu\vec{S}/\gamma$, and this relation survives the coarse-graining operation relating $\vec{S}$ to $\vec{M}$. Thus we have

$$d\vec{M}/dt = \gamma\vec{M} \times \vec{H} \tag{1}$$

The constant γ is known as the gyromagnetic ratio. Next, we must define an effective field $\vec{H}$. The change in energy dE due to an increment $d\vec{M}$ in magnetization is $d\mathrm{E} = -\int \vec{H} \cdot d\vec{M}\ dv$. The total magnetic energy E$[\vec{M}]$ is the sum of various contributions, that, in the field picture can be written as volume integrals of corresponding energy densities $E(\vec{r})$. The main contributions are

1. The electromagnetic energy $\mathrm{E}_{em} = -\int \vec{M} \cdot \vec{H}_{em} dv$.due to an imposed electromagnetic field.

2. The exchange energy, ultimately the result of electron-electron interaction plus the exclusion principle.These two effects conspire to make

a correlated state of two neighboring spin orientation the state of lowest energy (parallel orientation in ferromagnets, antiparallel in antiferromagnets.)

As a result, in a ferromagnet, the fully aligned state of all the spins is the ground state; any deviation from it costs energy. In terms of the field approach, this suggests defining the exchange energy as

$$\mathrm{E}_{ex} = \int J\ell^2 \left((\nabla M_x)^2 + (\nabla M_y)^2 + (\nabla M_z)^2 \right) dv \tag{2}$$

(usually abreviated as $\int J\ell^2 \left(\nabla\vec{M}\right)^2 dv$), where the so-called exchange constant J is a measure of stiffness of the fully aligned state. ℓ is a length, of the order of one lattice spacing. This form holds for sufficiently slow spatial variations of $\vec{M}$; terms with higher derivatives of $\vec{M}$ need to be included for fast variations. A constant value of J and only even derivatives evidently assumes that deviation of alignment in one spatial direction is exactly as costly as a deviation in the opposite direction. Well inside a uniform material, this is a good assumption. It must break down at a surface aswell

as at grain boundaries. Expression (2) for E_{ex} assumes cubic symmetry of the spin sites. For lower symmetries, the three different terms may have different coefficients.

3. Dipolar interaction energy

$$E_{dip} = \int\int \left\{ \vec{M}(\vec{r}) \cdot \vec{M}(\vec{r}')/|\vec{r} - \vec{r}'|^3 - 3\left(\vec{M}(\vec{r}) \cdot (\vec{r} - \vec{r}')\right)\left(\vec{M}(\vec{r}') \cdot (\vec{r} - \vec{r}')\right)/|\vec{r} - \vec{r}'|^5 \right\} dv d^3v' \tag{3}$$

This rather clumsy form of that interaction is often more profitably written as an interaction energy in an additional scalar field $\psi(\vec{r})$:

$$\mathrm{E}_{dip} = \int \left(\frac{1}{8\pi} (\nabla\psi(\vec{r}))^2 - \vec{M}(\vec{r}) \cdot \ \nabla\psi(\vec{r}) \right) dv \tag{4}$$

The 'equation of motion' of that field is obtained by minimizing this energy with respect to ψ. It is $\nabla^2\psi = 4\pi\nabla\cdot\vec{M}$. Writing down the usual integral that solves this Poisson equation, and substituting in the foregoing expression yields the more elaborate form for E_{dip}.

4. Crystalline anisotropy energy. Due to spin-orbit coupling, written $\lambda \sum \vec{L}_i \cdot \vec{S}_i$, the individual spins see the underlying lattice, and sense its symmetry.As discussed above, one does not wish to keep track of the orbital variables; one wishes to eliminate them in favor of the spin variables. In the detailed quantum theory,this is accomplished by treating the spin orbit energy as a perturbation. A simple and very successful (though not completely legitimate way) is to treat the individual spins in the expression $\vec{L}_i \cdot \vec{S}_i$ as though they were classical objects. Schrödinger perturbation theory then involves the matrix elements of the L_i,only, yielding polynomials in the spin components alone. The resulting energy expression is referred to as a spin Hamiltonian. Clearly, the symmetry of the spin Hamiltonian thus derived must be thesame as the lattice symmetry. (In Schrödinger perturbation theory, the non-locality in time mentioned above is lost. A time-dependent formulation will reveal it, but is of no importance as long as the ionic states connected to the ground state by the L operator are sufficiently high in energy.) In the coarse grained, continuum view, one will get corresponding polynomials in $\vec{M}(\vec{r})$. For example, in a cubic lattice, the dominant form will be

$$\mathrm{E}_{an} = \int K(M_x^4 + M_y^4 + M_z^4) dv \tag{5}$$

where K is a constant found from the microscopic theory, carried to fourth order in the spin-orbit parameter λ. In a uniaxial crystal, with axis along x_3, the lowest order term would be $\int K M_z^2 dv$, provided, of course that the underlying spin is greater than 1/2. In a crystal with inversion symmetry, the polynomial in the $M-$ components must be even.

Other forms of energy will be discussed as needed. We must now define an effective magnetic field for use in the equation of motion (1) This is defined as the functional derivative

$$\vec{H}(\vec{r}) = -\tilde{d}\mathrm{E}/\tilde{d}\vec{M}(\vec{r}) = -Lim_{\delta\vec{M}(x)\to 0}\left(\mathrm{E}[\vec{M}(\vec{r}) \ + \delta\vec{M} \ ] - \mathrm{E}[\vec{M}(\vec{r})]\right) / \left(\delta\vec{M}(\vec{r})dv\right) \tag{6}$$

For our purposes it suffices to note that it reduces to an ordinary derivative of the energy density $E(\vec{x})$, except in the expression for the exchange energy, where an integration by parts must be performed to isolate the increment $\delta\vec{M}(x)$. Then the effective exchange field becomes $\vec{H}_{ex} = -J\ell^2\nabla^2\vec{M}$, at least for increments that vanish on the sample boundaries.

The dimensions of the constants encountered so far are as follows: $\vec{M}^2$ has dimensions of energy per unit volume, as does $\vec{H}\cdot\vec{M}$. It follows that J must be dimensionless. Therefore, if $\tilde{J}$ is the exchange energy found in the microscopically calculated exchange coupling $-\tilde{J}\ \vec{S}_1\cdot\vec{S}_2$ of two neighboring spins (considered dimensionless), then $J = \tilde{J}\ \Omega^2/(g\mu_B)^2$, where Ω is the volume of a lattice cell. The anisotropy constant, K, for the quartic anisotropy in equation (6) must have dimensions of a volume, and can similarly be related to the microscopically calculated anisotropy constant $\tilde{K}$. (For anisotropy represented by a polynomial of order $2n$, the dimensions of of K will be Ω^{n-1}.) It will frequently be convenient to measure $\vec{M}$ in units of the saturation magnetization M_s, so that $|\vec{M}| = 1$. With that choice, the various terms in E must be multiplied by the appropriate power of M_s.

Evidently, the equation of motion (1) is nonlinear when H is the full effective field H_{eff}. Another awkward feature is that the three equations are subject to one constraint $(\vec{M}(\vec{r},t))^2 = M_s^2$, and therefore should be reducible to only two equations. In some fortunate cases such a step also helps to ameliorate the nonlinear aspects. Possibly the simplest reduction expresses the direction of $\vec{M}(\vec{r},t)$ in terms of its polar coordinates $\{\theta(\vec{r},t),\phi(\vec{r},t)\}$, using the normalization $|\vec{M}| = 1$. Resolving the increment $d\vec{M}$ along the

directions of increasing θ and ϕ results in the equations

$$\begin{aligned} \frac{d\cos\theta}{dt} &= \frac{\tilde{d}\mathrm{E}}{\tilde{d}\phi} \\ \frac{d\phi}{dt} &= -\frac{\tilde{d}\mathrm{E}}{\tilde{d}\cos\theta} \end{aligned} \tag{7}$$

(For all the good it does, these equations have Hamiltonian form, with conjugate variables $\cos\theta, \phi$. They are usually more conveniently written in the form $\dot{\theta} = -\frac{\tilde{d}\mathrm{E}}{\sin\theta \tilde{d}\phi}$, $\sin\theta\dot{\phi} = \frac{\tilde{d}\mathrm{E}}{\tilde{d}\theta}$). To evaluate the functional differentiations, E must be expressed in terms of θ and ϕ. In all except the exchange contributions to E, the components of $\vec{M}$ are simply written in polar form. Only the exchange term needs slightly more processing. It becomes

$$\mathrm{E}_{ex} = \int \frac{1}{2}\left((\nabla\theta)^2 + \sin^2\theta(\nabla\phi)^2\right) dv \tag{8}$$

As the result, the equations of motion may be written

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{1}{\sin\theta}\nabla\cdot(\sin^2\theta\nabla\phi) - \frac{\partial E}{\sin\theta\partial\phi} \\ \frac{\sin\theta d\phi}{dt} &= -\nabla^2\theta + \sin\theta(\nabla\phi)^2 + \frac{\partial E}{\partial\theta} \end{aligned} \tag{9}$$

Finally we briefly describe the small motions of the magnetization about a particular direction $\vec{M}_0$, allowing the equations to be linearized. Usually $\vec{M}_0$ is taken to be along the $z-$ axis, and the components m_x, m_yin the x and y directions are considered small. Then

$$M_z = \sqrt{M_s^2 - m_x^2 - m_y^2} = M_s - \frac{1}{2M_s}(m_x^2 + m_y^2) + o(m^4) \approx M_s \tag{10}$$

So to lowest order in m, the $z-$ component may be replaced by M_s wherever it occurs. There remain two coupled linear differential equations for m_x and m_y. In the infinite medium at least, the equations are translationally invariant; therefore they admit of solutions proportional to $e^{i\vec{k}\cdot\vec{x}}$, called spin waves or magnons. Even though these linear modes do not immediately help with solutions for large motions, they do sometimes provide some insight in cases in which linearization (and small improvements thereon) is not an option. Singling out the 'imperturbable' part of $\vec{M}$ to

be along the x_3- axis, *by fiat* rather than in a 'natural' way by an imposed magnetic field or by anisotropy, breaks symmetry, in the sense that any other direction could have been picked equally well. Such symmetry breaking implies that there must be a mode of excitation, the so called Goldstone mode[2], with lowest excitation energy equal to zero. Magnons are examples of this mode in the limit in which the 'natural' direction for the unperturbed magnetization is absent. When there is an applied steady field or a suitable anisotropy field, the lowest magnon energy is no longer zero (unless demagnetizing effects depress it to zero or beyond). Denoting the small transverse magnetization components (m_x, m_y) by the vector $\vec{m}$, we expand it in a Fourier series $\vec{m}(\vec{r},t) = \sum_k \vec{m}_k(t)e^{i\vec{k}\cdot\vec{r}}.$, along with the inverse $\vec{m}_k(t) = \frac{1}{V}\int \vec{m}(\vec{r},t)e^{-i\vec{k}\cdot\vec{r}}dv$ integrated through the sample volume V. In $k-$ space, the linearized equations are quite simple, especially if the form (5) of the dipolar energy is used; even before linearization,the dipolar energy gives an effective field whose Fourier component is

$$H_{dip,k} = -4\pi \ \frac{\vec{k}(\vec{k}\cdot\vec{M}_k)}{k^2} \tag{11}$$

where $\vec{M}_k$ is the Fourier component of the full $\vec{M}$. Similarly, the effective exchange field is $H_{ex,k} = -Jk^2\vec{M}_k$. In this brief survey we omit the anisotropy fields. Write $a_k = m_{x,k}+im_{y,k}$, (so that $a^*_{-k} = m_{x,k}-im_{y,k}$, since $\vec{m}^*_{-k} = \vec{m}_k$ to keep $\vec{m}(\vec{r},t)$ real). with this notation, the linearized equation of motion becomes

$$i\dot{a}_k = A_k a_k + B_k a^*_{-k} \tag{12}$$

where, for $k \neq 0$,

$$\begin{aligned} A_k &= \gamma(H_{app} - 4\pi N_z M_s) + 4\pi\gamma M_s \frac{k_x^2+k_y^2}{k^2} + \gamma M J k^2 \qquad (13)\\ B_k &= 2\pi\gamma M_s \frac{k_x^2+k_y^2}{k^2} \end{aligned}$$

and where it has been assumed that a uniform field H_{app} has been applied in the z direction. Equation (12) must break down for k values smaller than reciprocal sample dimensions. Then a full boundary value problem must be solved, which leads to the familiar demagnetizing fields; whereof the term $4\pi N_z M_s$ is an example for a simple ellipsoid of revolution about H_{app}. Equation (13a), together with its conjugate, have an oscillatory solution

with natural frequency ω_k, where

$$\omega_k^2 = (\omega_H - \omega_M N_z + \omega_{ex}\ell^2 k^2)(\omega_H - \omega_M N_z + \omega_{ex}\ell^2 k^2 + \omega_M \sin^2 \xi_k) \quad (14)$$

with $\omega_H = \gamma H_{app}$, $\omega_M = 4\pi\gamma M_s$, $\omega_{ex} = \gamma M_s J/\ell^2$ and $\sin^2 \xi_k = (k_{x_1}^2 + k_{x_2}^2)/k^2$. This dispersion relation is shown in Fig 1 for two angles: $\xi = 0$ and $\xi = \pi/2$. Note that for $k = 0$ and $\xi = 0$, the frequency is $\omega_H - \omega_M N_z$, so for $H_{app} < 4\pi M N_z$ it becomes negative. Viewing ω as an excitation frequency, a negative excitation frequency means that the unexcited state must be unstable. This agrees with magnetostatics: $H_{app} - 4\pi M N_z$ is the internal field; when negative, it opposes the magnetization direction, so that the state is unstable. The resulting collapse into a domain structure cannot be described by a linear theory. This result holds, even though for small k, analysis based on plane waves must be abandoned and replaced by a theory taking the boundary into account. Fortunately, for such small wavenumbers, the exchange term can be scrapped, and only the essentially scale independent dipolar term needs to be retained. Such an analysis was carried out by L.R. Walker, and the modes in that régime are called Walker modes[3].

There have been some modest successes in going beyond linear theory by expanding M_z to one more order, writing $M_z = M_s - \frac{1}{2} m^+ m^- / M_s$, with $m^\pm = m_x \pm i m_y$ This results in double and triple terms on the right hand side of the equations of motion (13a) and its conjugate. The interaction between these and the linear terms generally causes some degradation of the amplitude of the linear solution in ferromagnetic resonance experiments, as descibed in the next lecture. For example, the ferromagnetic resonance frequency (which is one of the Walker modes, and therefore not contained in expression (14)) is $\omega_{res} = \omega_H - \omega_M(N_z - N_T)$, where N_T is the transverse demagnetizing factor of the spheroidal sample. Except for absurdly large $H-$ fields, this frequency is degenerate (i.e. synchronous) with a whole manifold of spin wave frequencies. Such synchronism inevitably will cause a transfer of energy from the uniform precession frequency ω_{res} to the degenerate modes, and therefore broaden the resonance line. For sufficiently large signal amplitudes, the transfer may even become catastrophic, resulting, initially, in exponential growth of the degenerate waves.[6] In conclusion of this section, we note three more points. 1. There is an upper limit to the allowed values of wavenumbers: they cannot go beyond the zone boundary. The continuum picture already breaks down for somewhat smaller values,

particularly if only first derivative terms are retained in the exchange energy. 2. In samples smaller than a typical domain wall thickness, the excitation energies no longer form a continuum as in the above theory. The very lowest excitation energy (even in the absence of any applied or anisotropy fields) must be the energy of a domain wall (which may be regarded as an 'adult' form of localized magnon). For such small samples the entire spctrum must be viewed as discrete. This inhibits spatially non-uniform motion of $\vec{M}$, except in so far as non-uniformity may be enforced by certain boundary conditions, such as pinning. 3. In crystals with more than one magnetic ion per unit cell, there will be several additional branches of the spin wave spectrum located well above the one given by equation (14). These are the analogue of optical phonon modes, and involve large angles between the spins of the ions in each unit cell. Accordingly, their minimum frequency occurs at $k = 0$ and is of order $\gamma\times$ the exchange field between pairs within the cell. These branches can become interesting when their dipersion relations intersect with the dispersion relations of phonons or other modes, but they will not be considered further in this work

II. Relaxation Processes. Brief Review of Conventional Theory of Distributive Damping

II. 1. General observations.

Given the total magnetic energy, the precessional motion of the magnetization field $\vec{M}(\vec{r}, t)$ can be determined, but to descibe the manner in which that precession decays, a damping torque must be added to the right hand side of equation (1), Sect.I . Probably the oldest proposal for the form of such a term is that of Landau and Lifshitz [4]. They propose a form purely local in space and time, with a structure that forces the magnetization to seek complete alignment with the local effective field. The term

$$-\gamma\alpha_{LL}\vec{M} \times (\vec{M} \times \vec{H}_{eff}), \tag{15}$$

where α_{LL} is a constant, has that form. It also preserves the magnitude of $\vec{M}$ as it should, if the damping mechanism cannot change the saturation magnetization. An alternative form

$$-\gamma\alpha_G\vec{M} \times d\vec{M}/dt \tag{16}$$

was proposed by Gilbert [5]. For small values of the α 's, the two forms are equivalent. Substituting the undamped motion $d\vec{M}/dt = \gamma\vec{M} \times H_{eff}$ in the form (16) gives the form (15), with

$$\alpha_{LL} = \gamma\alpha_G$$

to lowest order. So, for sufficiently weak damping, and with the α 's treated as empirical constants, there are no *practical* differences between the two forms. On the other hand, when the physical origin of the loss torque is investigated, the Gilbert form usually wins out. However, as will be discussed later, not all relaxation mechanisms take the form of either (15) or (16)

Before examining some physical mechanisms of the loss torque, some limitations of either form should be noted.

a. Experiments on macroscopic samples usually (though not always) measure a spatial average $\left\langle \vec{M}(t) \right\rangle = \frac{1}{V}\int \vec{M}(\vec{r},t)dv$ over the sample volume V. Unless the sample dimensions are smaller than a typical domain wall thickness, the equation of motion for $< \vec{M}(t) >$ differs considerably from that for $\vec{M}(\vec{r},t)$. While the non-dissipative part of the total torque remains more or less intact, the dissipative part is quite different and does not conserve the magnitude of $\left\langle \vec{M}(t) \right\rangle$. The correct equation of motion for $\left\langle \vec{M}(t) \right\rangle$ will have dissipative terms similar to the Bloch terms found in nuclear resonance, in addition to the one of LL-G form.

b. For some mechanisms, the LLG part of the damping is neither local in time nor in space, except in appropriate limits. This is the result of eliminating degrees of freedom or other features one does not wish to carry along, such as lattice distortions, electromagnetic fields, imperfections, etc.

c. Even in the local limit, the damping term may have the more general form

$$-\gamma\alpha\vec{M} \times \vec{Q} \tag{17}$$

where $\vec{Q}$ is a function of the components of $\vec{M}$, dimensionally equivalent to $d\vec{M}/dt$.

In regard to a., it is helpful to distinguish two kinds of damping: *intrinsic* damping, which degrades magnetic energy by coupling to external degrees of freedom, and *distributive* damping, which degrades the interesting magnetic degree of freedom (such as the spatial average $\left\langle \vec{M}(t) \right\rangle$) only by scat-

tering to other magnetic degrees of freedom, without degrading the *total* magnetic energy. That the LL-G form is of the former kind is demonstrated as follows:

$$\begin{aligned}
\frac{d\mathrm{E}}{dt} &= \int \frac{\mathrm{E}}{\vec{M}(\vec{r},t)} \cdot \frac{\partial \vec{M}(\vec{r},t)}{\partial t} d^3x \qquad (18)\\
&= -\alpha_{LL} \int \frac{\mathrm{E}}{\vec{M}} \cdot \left(\vec{M} \times \left(\vec{M} \times \frac{\mathrm{E}}{\vec{M}} \right) \right) dv \\
&= -\alpha_{LL} \int \frac{\mathrm{E}}{\vec{M}} \cdot \left(\vec{M} \left(\vec{M} \cdot \frac{\mathrm{E}}{\vec{M}} \right) - M_s^2 \frac{\mathrm{E}}{\vec{M}} \right) dv \\
&= -\alpha_{LL} \int \left(\vec{M} \cdot \frac{\mathrm{E}}{\vec{M}} \right)^2 - M_s^2 \left(\frac{\mathrm{E}}{\vec{M}} \right)^2 dv
\end{aligned}$$

In the final, fully relaxed state, the effective -$\frac{\mathrm{E}}{\vec{M}}$ must be parallel to $\vec{M}$, and then the right hand side vanishes. Prior to that, the right hand side gives the rate of decay towards that state. For the common case in which E is a bilinear functional of $\vec{M}$, (i.e. homogeneous of degree 2), this is especially evident. In that case , the right hand side becomes approximately

$$-2\alpha_{LL} \int \left(\mathrm{E}^2 - \mathrm{E}_{eq}^2 \right) dv$$

where E_{eq} is the magnetic energy in the relaxed state, assuming its magnetization to be uniformly saturated. If it is assumed that the coupling between the space-averaged $\vec{M}$ and the spatially varying magnetic degrees of freedom is very weak, one is tempted to regard the latter as the external agency. One then attempts to write the total energy as the sum of an energy expressible in terms of $<\vec{M}>$ and an energy depending only on the spatially varying part of $\vec{M}$, and a coupling term. (That kind of damping of $<\vec{M}>$ is analogous to the decline of forward momentum of a ball on a horizontal pinball machine: The ball's total energy does not change, it is merely transferred from average, systematic forward motion to the 'non-uniform' degrees of freedom, in this example the transverse momenta. The coupling term in that case is furnished by the pins.)

II.2.. Brief Review of Conventional Theory of Distributive Damping.

We begin with a discussion of distributive damping. Ferromagnetic resonance linewidth is a case in point, but we shall discuss its theory only

to the extent that it furnishes hints to more general situations. The actually observed magnetic degree of freedom is a spatial average: the so-called uniform mode excited by a microwave field, applied at right angles to a steady magnetizing field. The theory of ferromagnetic resonance, as well as of antiferromagnetic resonance draws heavily on the spin wave approximation described in the previous section. As indicated there, terms just beyond the linear approximation couple the uniform mode to spin waves (aswell as the spin waves to one another.) This coupling transfers some of the uniform part of the magnetization into spin waves and thus is lost to direct observation.

We begin by re-stating the total energy in spin wave approximation expanded up to triple terms in the spin wave amplitudes: Up to triple terms, it will be convenient to write $\mathrm{E} = \mathrm{E}_1 + \mathrm{E}_2$, where

$$\begin{aligned} \mathrm{E}_1 &= \sum_{k\neq 0}\left(A'_k a^*_k a_k + B_k a^*_k a^*_{-k} + \sum_{k'\neq 0}\left(p_{k'} a_{k'} a^*_k a_{k-k'} + \; p^*_{k'} a^*_{k-k'} a_k a^*_{k'}\right)\right) \quad (19) \\ \mathrm{E}_2 &= \omega'_{res} a^*_0 a_0 + \sum_{k\neq 0}\left(p_k a_k a_{-k} a^*_0 + p^*_k \; a^*_0 \; a_k a^*_k\right) \end{aligned}$$

Here, the primed symbols

$$\begin{aligned} A'_k &= A_k + \frac{1}{2}\omega_M N_z \sum a^*_k a_k \\ \omega'_{res} &= \omega_{res} + \frac{1}{2}\omega_M (N_z - N_T) \sum a^*_k a_k \end{aligned}$$

have been renormalized by the prevailing spin wave excitations. The equations of motion for the a 's are obtained, either quantummechanically by Heisenberg's equations of motion, or classically by defining conjugate variables a_k and a^*_k, using $-i\mathrm{E}$ as Hamiltonian (so that $\dot{a}_k = -i\partial \mathrm{E}/\partial a^*_k$ etc) . We shall have no need to specify the structure of the constants p_k. The star on a symbol stands for complex conjugate (or Hermitian conjugate in the quantum formulation). Note that each term in equation (19) carries zero momentum, as appropriate for a perfect medium. However, in practice, translational invariance is broken by imperfections, irregular boundaries, etc. Hence the above energy expression must be supplemented by a term, which, in lowest order is bilinear, and has the general form

$$\mathrm{E}_{imp} = \sum_{k,k'} \left\{ q_{kk'} a_k a^*_{k'} + (r_{k,k'} a_k a_{k'} + \text{c.c.}) \right\} \quad (20)$$

This furnishes the simplest case of distributive decay. Haas and Callen[7] take the quantummechanical view of the decay of the number of quanta in the uniform mode , $n_0 = a_0^* a_0$, into spin wave quanta. The simplest process, which they call two-magnon decay, is due to the part of E_{imp} that involves one uniform mode and one spin wave. Keeping, for simplicity, only the first term in the sum (20), that part is $\sum_k (q_{0k} a_0 a_k^* + q_{k0} a_k a_0^*)$. Treating these terms as a perturbation, they apply Fermi's golden rule to calculate the transition rate from a state with n_0 quanta in the uniform precession and n_k quanta in states with various k numbers to new states with quanta $n_0 \mp 1$ and $n_k \pm 1,$. The net rate of this transition is $\lambda = \frac{2\pi}{\hbar} \sum_k |q_{k0}|^2 (n_0 - n_k) \delta(\hbar\omega_k - \hbar\omega_{res})$. If $|q_{k0}|^2$ can be replaced by some representative value $|q|^2$, this result may be written $\frac{2\pi}{\hbar} \rho |q|^2 (n_0 - n_k)$, where ρ is the density of $k-$ states with energies degenerate with $\hbar\omega_{res}$. So we have $\dot{n}_0 = -\lambda n_0$. (Usually n_k is close to its thermal equilibrium value and is much smaller than n_0). On the other hand, to treat the problem classically, it is necessary to supplement the equation of motion

$$i\dot{a}_0 = \omega'_{res} a_0 + \sum_k q_{k0} a_k \tag{21}$$

with an equation for a_k, namely

$$i\dot{a}_k = \omega_k a_k + \sum_k q_{0k} a_0 \tag{22}$$

Setting $a_0 = A_0(t) e^{-i\omega_{res} t}$ and $a_k = A_k(t) e^{-i\omega_k t}$, where A_0 and A_k vary slowly with time, we get

$$\begin{aligned} i\dot{A}_0 &= \sum_k q_{k0} A_k e^{i(\omega_{res} - \omega_k)t} \\ i\dot{A}_k &= \sum_k q_{0k} A_0 e^{i(\omega_k - \omega_{res})t} \end{aligned} \tag{23}$$

Assuming that A_k was zero a very long time ago, the second equation gives $iA_k(t) = \int_{-\infty}^{t} q_{0k} A_0(t') e^{i(\omega_k - \omega_{res})t'} dt'$, and substituting this result in the first gives

$$\begin{aligned} \dot{A}_0(t) &= -\sum_k |q_{0k}|^2 \int_{-\infty}^{t} A_0(t') e^{i(\omega_{res} - \omega_k)(t - t')} dt' \\ &= -\sum_k |q_{0k}|^2 \int_0^{\infty} A_0(t - t') e^{i(\omega_{res} - \omega_k)t'} dt', \end{aligned} \tag{24}$$

a simple demonstration that elimination of unfavored degrees of freedom causes history dependence in the equations of motion of the favored ones. If we now regard A_0 as very slowly varying and take it outside the integral, then, (noting that $\int_0^v e^{ixt}dt = iP(1/x) + \pi\delta(x)$) we find

$$\dot{A}_0(t) = -iA_0 \sum_k |q_{0k}|^2 P\left(\frac{1}{\omega_{res} - \omega_k}\right) - A_0\pi \sum_k |q_{0k}|^2 \delta(\omega_{res} - \omega_k) \quad (25)$$

This is the same result for the damping as in the Haas-Callen quantum treatment, except there they used energy units for the q, and here we use frequency units. Here we also get a bonus: the detuning of the resonance frequency by the coupling (the term involving the principal value). That extra term carries phase information of a_0, which, of course, is lost if only the decay of $n_0 = a_0^* a_0$ is considered. What has happened here is that part of the uniform mode has scattered into spin waves degenerate with it. The total magnetic energy, however, has not changed. As is readily verified, $n_0 + \sum n_k$, the total increment in $\vec{M}$ (aswell as the increment in total energy) due to the excitation is a constant of the motion under E_{imp}. A different kinds of damping that does not conserve $n_0 + \sum n_k$ results from triple and higher terms in the energy expression (19). For example, the triple term in E_2 gives

$$i\dot{a}_0 = \omega'_{res} a_0 + \sum_{k\neq 0} \left\{ \left(p_k a_k a_{-k} + p_k^* \; a_{-k}^* a_k^* \right) \right\} \quad (26)$$

The second term inside the sum amounts to a largely secular shift in a_0, and is removed by the contact transformation $a_0 \rightarrow a_0 + \sum_{k\neq 0} p_{-k}^* n_k / \omega'_{res}$. This small shift in the value around which the transverse magnetization oscillates will be neglected. It is clear that the very low order perturbation theory implied by Fermi's golden rule will involve creating two spin waves via $a_{-k}^* a_k^*$, and then destroying them with $a_k a_{-k}$. However to 'consummate' this process, it is necessary that energy is conserved. In frequency terms, this means that only those k 's can participate for which the frequency ω_0 on the left equals the frequency $\omega_k + \omega_{-k} = 2\omega_k$ on the right. The golden rule thus gives a decay rate $\sum |p_k|^2 n_k(n_k+1)\delta(\omega_0 - 2\omega_k)$. Using a similar lowest approximation in a purely classical equations of motion treatment gives the same result, except that $n_k(n_k+1)$ is replaced by n_k^2, because there is no such thing as classical excitation out of the vaccuum. Of course, except at extremely low temperatures, the difference is of no consequence. Here,

we again have distributive loss: the total magnetic energy does not change (although the total number of quanta is not conserved). However, one then finds a phenomenon that shows up the severe limitations of linearized theory, rendering it of dubious value in describing the large motions of $\vec{M}$ typical in magnetic recording. If coupling among the $k \neq 0$ waves is neglected, the first term in the sum of the last equation will run with time according to $\sum_{k\neq 0} p_k A_k A_{-k} e^{-2i\omega_k t}$ with constant amplitudes A_k, A_{-k}. Coupling among the waves will make these amplitudes functions of time, slowly varying only if the couplings are small. In addition, the phases of the A_k may be expected to be random. The time variation of this sum is sufficiently complex so that there is a temptation to replace it by a random function of time $f(t)$. Equation (26) then begins to resemble the Langmuir equation, except that it lacks a friction term. Anticipating that the ultimate effect of the random force will be frictional, one adds a damping term $-i\eta a_0$ to the right hand side. One then solves the equation, and determines η by demanding that the statistical properties of $f(t)$ be such that the statistics of the solution a_0 be Boltzmann statistics. The result is a form of the fluctuation-dissipation theorem. However, this procedure completely neglects the response of the spin waves to the motion of a_0. As is well known[6], this recoil becomes overwhelming for large enough a_0. Using E_1, eqtn (19), the interesting part of the equation of motion of a_k is found to be

$$i\dot{a}_k = \omega_k a_k + p_k^* a_{-k}^* a_0 + \sum_{k'\neq 0} p_{k'} a_{k'} a_{k-k'}$$

the response to a_0 being given by the second term on the right. Ignoring the sum on the right, which only gives rapidly varying terms, we have

$$i\dot{a}_k = \omega_k a_k + p_k^* a_{-k}^* a_0$$

Writing $a_k = A_k e^{i\omega_k t}$, $a_0 = A_0 e^{i\omega_{res} t}$ gives

$$i\dot{A}_k = p_k^* A_{-k}^* e^{i(2\omega_k - \omega_{res})t} A_0 \tag{27}$$

For spin waves such that $2\omega_k = \omega_{res}$, this, and the complex conjugate equation yield exponentially growing amplitudes A_k, A_{-k} if A_0 is considered fixed, no matter how small, unless these equations are respectively supplemented by damping terms$\mp i\eta_k A_{\pm k}$. (When there is such damping, its value sets a threshold that A_0 must exceed to result in this blow-up.) Naturally, this exponential growth does not go on for ever: the enhanced level of these

spin waves results in increased damping of A_0. For the case in which A_0 is driven by a microwave field, a compromise is reached, in which the spin waves are excited far above their thermal number, and the decay rate of A_0 is then large enough to keep its value just below the threshold beyond which spin waves satisfying $\omega_0 = 2\omega_k$ will blow up altogether. But these spin waves can in principle lead to exponential growth of other waves, unless these are adequately damped (it is not difficult to satisfy the condition $\omega_k = \omega_{k-k'} + \omega_{k'}$ for a whole manifold of k' 's. In fact, for all $|\vec{k}'| < |\vec{k}|$).

II. 3. Distributive damping of large motions. General Observations. We must now reexamine distributive damping for motions of $\vec{M}$ so large that linearized theory, even when extended to allow for mode-mode coupling, is not appropriate.This is obviously a serious matter in the case of magnetization reversal. Unless intrinsic damping is very heavy, the magnetization vector will precess relatively rapidly, with a more slowly, but steadily, opening angle $\theta(t)$ such that the corresponding a_0 reaches the threshold value discussed above, and then the further time evolution must be re-evaluated. As before, interest will center on the motion of the spatial average of $\vec{M}$. Only the case of uniaxial anisotropy will be considered. Filling in all the contributions to H_{eff}, the equations of motion are

$$\frac{d\vec{M}(\vec{r})}{\gamma dt} = \vec{M}(\vec{r})\times(\vec{H}+\vec{H}_K)+A\vec{M}(\vec{r})\times\nabla^2\vec{M}(\vec{r})+\vec{M}(\vec{r})\times\int dv'\left(\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|^3}-3\frac{(\vec{r}-\vec{r}')\,((\vec{r}-\vec{r}')\cdot M(r')}{|\vec{r}-\vec{r}'|^5}\right. \tag{28}$$

where dependence of $\vec{M}(\vec{r})$ on t is implied. In averaging over $\vec{r}$ we note that the exchange torque contributes nothing to that average. Neglecting possible boundary effects, integrating it by parts gives $-A\int\nabla\vec{M}(\vec{r})\times\nabla\vec{M}(\vec{r})dv = 0$. This is due to the fact that the exchange torque lies entirely within the spin system, so that the total spin is a constant of the motion. In the dipolar torque, write $\vec{r}' = \vec{r}-\vec{\rho}$. Using the notation $(a\times b)_i = \epsilon_{ijk}a_jb_k$ for the components of the vector product, and introducing the correlation tensor $C_{ij}(\vec{\rho}) = \frac{1}{\Omega}\int dv_r\vec{M}_i(\vec{r})\vec{M}_j(\vec{r}-\vec{\rho})$, the ith component of the averaged dipolar torque becomes

$$\epsilon_{ijk}\int dv_\rho\left(\frac{C_{jk}(\vec{\rho})}{\rho^3}-3\frac{\rho_k\rho_lC_{jl}(\vec{\rho})}{\rho^5}\right) \tag{29}$$

summation over repeated indices being implied. If now C depends only on the magnitude but not the direction of ρ, the result is zero, since l must

then equal k in the second term, and the angular integration gives 1/3, so that the first term is cancelled. Thus the conclusion is that if the pair autocorrelation of the magnetization depends only on the magnitude of the distance (a case of no practical importance), the dipolar torque cannot change the motion of the averaged $\vec{M}$. But in the general case, the dipolar torque, prior to averaging, is $\vec{M}(\vec{r}) \times \vec{Q}(\vec{r})$, where

$$\vec{Q}(\vec{r}) = \int dv_\rho \left(\frac{\vec{M}(\vec{r}-\vec{\rho})}{\rho^3} - 3\frac{\vec{\rho}\left(\vec{\rho}\cdot\vec{M}(\vec{r}-\vec{\rho})\right)}{\rho^5} \right) \tag{30}$$

and the average torque is not zero. The reason is that the underlying lattice can absorb some of the total spin angular momentum. An averaged torque of zero is characteristic of equilibrium in the absence of external fields. In that state the correlation function should indeed be independent of direction, at least if boundary effects are neglected. The same remark applies to the anisotropy torque $\vec{M} \times \vec{H}_K = (KM_y(\vec{r})M_z(\vec{r}), -KM_x(\vec{r})M_z(\vec{r}), 0)$ whose average is obviously zero in equilibrium. In the general case, (28) and (30) are hard to handle for large signals, except in one rather trivial case, which does, however provide some insight: Suppose that the spatial variation of the magnetization vector is restricted to a particular direction, $\vec{D}$, which makes an angle β with the applied field and has azimuthal angle 0. If θ, ϕ are the polar angles of the magnetization vector, the component of $\vec{M}$ along $\vec{D}$ is

$$M_D = M_s(\cos\theta\cos\beta - \sin\theta\sin\beta\cos\phi)$$

In this case we benefit greatly from the alternative formulation from the potential field formulation of the dipolar interaction discussed earlier. That interaction was $\int \left((\frac{1}{2}\nabla\psi)^2 + \nabla\psi\cdot\vec{M} \right) dv$, where $\nabla^2\psi = -4\pi\ \nabla\cdot\vec{M}$, (or $\mathrm{div}\vec{B} = 0$). In the present case, this means, writing $\nabla\psi = h_{demag}$, and denoting by ζ the coordinate along $\vec{D}$,

$$\frac{dh_{demag}}{d\zeta} = -\frac{dM_D}{d\zeta}$$

the simplest solution of which (though by no means the only one) is $h_{demag} = -4\pi M_D$. So the dipolar energy becomes simply $-\frac{1}{2}\int dv M_D^2$. If in the form (16) for Q one could express $\vec{M}$ as a Fourier series, that formula would become very simple; unfortunately the constraint $\vec{M}^2(\vec{r}) = M_s^2$, all $\vec{r}$, would

be extremely hard to handle (except, of course, in the linearized limit). Allowing for the full nonlocal character of (30) is very hard. However, it may be argued that the essential effects lie not so much in the nonlocality, but rather in the dipolar symmetry of the expression. So we replace it by a local term in the following manner: Write, with $r_{1,2,3} = x, y, z$, respectively,

$$\vec{Q}'_i(\vec{r}) = -\nabla^2 M_i - \frac{\partial}{\partial r_i}\nabla\cdot\vec{M} - \frac{3}{10}\frac{\partial^2 M_i}{\partial r_i^2} \tag{31}$$

This is obtained by expanding the integrand in (30) up to the fourth power of the components of $\vec{\rho}$, and performing the angular integrations, ignoring the fact that the integral over $|\rho|$ then diverges. The divergence notwithstanding, the form 31 is still reasonable, if $\vec{M}$ does not vary too rapidly with position. This is seen by making use of Rolle's theorem. Even with this simplification, the constraint on $\vec{M}^2$ is still hard to handle.

A formalism should be used that satisfies the constraint $\vec{M}(\vec{r},t)^2 = M_s^2$ as an identity. The simplest way to do this is to write the equations of motion in polar coordinates (θ,ϕ), with polar axis along the field $\vec{H}$, longitude ϕ and latitude θ. We recapitulate the way this is done. Chose a set of rectangular coordinates along θ increasing, ϕ increasing, and the third axis along $\vec{M}$ (see figure). Chosing M_s to equal unity, the components of $\vec{M}$ in this system are (0,0,1). On the other hand, the components of the increment $d\vec{M}$ are ($d\theta, \sin\theta d\phi, 0$). The components of H_{eff} are $-(\mathrm{E}/\theta, \mathrm{E}/(\sin\theta\phi), 0)$, where we have set $M_s = 1$. A small amount of work shows that in this coodinate system

$$\begin{aligned} \mathrm{E}_{ex} &= \frac{1}{2}\int dvA\left((\nabla\theta)^2 + \sin^2\theta\,(\nabla\phi)^2\right) \\ \mathrm{E}_H &= -\int dvH\cos\theta \end{aligned} \tag{32}$$

and, restricting ourselves to the uniaxial case with anisotropy axis along $\theta = 0$,

$$\mathrm{E}_{ani} = -\frac{1}{2}\int dvK\cos^2\theta \tag{33}$$

The dipolar energy, on the other hand, looks ugly in all systems of coordi-

nates. In polar coordinates it reads

$$\mathrm{E}_{dip} = \int \frac{dv'}{|\vec{r}-\vec{r}'|^3} \left(\begin{array}{c} \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') - \\ -3\ \ (\sin\theta(c_1\cos\phi + c_2\sin\phi) + c_3\cos\theta)) \left(\begin{array}{c} \sin\theta'(c_1\cos\phi' + c_2\sin\phi') + \\ +c_3\cos\theta' \end{array} \right) \end{array} \right) \tag{34}$$

Here, the undashed θ, ϕ are taken at position $\vec{r}$, and the dashed symbols are taken at $\vec{r}'$. The c 's denote the direction cosines $\frac{x-x'}{|r-r'|}$, etcetera. Then

$$\frac{\partial E_{dip}}{\partial\theta} = 2\int \frac{dv'}{|\vec{r}-\vec{r}'|^3} \left(\begin{array}{c} (-\sin\theta\cos\theta' + \cos\theta\sin\theta'\cos(\phi-\phi')) - \\ -3\,(\cos\theta(c_1\cos\phi + c_2\sin\phi) - c_3\sin\theta)) \left(\begin{array}{c} \sin\theta'(c_1\cos\phi' + c_2\sin\phi') + \\ +c_3\cos\theta' \end{array} \right) \end{array} \right) \tag{35}$$

$$\frac{\partial E_{dip}}{\partial\phi} = 2\int \frac{dv'}{|\vec{r}-\vec{r}'|^3} \left(\begin{array}{c} \cos\theta\cos\theta' - \sin\theta\sin\theta'\sin(\phi-\phi') - \\ -3\ \ (\sin\theta(-c_1\sin\phi + c_2\cos\phi) + c_3\cos\theta)) \left(\begin{array}{c} \sin\theta'(c_1\cos\phi' + c_2\sin\phi') + \\ +c_3\cos\theta' \end{array} \right) \end{array} \right) \tag{36}$$

In some fortunate circumstances , these expressions simplify greatly. For example, consider a plane slab of ferromagnetic material, magnetized within the plane by a field H, which for this geometry, evokes an only negligible demagnetizing field. Suppose we wish to study a disturbance that depends only on the coordinate, x, say, normal to the plane. Then, rather than carry out the integrations over y, and z in the last two equations, it is preferable to proceed via the fact that in the interior, $\mathrm{div}\vec{B} = 0$, which, in this case of x variation only, gives a dipolar field with x - component only, such that

$$\begin{aligned} h_{dip} &= -4\pi M_x \\ &= -4\pi\sin\theta\cos\phi \end{aligned}$$

where boundary conditions on the faces of the slab have been ignored.The associated dipolar energy is $\frac{1}{2}h_{dip}^2 + 4\pi M_x h_{dip} = -\frac{1}{2}(4\pi)^2\sin^2\theta\cos^2\phi$, so that

$$\frac{\partial E_{dip}}{\partial\theta} = -(4\pi)^2\sin\theta\cos\theta\cos^2\phi,\ \ \frac{\partial E_{dip}}{\partial\phi} = (4\pi)^2\sin^2\theta\sin\phi\cos\phi$$

To begin with, we leave this field out altogether, except for its uniform, demagnetizing part, where needed. As noted before, the equations of motion become

$$\frac{\partial\cos\theta}{\partial t} = -\frac{\partial E}{\partial\phi}$$

$$\frac{\partial \phi}{\partial t} = -\frac{\partial E}{\partial \cos\theta}$$

for two conjugate variables ϕ and $\cos\theta$. Explicitly, we have

$$\begin{aligned} \frac{\partial \theta}{\partial t} &= \frac{A}{\sin\theta}\nabla\cdot\left(\sin^2\theta\nabla\phi\right) \\ \frac{\partial \phi}{\partial t} &= A(\cos\mathrm{ec}\theta)\nabla^2\theta + A\ \cos\theta(\nabla\phi)^2 + H\sin\theta + H_K\sin\theta\cos\theta \end{aligned} \tag{37}$$

where $H_K = K - 4\pi N_z$. If there is intrinsic damping, (local in time), and it has Landau Lifshitz form, then terms $-\alpha\frac{\partial E}{\partial\theta}$ and $-\alpha\frac{\partial E}{\sin\theta\partial\phi}$ must be added to the first and second of eqtns 23 respectively. From the first of eqtns 37, we immediately see why the equation satisfied by the spatial average $\int dv\vec{M}(\vec{r},t)$ cannot possibly be tortured into LL-G form: We have

$$\begin{aligned} \frac{d}{dt}\int dv M_z(\vec{r},t) &= \int dv\frac{\partial}{\partial t}\cos\theta(\vec{r},t) \\ &= -\int dv\nabla\cdot\left(\sin^2\theta\nabla\phi\right) \\ &= 0, \end{aligned}$$

if the integral over the bounding surface is assumed to vanish (for example by pinning θ to be zero). This is because, under exchange and $z-$ directed fields alone $\int dv M_z$ is a constant of the motion. Thus, the only way $<M_z>$ can (and will) decay is by *intrinsic* damping. As for the averaged transverse components, $<\vec{M}_T>$, they likewise are not affected by the exchange coupling. But they are more 'sensitive' than $<M_z>$ to distributive disturbances. This is a consequence of the priviledged position of M_z along the axial magnetic fields that are tacitly assumed to dominate. Distributive disturbances can cause changes in M_z (i.e.θ) that are slow, plus small rapidly oscillating variations that time-average to zero. On the other hand, the motion of $M_T(\vec{r},t)$ is dominated by uniform precession in the longitudinal fields, and disturbances will dephase the precession, generally by different amounts at different positions. This causes the spatial average $<\vec{M}_T>$ to decay. No comparable mechanism affects $<M_z>$. Some modest further progress, even for the formidable looking expressions 35 and 36, is possible, if the axial fields (applied and anisotropy) are larger than the dipolar field. This will be discussed later.

II. 4. Some Propagation effects. Breaking Spin Waves. Moving Domain Walls as Shock Fronts.

Before discussing distributive relaxation mechanisms beyond the spin wave approximation, we digress to discuss a nonlinear aspect of spin waves that uses a simple version of the mathematics needed there. In the absence of intrinsic damping, the magnetic system is its own microcanonic ensemble, since coupling to external degrees of freedom is then absent. This means that the magnetic energy E is a constant. So we can classify the states of the system by the value of this constant. . A particular one of these is : $\theta = \theta_0 =$ constant (although ϕ will vary in time: $\phi = (H + H_K \cos\theta_0)t$. Its energy is a function of θ_0 only, and , for the right sign of H, the lowest energy has $\theta_0 = 0$. Furthermore, it is also a stationary state. Another group of *exact* states with fixed energy has $\theta = \theta_0$ as before, but $\phi = \vec{k}\cdot\vec{r} - \omega_k t$, a group that has an extra energy $Ak^2\sin^2\theta_0$. These states are precisely the usual spin wave states, except that in most of the literature, θ_0 is set equal to zero, because the deviation of M_z from 1 is only second order in the spin wave amplitudes. So the first, quite trivial, gesture towards a fully nonlinear theory, is to observe that, in addition to a $k-$ number to characterize a spin wave, one needs to specify a constant angle of precession θ_0. But this group of states is not adequate to describe general excitations of the medium. In the usual spin wave description, a general (small) excitation is described by a superposition of spin waves. The coefficients in the superposition are determined by initial conditions. So we must determine how to do this for large motions, for which linear superposition is not allowed. A way to do this is as follows: With θ still constant at θ_0, note that the so-called 'Complete Integral' $\phi = \vec{k}\cdot\vec{r} - \omega_k t + g$ of the second of equations (37), where g is an integration constant, can be extended by making g an arbitrary function of k, differentiating this solution with respect to $\vec{k}$, equating the result to zero, and then forming the envelope of this group of solutions by eliminating $\vec{k}$ between the four equations

$$\begin{aligned} \phi &= \vec{k}\cdot\vec{r} - \omega_k t + g(\vec{k}) \\ 0 &= \vec{r} - t\nabla\omega_k + \nabla g(k) \end{aligned} \tag{38}$$

making ϕ a correspondingly arbitrary function of $\vec{r}$ and t .This so-called envelope is also a solution of the second of (37). The nonlinear analog of the arbitrary superposition of small amplitude spin waves is the arbitrary choice of the function $g(\vec{k})$. Further, an arbitrary superposition of small amplitude

states, inasmuch as the components were all calculated on the assumption $M_z = 1$, reveals no information on the inevitable change in M_z implied by the superposition. One optimistically derives that change only by squaring the superposition, (call it s), and writing $M_z = \sqrt{1-s^2} \approx 1 - \frac{1}{2}s^2$, with no regard to the actual dynamics of M_z. Our nonlinear analog of this cavalier treatment of M_z is to treat θ as fixed at θ_0. This is inconsistent with the first of (37) (except, of course, for the Complete Integral). That defect can be remedied by an iterative procedure. Successive steps in the procedure involve solving only *first order* partial differential equations, for which a standard solution method is available[8] . The iterants satisfy

$$\begin{aligned} \frac{\partial\theta^{(n+1)}}{\partial t} &= 2A\ \cos\theta^{(n+1)}\Big(\ \nabla\theta^{(n+1)}\cdot\nabla\phi^{(n)}\Big) + A\sin\theta^{(n+1)}\nabla^2\phi^{(n)} \qquad (39)\\ \frac{\partial\phi^{(n)}}{\partial t} &= A(\cos\mathrm{ec}\theta^{(n)})\nabla^2\theta^{(n)} + A\ \cos\theta^{(n)}(\nabla\phi^{(n)})^2 + H\ + H_K\ \cos\theta^{(n)} \end{aligned}$$

where $n = 0, 1, 2, ...$ and $\theta^{(0)} = \theta_0$, with $\phi^{(0)}$ the envelope solution of (38). Substituting $\phi^{(0)}$ in the first of (39) makes it a first order partial differential equation for $\theta^{(1)}$,whose solution is substituted in the second of (39), which becomes a first order equation for $\phi^{(1)}$, and so on.

We try this out on the simple case of a single spin wave $\phi^{(0)} = \omega_k t - kx$ traveling in the $x-$ direction, i.e. we use only the Complete Integral, without envelope formation. Then the first of equations 25, with $n = 0$, becomes

$$\frac{\partial\theta^{(1)}}{\partial t} = -2Ak\sin\theta^{(1)}\frac{\partial\theta^{(1)}}{\partial x} \qquad (40)$$

(Incidentally, equation 40 can be written as a conservation law

$$\frac{\partial\theta^{(1)}}{\partial t} + \frac{\partial(-2Ak\cos\theta^{(1)})}{\partial x} = 0$$

connecting the 'charge' θ with a 'current' $-2Ak\cos\theta^{(1)}$.) The solution $\theta^{(1)} =$ constant is of no interest, since it has already been assigned to $\theta^{(0)}$. The general solution of (40) is found by the methods of characteristics[8], whose equations in this case are:

$$\frac{dt}{ds} = 1,\ \frac{dx}{ds} = 2Ak\sin\theta^{(1)},\ \frac{d\theta^{(1)}}{ds} = 0 \qquad (41)$$

Since the first of these three is trivial, we can instead write these as

$$\frac{dx}{dt} = 2Ak\sin\theta^{(1)},\ \frac{d\theta^{(1)}}{dt} = 0$$

To get a solution of (40) it is necessary to specify $\theta^{(1)}$ at time zero as some function of initial position x_0, say $\psi(x_0)$. Then since $d\theta^{(1)}/dt = 0$,

$$\begin{aligned} \theta^{(1)} &= \psi(x_0) \\ x &= 2Akt\sin\psi(x_0) + x_0 \end{aligned} \tag{42}$$

Eliminating x_0 between these two equations gives $\theta^{(1)}$ as a function of x and t. Suppose that $\theta^{(1)}$ at time zero is equal to $\theta_0 + \theta_1 x_0 \exp(-x_0^2/\xi^2)$. At large x_0, equation (40) describes a wave propagating with a speed $2Ak\sin\theta_0$. But near the origin, the speed can be larger (for suitable θ_1, θ_0) , and then the large values of θ propagate faster than the small values. Just as in a similar situation in fluid dynamics, the wave will break (see fig.II 1). (A slightly more sophisticated theory also applies if there is a large, discontinuous change in θ (see Whitham's book, reference 8.) such as $\theta = 0$, $x < 0$, and $\theta = \pi$, $x > 0$. This propagates in a similar way, but the break occurs immediately. The term cosec$\theta^{(1)}$in equation 39 remains finite everywhere except at zero and π, but there,$\nabla^2\theta^{(1)}$ is strongly zero, so there is no problem.) From the figure it is obvious that $\phi^{(1)}$,the next approximation to the spin wave, will differ from $\phi^{(0)}$ only around the switching region. Apparently $\phi^{(1)} - \phi^{(0)}$ is 'surfing' on top of the wave. This strongly suggests, though it does not prove, that this state of affairs will continue to apply for higher iterants. In the range of $x-$ values in which θ is triplevalued, the system avoids the inherent instability by means of a discontinuous jump from one stable branch to the other. The nature of this shock is elucidated by means of the Whitham-Kruskal equal-area construction[8] (reminiscent of the Maxwell construction for van der Waals fluids). We briefly describe this construction, closely following reference 8, for our example, (disturbance on top of θ_0). Consider the area under the curve of $\psi(x_0)$ vs. x_0. Draw a chord connecting the points $\mathrm{P}_1 = (x_{01}, c(x_{01}))$, $\mathrm{P}_2 = (x_{02}, c(x_{02}))$, where c denotes the speed $c(x_0) = 2Akt\sin\psi(x_0)$ (fig.II2a). Clearly the points P_1 ,P_2 can be chosen in such a way that the area $\frac{1}{2}\left(c(x_{01}) + c(x_{02})\right)(x_{02} - x_{01})$ under the rectilinear figure with corners at x_{01}, x_{02},P_1,P_2 is equal to $\int_{x_{01}}^{x_{02}} c(x_0)dx_0$, . This means that the shaded regions in fig(4a) must have equal areas. Next, consider the mapping $(x_0, c(x_0)) \to (x, c(x,t))$, which can be shown

to preserve areas. Also, $c(x,t)$, for reasonable values of x, has the same foldover as $\theta^{(1)}$. So we translate the points P_1, P_2 to a time t at which $x(x_{01}) = x(x_{02})$, as in figure II2b, the two shaded lobes still have equal area. Therefore any pair of values of x_0 that satisfy the equation

$$\frac{1}{2}\left(c(x_{01}) + c(x_{02})\right)(x_{02} - x_{01}) = \int_{x_{01}}^{x_{02}} c(x_0)dx_0, \tag{43}$$

describe a shock. The position of the shock, X(t), say, satisfies

$$X(t) = c(x_{01})t + x_{01} = c(x_{02})t + x_{02} \tag{44}$$

The three equations (29)&(30) in principle can be solved for x_{01}, x_{02} and X as functions of t . The time at which the shock first forms is found from the fact that at that time the slope of the $\theta^{(1)}$ vs. x curve must be infinite. But from equation 28, $\partial\theta^{(1)}/\partial x = c'(x_0)\partial x_0/\partial x = c'(x_0)/\left(1 + tc'(x_0)\right)$, which is infinite when $t = -1/c'(x_0)$. This requires that c' is negative. The shortest time needed to form the shock is $1/|c'(x_0)|_{\max}$. In our example,

$$|c'(x_0)| = 4Ak\theta_1 \frac{x_0}{\xi^2}\exp(-x_0^2/\xi^2)|\cos\left(\theta_0 + \theta_1\exp(-x_0^2/\xi^2)\right)|$$

For small θ_1, the maximum is attained approximately at $x_0 = \xi/\sqrt{2}$ giving a first formation time approximately equal to

$$t_{\min} = \frac{\sqrt{2}\xi\exp(1/2)}{4Ak\theta_1\cos\theta_0}$$

Note that the broader and smaller the disturbance (larger ξ, smaller θ_1) the longer the time to first formation. The larger the exchange constant and the larger k, the shorter the time to formation. For $\theta_0 = \pi/2$, the formation time would seem to be infinite, but this is not correct, because we have neglected the θ_1 term inside the trigonometric function. However, the time becomes large, because at $\theta_0 = \pi/2$, the speed c changes only very slowly with θ_1 , and if the speed were unchanging altogether, no shock would form at all.

The jump in θ at the shock front amounts to a domain wall, although the jump is not very large (it cannot be much bigger than θ_1, the size of the original disturbance). The size of the jump is $\psi(x_{01}) - \psi(x_{02})$, evaluated at the time and place that solve equations 43 and 44. Evidently that evaluation requires intervention by a computer.

I suspect, but have not been able to prove so far, that this effect is the large amplitude analogue of the spin wave instability discussed at the end of section II.2.

.Finally it should be noted that in the presence of intrinsic damping, the equation for θ acquires a term $-\alpha\partial E/\partial\theta$, part of which is proportional to $\nabla^2\theta$. This probably has the same effect here as the viscous damping term has on the profile of the shock front, smoothing out the discontinuous jump.

II. 5. Large amplitude scattering by imperfections. T_1,T_2 and all that.

We are now ready to resume the discussion of distributive damping. As in the spin wave approximation, the easier case is that of inhomogeneities of the medium. Depending on the extent to which the magnetic energy density is anisotropic, due to crystalline anisotropy, demagnetizing, etc., different components of the spatially averaged magnetization vector will revert to equilibrium at different rates. Consider the following case: non-uniform uniaxial anisotropy, with anisotropy field of the form $H_K M_z = \left(H_{K0} + \sum_i h(\vec{r} - \vec{R}_i)\right) M_z$, where h is a short-range function, and $\vec{R}_i$ are random positions of imperfections. Leaving aside dipolar forces, $<M_z>$ is a constant of the motion. Therefore $<M_z>$ can decay only as the result of intrinsic damping, customarily denoted by a decay rate $1/T_1$. So $<\cos\theta>$ is a constant, and we begin by assuming that θ itself is also constant, $= \theta_0$. On the other hand, we anticipate that the average transverse components of $\vec{M}$ will decline as the result of cumulative dephasing by 'collisions' with the imperfections. It will be shown that this problem reduces to the scattering of a particle of unit mass in a potential $\sum_i h(\vec{r} - \vec{R}_i)$. Consider the equation satisfied by ϕ :

$$\frac{\partial\phi}{\partial t} = A\ \cos\theta_0(\nabla\phi)^2 + H\ + \left(H_{K0} + \sum_i h(\vec{r} - \vec{R}_i)\right)\ \cos\theta_0 \qquad (45)$$

In rotating coordinates $\phi = \psi(\vec{r}, t) + (H + H_{K0}\cos\theta_0)t$, the equation reads

$$\frac{\partial\psi}{\partial t} = A\ \cos\theta_0(\nabla\psi)^2 + \sum_i h(\vec{r} - \vec{R}_i)\ \cos\theta_0 \qquad (46)$$

This equation, too, can be solved using slightly more complicated characteristics [8], which are no longer straight lines, like in the previous section.

They satisfy

$$\frac{d\vec{r}}{dt} = 2A\cos\theta_0\vec{p}, \frac{d\vec{p}}{dt} = -\cos\theta_0\sum_i \nabla h(\vec{r}-\vec{R}_i),\ \frac{d\psi}{dt} = q+2A\cos\theta_0(\vec{p})^2,\ \frac{dq}{dt} = 0, \tag{47}$$

where q stands for $\partial\psi/\partial t$, and $\vec{p}$ for $\nabla\psi$. From the first and second of these follows Newton's equation:

$$\frac{d^2\vec{r}}{dt^2} = 2A\cos^2\theta_0\sum_i \nabla h(\vec{r}-\vec{R}_i) \tag{48}$$

subject to $\vec{r}$, $\vec{p}$ having assigned initial values at $t = t_0$. The fourth equation gives $q = q_0$ independent of t. Then the third of equations (47) gives $\psi = \psi(t, t_0)$, and, eliminating t_0 between $\psi(t, t_0)$ and the solution of (48), gives ψ as a function of t and $\vec{r}$. Equation (48) describes motion in a random potential and cannot be solved completely. However, we can write down the energy integral

$$\begin{aligned} \frac{1}{2}\left(\frac{d\vec{r}}{dt}\right)^2 - 2A\cos^2\theta_0\sum_i h(\vec{r}-\vec{R}_i) &= \text{constant} \qquad (49)\\ &= 2A^2\cos^2\theta_0\left(\vec{p}(t_0)^2\right) - 2A\cos^2\theta_0\sum_i h(\vec{r}(t_0)-\vec{R}_i) \end{aligned}$$

We chose the constant value q_0 to be zero. Then

$$\begin{aligned} \frac{d\psi}{dt} &= \left(\frac{1}{2A\cos\theta_0}\right)\left(\frac{d\vec{r}}{dt}\right)^2 \qquad (50)\\ &= 2\cos\theta_0\left(\sum_i h(\vec{r}-\vec{r}_i) - \sum_i h(\vec{r}(t_0)-\vec{R}_i)\right) + \\ &\quad +2A\cos\theta_0 p^2(0) \qquad (51) \end{aligned}$$

and finally, $\vec{r}(t_0)$ must be expressed in term of $\vec{r}$ and t by inverting the solution of equation (48). If h is a short range potential, and the scattering is elastic, that solution must have the form $\vec{r} = \Gamma(t-t_0, \{\vec{R}\})\vec{r}(t_0)$, where Γ is a three dimensional rotation matrix depending on the form of h and on the positions $\vec{R}$ of all the impurities. Therefore $\vec{r}(t_0) = \Gamma^{-1}\vec{r}$. Since the positions $\vec{r}_i$ are random, there is no statistical distinction between Γ and

its inverse, so that 50 may be written

$$\frac{d\psi}{dt} = p_0 + 2\cos\theta_0 \sum_i \Big(h(\vec{r} - \vec{R}_i) - h(\Gamma\vec{r} - R_i) \Big) + 2A\cos\theta_0 p^2(0) \qquad (52)$$

with t_0 set equal to zero. Equation (50) cannot be solved for general h. However, Further progress is possible, if h is so short range that there is no overlap between the h 's around neighboring impurities. Then 48 describes successive elastic scatterings. Suppose first that h is isotropic. Then the square of the velocity vector $\vec{q}$ does not change in any of the collisions, so that $\vec{q}^2$ is constant, and equal to $\vec{q}(0)^2$. This agrees with equation 35, since in any particular collision with the impurity at R_t we may chose R_t as origin, and h is assumed isotropic, so the dependence of the right hand side of 48 on $\vec{r}$ disappears . In this case, then, equation 52 shows that the impurities can only shift the phase by a fixed amount times t, which can be relegated to the rotating axis frequency, and which does not contribute to damping. Consider now anisotropic scattering from a particular imperfection, at $R_i = 0$, say. Suppose that, in this particular collision, the $\vec{r}$ vector is rotated through angles $\chi_x^i, \chi_y^i, \chi_z^i$ around the x, y, and z axes. Then the collision changes $\vec{r}$ to $\vec{r}' = \vec{r} + \vec{\omega}^i \times \vec{r}$, where $\vec{\omega}^i = (\chi_x^i, \chi_y^i, \chi_z^i)$. For simplicity, assume that the collisions are all small-angle collisions, so that $h(\vec{r}\,) - h(\Gamma\vec{r}\,) = -(\vec{\omega}^i \times \vec{r}) \cdot \nabla h(\vec{r})$ for the ith collision. In this expression, $\vec{r}$ is to be interpreted as the direction $\vec{e}^i = \vec{q}_i/|\vec{q}_i|$ of the incoming momentum vector; its magnitude is irrelevant. Thus if $h(\vec{r}) = h(x/r, y/r, z/r)$, then

$$\begin{aligned}
(\vec{\omega}^i \times \vec{r}) \cdot \nabla h &= \left((y^2+z^2)/r^3 \frac{\partial h}{\partial(x/r)}, (z^2+x^2)/r^3 \frac{\partial h}{\partial(y/r)}, (z^2+x^2)/r^3 \frac{\partial h}{\partial(z/r)} \right) \qquad (53) \\
&= (\vec{\omega}^i \times \vec{e}^i) \cdot \left\{ \left(1-(e_x^i)^2\right) \frac{\partial h}{\partial(e_x^i)}, \left(1-(e_y^i)^2\right) \frac{\partial h}{\partial(e_y^i)}, \left(1-(e_z^i)^2\right) \frac{\partial h}{\partial(e_z^i)} \right\} \\
&= \epsilon_{lmn} \omega_m^i e_n^i \left(1-(e_l^i)^2\right) \frac{\partial h}{\partial(e_l^i)}
\end{aligned}$$

where the summation indices l, m, n stand for x, y, z. After N collsions, equation 52 gives

$$\psi(N) = 2NA\cos\theta_0 q(0)^2 + 2\cos\theta_0 \sum_{i=1}^{N} \epsilon_{lmn} \omega_m^i e_n^i \left(1-(e_l^i)^2\right) \frac{\partial h}{\partial(e_l^i)} \qquad (54)$$

The first term gives just a constant phase shift. The second term is responsible for T_2- type damping. N is related to the elapsed time by the

relation $N = tp(0)\times$ density of imperfections $\times$ collision cross section, and the $\vec{e}^{\prime}$ are random variables. Thus, in a system of rotating coordinates

$$<M_x> + i<M_y> \ = \sin\theta_0 \left\langle \exp i\psi(N) \right\rangle$$

and any kind of reasonable distribution of the $\vec{e}^{\prime}$ will give a decaying average for the exponential; the decay constant is known as $1/T_2$. In this incoherent scattering model, $\nabla\psi$ changes only by collisions, which change its direction but not its magnitude. Therefore the hypothesis of a constant value for θ is consistent with the equation of motion of that quantity between the collisions. Collisions will change the value by positive and negative increments that average to zero. In any case, for this model, θ does not change *directly* by the imperfections, only through the spatial rate of change of ϕ, so that, if one wishes to pursue the course of θ in detail, one can do so by the iterative procedure discussed in the previous section. Also note that the above solution is not the only possible one, although it is the most appealing one because of its reduction to particle dynamics. As in the previous section, a constant can be added to ψ. That constant can be made an arbitrary function of one or more of the constant parameters entering ψ. Differentiating the new solution with respect to each such parameter, equating the results to zero, and eliminating the parameters between all the resulting relations gives an envelope that again is a solution of the original equation. Ultimately, the boundary conditions will decide which solution is the right one. The simple one that we discussed in detail is probably the relevant one for the infinite medium.

To conclude this section, we indicate how to make contact with the rear guard of spin wave theory of this kind of damping. We expand our solution in spin waves: In rotating coordinates,

$$\sin\theta_0 \exp i\psi(x,t) = \sin\theta_0 \sum_k \left[\left(\int dv' dt' e^{i\left(\psi(\vec{r}',t') - \vec{k}\cdot\vec{r}' + \omega_k t'\right)} \right) e^{i(\vec{k}\cdot\vec{r} - \omega_k t)} \right].$$

We evaluate the integral by the principle of stationary phase: at given $\vec{k}$, the phase is stationary at values $\vec{r}^{\prime}$ and t' obtained by solving the four equations

$$\begin{aligned} \nabla_{r'}\psi &= \vec{k} \\ \frac{\partial\psi}{\partial t'} &= -\omega_k \end{aligned}$$

for $\vec{r}', t'$, giving $\vec{r}_0(\vec{k}), t_0(\vec{k})$, say. Expanding the exponent up to terms bilinear in the deviations $\vec{\xi} = \vec{r}' - \vec{r}_0(\vec{k})$ and $t = t' - t_0(\vec{k})$, and performing the integration, gives the Fourier coefficient

$$f_k = \frac{4\pi^2}{\sqrt{\lambda_1(\vec{k})\lambda_2(\vec{k})\lambda_3(\vec{k})\frac{\partial^2\psi(\vec{r}_0(k),,t_0(\vec{k}))}{\partial t_0^2}}}$$

where the λ 's are the eigenvalues of the bilinear form

$$\xi_1^2\frac{\partial^2\psi(\vec{r}_0(k),,t_0(\vec{k}))}{\partial x_0^2} + \text{ etcetera}$$

As usual, the Fourier series $\sum f_k e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}$ gets its greatest contribution from the region of large state density $d\omega_k/dk$. The notion that this region is around $\omega_k = \omega_0$ (the uniform precession mode) requires some comment. The uniform precession is a Walker mode normally calculated without taking exchange into account. How can it 'reach out' all the way to k values implied by $\omega_k = \omega_0$? The answer is that, in the presence of coupling furnished by the imperfections, the original excitation spectrum is changed. The original levels cross and therefore a gap opens up. ω_0 originally was calculated taking into account boundary conditions that become unimportant at short wavelength and are ignored in the usual s.w. spectrum. When exchange is included in its calculation ω_0 does become a function of k, initially rising slowly, but more rapidly as the split is approached. As is usual in such cases, the flattening of the lower branch amounts to a large density of states (Fig I.3).

Up to this point, the damping was deemed to come from randomly placed imperfections. But we know from spin wave theory that even in the perfect sample, dipolar forces can give distributive damping. As already indicated, this is hard to evaluate for arbitrarily large motions. However, in the relatively simple case of variation in $\vec{M}$ limited to one direction only, further progress is possible. From the earlier results we see that if the angle β between $\vec{H}$ and the direction of variation $\vec{D}$ is zero, then the added dipolar energy just gives an extra uniaxial 'hard' anisotropy. This is not surprising. the biggest effect comes when $\beta = \pi/2$. Then the added energy (with M_s taken to be 1) is $\frac{1}{2}\sin^2\theta\cos^2\phi$.

A more general formulation is obviously needed. We restate equation

$$\mathrm{E}_{dip} = \int \frac{dv'}{|\vec{r}-\vec{r}'|^3} \Big(\begin{array}{c} \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') - \\ -3\ (\sin\theta(c_1\cos\phi + c_2\sin\phi) + c_3\cos\theta))\,(\sin\theta'(c_1\cos\phi' + c_2\sin\phi') + c_3\cos\theta')) \end{array}$$

Set $\phi = (Ht + H\int_0^t dt\cos\theta) + \psi(\vec{r},t)$, and, using trigonometric formulae, discard all terms of the form $\cos(\phi+\phi')$ on the grounds that they would oscillate rapidly. Then the energy becomes

$$\mathrm{E}_{dip} = \int \frac{dv'}{|\vec{r}-\vec{r}'|^3} \left\{ \cos\theta\cos\theta' + \sin\theta\sin\theta' \left(1 - \frac{3}{2}\left(c_1^2 + c_2^2\right)\right)\cos(\psi-\psi') - 3c_3^2\cos\theta\cos\theta' \right\}$$

Recall that the damping from the dipolar terms in spin wave approximation occured only at finite temperatures, with all spin waves in thermal equilibrium, except, of course, the excitation whose decay is being studied. Further, the magnetic energies in this problem are much greater than k_BT; therefore the deviation of θ from 0 is small every where. So the dipolar energy can be expanded to second order in θ. On the other hnd, the phase angle ψ cannot be treated in this manner; it need never be of any particular size, (unless there is anisotropy or demagnetization dependent on ϕ) . Bearing this in mind, we see that it is sufficient to write

$$\begin{aligned} \mathrm{E}_{dip} &= C + \int \frac{dv'}{|\vec{r}-\vec{r}'|^3} \left\{ -\frac{1}{2}(\theta^2+\theta'^2) + \ \theta\ \theta' \left(1 - \frac{3}{2}\left(c_1^2+c_2^2\right)\right)\cos(\psi-\psi') + \frac{3}{2}c_3^2(\theta^2+\theta'^2) \right\} \\ &= C + \int \frac{dv'}{|\vec{r}-\vec{r}'|^3} \left\{ \frac{3c_3^2-1}{2}(\theta^2+\theta'^2) + \ \theta\ \theta' \left(1 - \frac{3}{2}\left(c_1^2+c_2^2\right)\right)\cos(\psi-\psi') \ \right\} \end{aligned}$$

where C is a constant. The part involving θ^2 zero, because the angular integral os $c_3 = 1/3$. Finally, (using $\sum c_1^2 = 1$), there remains 'only'

$$\mathrm{E}_{dip} = C + \int \frac{dv'}{|\vec{r}-\vec{r}'|^3} \frac{3c_3^2-1}{2}\ \left\{\theta'^2 + \ \theta\ \theta'\ \cos(\psi-\psi')\ \right\}$$

At this point, it probably becomes necessary to make a local approximation similar to the one discussed earlier for cartesian coordinates. Then a solution for the dephasing of ϕ should become possible, but this work is still incomplete.

III. Intrinsic Damping.

III. 1. General Aspects. If the magnetic system is coupled to other *dynamic* degrees of freedom, these may carry away energy from excited magnetic degrees of freedom of interest, resulting in damping of the latter that cannot be described as a redistribution of the excitation energy to other, uninteresting, magnetic motions. for example, spin orbit coupling results in coupling to the underlying lattice. The *static* part of that coupling, calculated for fixed lattice positions, gives rise to the magnetic anisotropy constants, but there is also a *dynamic* part; the coupling to the lattice displacements, i.e. the phonons. As for the static part of the anisotropy, the orbital angular momenta that enter the expression for spin orbit coupling, are eliminated in favor of the magnetization field, usually by means of perturbation theory. It is not common practice to similarly eliminate the phonons in favor of the magnetization field. If the phonons have a finite lifetime, the result of the elimination will reflect this finite lifetime in a loss term in the equations of motion of the magnetization field. Another example is furnished by magnetic metals. Elimination of the electromagnetic field will result in the appearance of the resistivity losses in the resulting damping term for the magnetic field. Also, in magnetic compounds whose ions undergo slow valence fluctuations, elimination of these leaves behind a dissipative torque on the magnetization field of a rather unusual form. Finally, there are cases in which the magnetization field is coupled to impurities that have their own internal dynamics. Again, these may be disposed of in favor of the magnetization field, which acquires the appropriate loss torque.Three of these examples are already in the literature [9,10,11]; the remaining one will be discussed in detail; the other two only briefly. It will appear that a reliable reduction of the motion to describe magnetic degrees of freedom alone will be critically dependent on the nature of, and the coupling to, the degrees of freedom to be eliminated. Because this is not always easy to do, it is tempting to discuss the problem for a model of the reservoir of external motions: a scalar field coupled linearly to the magnetization components. This was done in reference 11 for a uniform magnetization in the linearized limit of small transverse magnetization components. In that model, the reservoir field, f, say, is characterized by random fluctuations with delta-function autocorrelation. It is supposed to be in thermal equilibrium, and its mean square value is consistent with the fluctuation-dissipation theorem. It is assumed sufficiently robust, so that the back-reaction of the magnetic motions on f can be neglected. This procedure works well in the linearized limit, provided one neglects certain

stochastic features of the random field, to be dicussed later in this section.

III. 2. Damping torque in a ferromagnetic metal. We discuss this case only in the limit in which electromagnetic propagation effects are negligible, i.e. for electromagnetic wavelengths much larger than the sample dimensions. Then the equation of motion of $\vec{M}$, in component form, may be written

$$\dot{M}_i = \epsilon_{ijk} M_j (H_k + h_k \ + A\nabla^2 M_k) \tag{55}$$

where $H_k is$ the steady applied magnetic field, h the magnetic field satisfying Maxwells equations. Crystalline anisotropy fields have been neglected, since they do not affect the results derived here We wish to eliminate $\vec{h}$ and the electric field $\vec{e}$ between equations (1) and Maxwell's equations (2)

$$\begin{aligned} \nabla \times \vec{h} &= 4\pi\sigma\vec{e}/c \\ \nabla \times \vec{e} &= -\left(\frac{\partial\vec{h} + 4\pi\vec{M}}{c\partial t}\right) \end{aligned} \tag{56}$$

where σ is the conductivity. Eliminating $\vec{e}$ between the last two equations gives

$$\nabla \times \nabla \times \vec{h} + \frac{4\pi\sigma}{c^2}\left(\frac{\partial\vec{h} + 4\pi\vec{M}}{\partial t}\right) = 0$$

and since $\nabla \times \nabla \times \vec{h} = \nabla(\nabla \cdot \vec{h}) - \nabla^2\vec{h}$, and (from the second of equation 2) $\nabla \cdot \vec{h} = -4\pi\nabla \cdot \vec{M}+$ some function of $\vec{r}$ (which we discard) , we get

$$\nabla^2\vec{h} = \frac{4\pi\sigma}{c^2}\frac{\partial(\vec{h} + 4\pi\vec{M})}{\partial t} + \ 4\pi\nabla(\nabla \cdot \vec{M}) \tag{57}$$

To solve this equation for $\vec{h}$ in terms of $\vec{M}$, we need the retarded Green's function that satisfies the equation

$$\nabla^2 G - \frac{4\pi\sigma}{c^2}\frac{\partial G}{\partial t} = -4\pi\delta(\vec{r} - \vec{r}')\delta(t - t') \tag{58}$$

In infinite space, G is the source-solution of the diffusion equation:

$$G = \left(\frac{\sigma}{(t - t')c^2}\right)^{3/2} e^{-\frac{(\vec{r}-\vec{r}')^2\pi\sigma}{c^2(t-t')}} \tag{59}$$

The solution of equation 3 is then

$$h = \int dv' \int_{-\infty}^{t} dt' G\left(|\vec{r}-\vec{r}'|,(t-t')\right)\left(\frac{16\pi^2\sigma}{c^2}\frac{\partial\ \vec{M}(\vec{r}',t')}{\partial t'} + 4\pi\nabla(\nabla\cdot\vec{M}(\vec{r}',t')\right) \quad (60)$$

$$= \int dv \int_{0}^{\infty} d\tau G\left(\rho,\tau\right)\left(\frac{16\pi^2\sigma}{c^2}\frac{\partial\ \vec{M}(\vec{r}+\vec{\rho},t+\tau)}{\partial \tau} + 4\pi\nabla(\nabla\cdot\vec{M}(\vec{r}+\vec{\rho},t+\tau)\right) \quad (61)$$

First, consider some special cases:

a. $\vec{M}$ is independent position. Then in the first term on the right of 60, the volume integral only affects G, and gives unity. Then the time integral gives the value of $\vec{M}$ at time t, which contributes nothing to the torque $\vec{M}\times\vec{M}$. This may seem strange, in as much as the time derivative of $\vec{M}$ in (6) seems to be the only candidate for a Gilbert time damping. The reason is quite simple: if $\vec{M}$ does not depend on position, it cannot produce a current, and without a current, there can be no dissipation. This is seen easily by eliminating $\vec{h}$ in favor of $\vec{e}$, resulting in a diffusion equation for $\vec{e}$ that is driven by a current proportional to $\nabla\times\vec{M}$. Now if $\vec{M}$ is independent of position, the second term on the right of (60) would seem to be zero. However, this conclusion ignores boundary conditions that involve a surface divergence of $\vec{M}$. The correct procedure is to put the burden of the differentiations on G by integration by parts. The result becomes familiar when

b. $\vec{M}$ is independent of time. Then only the second term in (6) survives. The time integration involves only G, and(using the first form of (60)), it gives essentially $1/|\vec{r}-\vec{r}'|$. Then, two integrations by parts result precisely in the dipolar field due to $\vec{M}$, which yields the familiar demagnetizing field. When $\vec{M}$ is not independent of time, then the second term tells us what becomes of the dipolar field under the diffusion-like propagation in this model. (Actually, for consistency, we should have used the Greens function appropriate to the finite sample for which the demagnetizing is calculated, not the infinite medium Greens function).

Hopefully, in the general case of a both $\vec{r}$ and t dependent $\vec{M}$, the non-local expression can be expanded in an at least asymptotically convergent series of purely local terms. This procedure is very successful in the case of interaction of $\vec{M}$ with elastic lattice displacements via magnetostriction. In its simplest form, it fails completely in the present case. Write $t'=t-\tau$

and $\vec{r}' = \vec{r} - \vec{\rho}$. Then, for example,

$$\frac{\partial \vec{M}(\vec{r}', t')}{\partial t'} = \frac{\partial \vec{M}(\vec{r}, t)}{\partial t} - \left((\vec{\rho} \cdot \nabla) + \tau \frac{\partial}{\partial t} \right) \frac{\partial \vec{M}(\vec{r}, t)}{\partial t} + \frac{1}{2!} \left((\vec{\rho} \cdot \nabla) + \tau \frac{\partial}{\partial t} \right)^2 \frac{\partial \vec{M}(\vec{r}, t)}{\partial t} + ... \tag{62}$$

so to evaluate (6), we need the successive moments

$$I_{nm} = \int dv \int_0^\infty d\tau \rho_i^n \tau^m G(\rho, \tau) \tag{63}$$

The case $n = m = 0$ requires evaluation of

$$I_{00} = 4\pi \int_0^\infty \frac{b^{3/2} d\tau}{\tau^{3/2}} \int_0^\infty \rho^2 d\rho e^{-\frac{\pi b \rho^2}{\tau}} \tag{64}$$

where $b = \frac{\pi\sigma}{c^2}$. This integral diverges, and higher moments even more so. Thus damping cannot be local in the sense of the above expansion. However, there is one highly unattractive possibility of saving the moment expansion, and that is to use the Greens function for a finite sample. Suppose that $\vec{M} = \vec{M}_u + \vec{m}$, where $\vec{M}_u$ is the spatially uniform part of $\vec{M}$ and $\vec{m}$ the rest. If the sample is saturated, $\vec{h}$ may be similarly decomposed, its uniform part calculated as the internal field (the external field minus the demagnetizing field). If the sample is not saturated, the internal field and the uniform part are zero. For simplicity, assume that the non-uniform part of $\vec{h}$ vanishes on the boundary. Suppose that $\{ \zeta_n \}$ is the set of eigenfunctions of $\nabla^2 \zeta + \lambda \zeta = 0$, vanishing on the boundary, with eigenvalues $\{\lambda_n\}$. Then the Greens function is

$$G(\vec{r}, \vec{r}', t - t') = \sum_n \zeta_n(\vec{r}) \zeta_n(\vec{r}') e^{-\frac{\lambda_n c^2}{4\pi\sigma}(t - t')} \tag{65}$$

Noting that no eigenvalue can be zero (because the only solution of Laplace's equation that vanishes on the boundary is identically zero), it is now clear that we *can* make a moment expansion. For example, because $\int_0^\infty G(\vec{r}, \vec{r}', \tau) d\tau$ =1 for all $\vec{r}$ and $\vec{r}'$, the first term on the right of eq.(6) gives

$$\frac{16\pi^2\sigma^2}{c^4} S(\vec{r}) \frac{\partial \ \vec{M}(\vec{r}, t)}{\partial t} \tag{66}$$

that is to say, simple Gilbert damping. Here, $S(\vec{r}) = \sum_n \int dv_\rho \frac{\zeta_n(\vec{r}) \zeta_n(\vec{r} + \vec{\rho})}{\lambda_n}$Recall, however, that a spatially uniform $\vec{M}$ will give zero. That can be seen by

noting that, for uniform $\vec{M}$, the summation that involves only powers of $\tau\partial/\partial t$, and adds up to $\frac{\partial\ \vec{M}(t+\tau)}{\partial t} = \frac{\partial\ \vec{M}(t+\tau)}{\partial \tau}$. So for almost uniform $\vec{M}$, at least as much of the damping will come from terms involving powers of $\rho \cdot \nabla$, as will come from powers of $\tau\partial/\partial t$. For example, the lowest term in the expansion that involves one spatial gradient is

$$\frac{64\pi^3\sigma^3}{c^6}\vec{T}\cdot\nabla\frac{\partial^2}{\partial t^2}\ \vec{M}(\vec{r},t) \tag{67}$$

where $\vec{T} = \left(\int dv_\rho \sum_n \frac{\zeta_n(\vec{r})\zeta_n(\vec{r}-\vec{\rho}\ )}{\lambda_n}\vec{\rho}\right)$. However, terms involving only even time derivatives presumably do not contribute to damping, but only renormalize the effective field. The lowest term giving position dependent damping would involve the moment of $(\rho\cdot\nabla)(\tau\frac{\partial}{\partial t})^2\frac{\partial\ \vec{M}(\vec{r},t)}{\partial t}$The reason why these results diverge in the case of the infinite medium is that the eigenfunctions are then $\exp[i\vec{k}\vec{r}]$, with lowest eigenvalue $k^2 = 0$, causing the sums over k to diverge. In a finite sample, the lowest eigenvalue is finite (recall the zero point energy in quantum mechanics of a potential well).

III. 3. Damping torque due to magnetoelastic coupling. When there is substantial magnetostriction, the viscous damping of the lattice manifests itself as a loss torque on the magnetization. This torque is found by eliminating the lattice strain components in favor of the magnetization. The full treatment is found in reference 9; here we only discuss the case of magnetic samples small compared with both the wavelength of sound and small compared with domain wall width. In that case the resulting loss torque is independent of position. With the sound velocity effectively infinite in this limit, the magnetoelastic energy is E$= \int dv \left(F_e + F_{em} + E(\vec{M})\right)$, where[12]

$$F_e = \mu e_{ij}^2;\ F_{em} = Be_{ij}M_iM_j$$

with repated subscripts summed over. F_e is the shear part of the elastic energy due to strain components $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, where the u_i is the ith component of the lattice displacement and $x_{1,2,3}$ strand for x, y, and z. Since the sound velocity is taken to be infinite, the e_{ij} may be considered uniform throughout the sample. B is the magneto-elastic coupling coefficient. μ is the shear modulus. (Compressive strain energy is not included. If it were, there would be the added complication of the saturation magnetization being affected. A complete theory would have to take it into account). The

elastic stress tensor is $-\frac{\partial F}{\partial e_{ij}} = 2\mu e_{ij} + BM_iM_j$. When the e_{ij} functions of the time, this stress tensor must be balanced by a viscous stress tensor $2\eta\dot{e}_{ij}$. So the equation of motion for e_{ij} becomes[13]

$$\eta\dot{e}_{ij} + \mu e_{ij} + \frac{1}{2}BM_iM_j = 0 \tag{68}$$

On the other hand, the equation of motion for $\vec{M}$ is

$$\begin{aligned} \dot{M}_i &= \gamma\epsilon_{ijk}M_j\frac{\partial E}{\partial M_k} + \gamma\epsilon_{ijk}M_j\frac{\partial F_{em}}{\partial M_k} \\ &= \gamma\epsilon_{ijk}M_j\frac{\partial E}{\partial M_k} + 2B\epsilon_{ijk}M_jM_le_{lk} \end{aligned} \tag{69}$$

Equation 13 may be formally solved for e_{ij} and the result substituted in 14. This gives

$$\dot{M}_i = \gamma\epsilon_{ijk}M_j\left(\frac{\partial E}{\partial M_k} - \frac{B^2}{2\eta}\int_0^\infty M_k(t-\tau)\left(M_l(t)M_l(t-\tau)\right)e^{-\lambda\tau}d\tau\right) \tag{70}$$

where $\lambda = \mu/\eta = \frac{E}{2\eta(1+\sigma)}$ (E = Young's modulus, σ = Poisson's ratio). The last term depends on prior history of $\vec{M}$. As η goes to zero, the term vanishes in a singular manner. Nevertheless a moment expansion in powers of η is possible, even though it is only asymptotic. This is done by expanding $M(t-\tau)$ in powers of τ and integrating, making use of the fact that $M_l(t)M_l(t) = M_s^2$ is taken to be 1. The result is

$$\gamma\epsilon_{ijk}M_j\frac{B^2}{2\eta}\int_0^\infty M_k(t-\tau)\left(M_l(t)M_l(t-\tau)\right)e^{-\lambda\tau}d\tau = \gamma\epsilon_{ijk}M_j\left(\alpha_1\dot{M}_k + \alpha_2\ddot{M}_k + \alpha_3 d^3M_k/dt^3 + ...\right) \tag{71}$$

where

$$\alpha_1 = 2B^2\eta(1+\sigma)^2/E^2,\ \alpha_2 = -8B^2\eta^2(1+\sigma)^3/E^3,\ \alpha_3 = 8B^2\eta^3(1+\sigma)^4/E^4 \tag{72}$$

Evidently trhe leading term in the expansion is unequivocally of Gilbert type. Using the constants in various papers on elastic constants one finds that the terms in higher derivative should become significant at frequencies above a few gigahertz.

The wavelength of sound in typical materials of interest should be of order of one micrometer, which is also of the order of typical domain wall

widths. Therefore in samples larger than abot one micrometer. It is then necessary to include spatial variation aswell. This is done in reference 9. Here we mention only that the result becomes nonlocal in space aswell as time, but (in contrast with the electromagnetic case in the unbounded medium) a moment expansion in both space and time is possible here. It generates various new terms, the simplest example of which is proportional to $\gamma\epsilon_{ijk}M_j\nabla^2\dot{M}_k$. The possible existence of space dependent damping term had been anticipated by Baryakhtan et al.[14]

IV. Dissipative Damping and Fluctuation-Dissipation Relations

IV. 1.. Fluctuation-Dissipation Relations. Up to this point, the magnetic system plus the environment to which it is coupled were considered on a purely deterministic basis. Thus, the environment was not treated as a 'heat bath' in the usual sense. Our purpose was to have the environment's effect on the motion of the magnetization appear as parameters or functional forms in the equations of motion of the magnetization only. However, this is not enough to lead to a solution of some important problems, particularly problems of magnetization reversal. When there is a barrier to overcome, a genuine heat bath is required to supply random impulses to nudge the system. Because the equations of motion of $\vec{M}$ are first order in time, there is no inertial kinetic energy to carry $\vec{M}$ over the barrier. Not only that, but since its first order derivative depends on $\partial E/\partial\vec{M}$, which vanishes at *all* stationary points, the magnetization vector, without thermal agitation, can neither depart from an (unstable) maximum, nor overcome a maximum in a finite time. How is this Brownian motion of $\vec{M}$ to be described ? Recall the simplest case of the Langevin equation of a free particle subject to a random force exerted by the reservoir. The particle can exchange momentum, and perhaps also energy, with the reservoir. This leads to the degradation of any initial forward motion of the particle, i.e. a damping. Assuming the reservoir to be in thermal equilibrium and sufficiently robust to be insensitive to the motion of the particle, the velocity of the latter must ultimately reach a Maxwell distribution. The equation of motion for the speed u, with friction constant η, unit mass, and random force $f(t)$ is

$$\dot{u} + \eta u = f(t) \tag{73}$$

and its solution is

$$\begin{aligned} u(t) &= u_0 e^{-\eta t} + \int_{-\infty}^{t} f(t') e^{-\eta(t-t')} dt' \\ &= u_0 e^{-\eta t} + \int_{0}^{\infty} f(t-\tau) e^{-\eta\tau} d\tau \end{aligned} \tag{74}$$

So, after the transient has died out,

$$\left(u(t)^2\right) = \int_0^\infty \int_0^\infty f(t-\tau) f(t-\tau') e^{-\eta(\tau+\tau')} d\tau d\tau' \tag{75}$$

We know that thermal averaging is supposed to give $\frac{1}{2}k_B T$ on the left. On the right, it gives

$$\int_0^\infty \int_0^\infty C(\tau-\tau') e^{-\eta(\tau+\tau')} d\tau d\tau'$$

where $C(t)$ is the autocorrelation function of f. At this point, it is usually assumed that $C(t) =< f^2 > \delta(t)$, in which case averaging (3) gives

$$\eta = \frac{1}{k_B T} < f^2 > \tag{76}$$

Also, the autocorrelation function describing the fluctuations of of u is

$$\begin{aligned} G(\sigma) &= < u(t) u(t-\sigma) > \\ &= e^{-\eta\sigma} < f^2 > /(2\eta) \\ &= \left\langle u(t)^2 \right\rangle e^{-\eta\sigma} \end{aligned} \tag{77}$$

The dissipative response to an initial impulse can also be written in this form; hence the name 'fluctuation-dissipation theorem'. Note that f no longer appears in that *relation*, but the constant η does require information on f. That calculation can be extended to the case of a particle in a harmonic well. If the particle moves in a general potential, its equation of motion is no longer linear and no general solution of the Langevin equation exists. However, one can prove, both classically and quantum mechanically, that for a *linear* response, the fluctuation-dissipation relation still holds. (In the quantum proof, one still has to make the charitable assumption that the average of two non-commuting quantities may be replaced by the

average of their anticommutator). But to determine either the fluctuations or the response *seperately*, one still needs information on the coupling to the bath and its statistical properties.

Not surprisingly, an ab initio derivation of the Langevin equation from full-scale dynamics has not so far been achieved except in very simple cases. Consider, for example, the Brownian motion of a harmonic oscillator, coupled linerarly to a large reservoir of other harmonic oscillators. This is like the problem of distributive damping discussed in Section II. There, the Brownian particle was the uniform part of the magnetization, and the reservoir the spatially varying part of the magnetization. When that problem was linearized, and the coupling mechanism was taken to be lowest order spin wave scattering from imperfections, a damping constant was found. The reason for success in that case was that the character of the bath (the synchronous manifold of spin waves, in the case of small motions) was carefully included in the calculation (in contrast to the Langevin procedure). For a multidimensional harmonic oscillator that calculation will in general furnish different damping constants for the different components. As demonstrated in the Safanov-Bertram[11] paper, the result can be very different from that using an a priori form for the damping term in the equation for $\vec{M}$ and simply adding a random term to the right hand side of the equation. However, we note here that one can circumvent the need to involve the direct effect of the random reservoir motions on $\vec{M}$, if it is possible to relegate fluctuation-dissipation questions to a lossy buffer between the equation for $\vec{M}$ and the reservoir. In sections III. 3 and III. 4, the buffers were respectively the electromagnetic field with resistivity losses, and the viscous magnetoelastic medium. The recoil was calculated from the deterministic part of the motion of the buffer and, as long as we were interested only in the mean motion of $\vec{M}$, there was no need to mention any random force. But this is not good enough if there is a barrier to overcome. Consider the coupling to the electromagnetic case, discussed in section 1. The heat bath is composed of random currents of fluctuating charge carriers, and we must determine how these fluctuations affect the motion of $\vec{M}$, which has no random torque acting on it *directly* . The random currents must be added to the right hand side of the first of Maxwell's equation, and we will neglect their recoil, uncritically accepting a Langevin equation

$$\nabla \times \vec{h} = 4\pi\sigma\vec{e}/c + 4\pi\sigma\vec{e}_{rndm}/c \tag{78}$$

with $\vec{e}_{rndm}$ an electric field random in both time and space. Proceding exactly as in III.2, we now find equation III(6) for $\vec{h}$, but with an additional term on the right hand side:

$$\vec{h}_1 = \frac{4\pi\sigma}{c^2} \int dv_{r'} G(\vec{r}, r', t - t') \vec{h}_{rndm}(r', t') \tag{79}$$

where $\partial h_{rndm}(r,t)/\partial t = c\, \nabla \times \vec{e}_{rndm}$ .The crucial point is that h_i in (79) is the *same* function of $.h_{rndm}$ as h is of $\partial \vec{M}/\partial t$ in equation 61. As in Langevin's approach, we need not know anything about the statistical properties of h_{rndm}, except that they must be such that, leaving aside the magnetization, the solution of the equation for $\vec{h}$,

$$\nabla \times \nabla \times \vec{h} + \frac{4\pi\sigma}{c^2} \frac{\partial \vec{h}}{\partial t} = \frac{4\pi\sigma}{c^2} \frac{\partial \vec{h}_{rndm}}{\partial t} \tag{80}$$

gives the magnetic field of the black body spectrum, with the displacement current neglected. Thus, in the equation of motion for $\vec{M}$ appears in addition to the systematic Gilbert-like loss term (section2), an additive random torque $\vec{M} \times \vec{h}_{rndm}$. Thus we have almost the pristine Langevin equation, escept that, because of the $\vec{M}\times$ operation, it is multiplicative noise, in contrast with the purely additive noise term of the original Langevin equation. This happy state of affairs is not likely to continue in the general case of a lossy medium.

Multiplicative noise can give rise to systematic effects, like so called 'noise induced drift', and it seriously complicates the Langevin equation. In addition, nonlinear terms in the deterministic part of the equation force one to resort to an iterative solution procedure, not an easy exercise. A ray of hope in this regard is provided by an equivalent formulation, the Fokker-Planck equation. This is a partial differential equation for the probability distribution of the dependent variables, and it is linear. Actually, this is no great help , for the same reason that the Schroedinger equation for a particle in some general potential is no easier to solve than Newton's equation, even though the former is linear, and the latter nonlinear. Nevertheless we shall use the Fokker-Planck equation, mainly because in that formulation all the effects of the random force, including multiplicative noise, are buried once and for all in the diffusion coefficient. There is a derivation of the Fokker-Planck equation to suit every taste, from severely mathematical to user friendly. Among the latter, very physical derivations are those of

Chandrasekhar (Rev, Mod. Phys, April 1943) and of van Kampen (book on Stat. Mech.). Somewhat more mathematical derivations appear in a recent book by Coffey et al (*TheLangevin Equation*). We shall not derive the Fokker-Planck equation here. Also, for simplicity we will assume that the deterministic part of the damping resulting from the recoil of the lossy environment to the magnetization has simple Gilbert or equivalent Landau-Lifschitz form, with memory terms neglected. Further, the equation we use will be for the probability distribution of polar angles of the magnetization, for reasons explained earlier. In a small enough single-domain particle, the two polar angles are independent of position, so the equation likewise involves no position dependence and describes the diffusion of the variables in θ, ϕ space. For larger particles, the two angles are functions of position, and then we are dealing with a diffusion equation in the function space of $\theta(\vec{r})$ and $\phi(\vec{r})$.

IV. 2. The Fokker-Planck equation. All derivations of the F.P. equation are based on the notion that the fluctuations in the variables of interest induced by random additive or multiplicative noise are very rapid compared with the motion along their average trajectory, and that their autocorrelation functions also decay rapidly, which is very likely if the random field's autocorrelation decays rapidly. Under these conditions, double integrals such as $\int_0^{\Delta t}\int_0^{\Delta t}\Delta\theta(t)\ \Delta\theta(t')dtdt'$, $\int_0^{\Delta t}\int_0^{\Delta t}\Delta\theta(t)\ \phi(t')dtdt'$ etc., when averaged over all values of the random force, turn out to be proportional to Δt only, rather than to Δt^2. This assumes that autocorrelation time $<<$ $\Delta t <<$ time scale of the average motion. Under these conditions the F.P. equation becomes:

$$\frac{\partial W}{\partial t}+\frac{\partial W}{\partial \theta}\frac{\partial E}{M\sin\theta\partial\phi}-\frac{\partial W}{\sin\theta\partial\phi}\frac{\partial E}{\partial\theta}-\left(\frac{\partial}{\partial\theta}\gamma\alpha W\frac{\partial E}{M\partial\theta}+\frac{\partial}{\sin\theta\partial\phi}\gamma\alpha W\frac{\partial E}{M\sin\theta\partial\phi}\right)=L(W) \tag{81}$$

where

$$L(W)=\frac{1}{2}\left(\frac{\partial^2(W\left\langle\Delta\theta^2\right\rangle/\Delta t}{\sin^2\theta\partial\phi^2}+2\frac{\partial^2 W\left\langle\Delta\theta\Delta\phi\right\rangle/\Delta t}{\sin\theta\partial\phi\partial\theta}+\frac{\partial^2(W\left\langle\Delta\phi^2\right\rangle/\Delta t}{\partial\theta^2}\right) \tag{82}$$

and where, for brevity, we have written $\int_0^{\Delta t}\int_0^{\Delta t}<\Delta\theta(t)\ \Delta\theta(t')>dtdt'=\left\langle\Delta\theta^2\right\rangle$, etc. (The deterministic $\gamma\alpha\vec{M}\times\vec{M}\times\nabla_M E$ term simply becomes $-\gamma\alpha\partial E/\partial\theta$ and $-\gamma\alpha\partial E/(\sin\theta\partial\phi)$ when resolved along M_θ and M_ϕ. increasing).

To start with, consider the model:

$$\begin{aligned} \left\langle \Delta\theta^2 \right\rangle / \Delta t &= \left\langle \Delta\phi^2 \right\rangle / \Delta t = \left\langle f^2 \right\rangle \text{ , independent of } \theta, \phi \\ \left\langle \Delta\theta \Delta\phi \right\rangle / \Delta t &= 0 \end{aligned}$$

which actually turns out to be correct. In equilibrium, $W \propto e^{-Ev/kT}$ and $\partial W/\partial t = 0$, and the F.P. equation can be written

$$\frac{\partial}{\partial\theta}\left(\gamma\alpha W \frac{\partial E}{M\partial\theta} + \frac{1}{2}\left\langle f^2 \right\rangle \frac{\partial W}{\partial\theta}\right) + \frac{\partial}{\sin\theta\partial\phi}\left(\gamma\alpha W \frac{\partial E}{M\sin\theta\partial\phi} + \left\langle f^2 \right\rangle \frac{\partial W}{\sin\theta\partial\phi}\right) = 0$$

But

$$\begin{aligned} \frac{\partial W}{\partial\theta} &= -\frac{1}{kT}\frac{\partial vE}{\partial\theta}W, \\ \frac{\partial W}{\sin\theta\partial\phi} &= -\frac{1}{kT}\frac{\partial vE}{\sin\theta\partial\phi}W \end{aligned}$$

Therefore the simplest solution has both brackets equal to zero, giving

$$\gamma\alpha = \frac{vM}{2kT}\left\langle f^2 \right\rangle,$$

the fluctuation-dissipation theorem for this case. The reason why this is correct, is that the energy in the random field is (restoring the saturation magnetization M to facilitate comprison of dimensions)

$$E_{rndm} = M\vec{h}_{rndm} \cdot \vec{u}$$

where $\vec{u}$ is a unit vector with components $\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta$, and the resulting averages $\left\langle \Delta\theta^2 \right\rangle / \Delta t$, $\left\langle \Delta\phi^2 \right\rangle / \Delta t$ both turn out to be proportional to $|\vec{h}_{rndm}|^2$ (without any angular dependence) and $\left\langle \Delta\theta\Delta\phi \right\rangle / \Delta t = 0$. This follows from integrating the equations

$$M\dot{\theta} = -\gamma\frac{\partial E_{rndm}}{M\sin\theta\partial\phi}, \; M\sin\theta\dot{\phi} = \gamma\frac{\partial E_{rndm}}{M\partial\theta}$$

over a time interval Δt, and averaging, it being assumed that different cartesian components of $\vec{h}_{rndm}$ are uncorrelated. So the fluctuation dissipation theorem holds for arbitrarily large motions of $\vec{M}$. The reason for success in this case lies in the fact that $\vec{M}$ and $\vec{h}_{rndm}$ enter Maxwell's equations in

essentially the same linear way. For other loss mechanisms, such as magnetoelasticity, this simple result is not guaranteed. Equation 6 may now be written

$$\frac{\partial W}{\partial t}+\frac{\partial W}{\partial \theta}\frac{\gamma \partial E}{M\sin\theta\partial\phi}-\frac{\partial W}{\sin\theta\partial\phi}\frac{\gamma\partial E}{M\partial\theta}-$$

$$-\gamma\alpha\left(\frac{\partial}{\partial\theta}W\frac{\partial E}{M\partial\theta}+\frac{\partial}{\sin\theta\partial\phi}W\frac{\partial E}{M\sin\theta\partial\phi}\right) \;=\; D\left(\frac{\partial^2 W}{\sin^2\theta\partial\phi^2}+\frac{\partial^2 W}{\partial\theta^2}\right)$$

where the diffusion coefficient $D=\frac{1}{2}\left\langle f^2\right\rangle=\frac{kT}{vM}\gamma\alpha$ is considered is independent of angle. (for example, in the case of a metal magnet, we found above that this is indeed correct). In the general case, the right hand side must be written

$$\frac{\partial}{\sin\theta\partial\phi}D\frac{\partial W}{\sin\theta\partial\phi}+\frac{\partial}{\partial\theta}D\frac{\partial W}{\partial\theta}$$

and similarly, in the second bracket on the left, $\gamma\alpha$ must be pulled inside the outer differentiations.

V. Magnetization Reversal in Small Particles

V.1. Overview. Most recording media are composed of more or less dense planar assemblies of small magnetic particles. In the typical case, these particles are too small to support magnetic domain walls, yet large enough to be outside the range of superparamagnetic effects. Imprinting information on them involves reversal of their magnetization directions from one stable position to another by means of a magnetic field supplied by the recording head. A fairly detailed understanding of the switching process should help in delineating some of the material parameters favoring a desired recording performance (such as high recording rate and efficiency). In its full generality, the problem must deal with magnetic interactions of the particles (dipolar, and, in some cases, exchange couplings), possible random spread of their anisotropy axes, and surface anisotropy. As a first step, we consider the problem of switching a small, single domain particle with uniaxial anisotropy. Initially, its magnetization vector is along the anisotropy axis, at angle zero, say. A magnetic field H is now applied along the same axis, but in the opposite direction, with the object of forcing the magnetization vector to reverse, so that it ends up at angle π, at the opposite end of the anisotropy axis. Opposing this switch is the anisotropy

barrier. Evidently, to achieve rapid switching, the applied field should substantially exceed the anisotropy field H_k ,in which case the barrier has disappeared, but even when the field is smaller, thermal agitation can help the magnetization over the barrier. Customarily, the experimenters plot the applied field, versus a 'switching time', defined somewhat loosely, but precise enough for practical purposes. There is a voluminous literature dealing with the escape of particles over potential barriers. Essentially all of it uses a form of either the Fokker-Planck equation or (where appropriate) the Smoluchowski diffusion equation as starting point. Since the original classic paper of Kramers on this subject, there have been various significant extensions in the details, if not in the concepts. An exceptionally careful application of the traditional approach to the switching problem problem was made by Safanov and Bertram[18]. These authors match the barrierless diffusion for $-H > H_k$ to the barrier regime for $-H < H_k$. Their results agree with the experimental field vs. switching time curve over most of the measured range. However, one puzzle remained: the theoretical curve has a kink at $-H = H_k$. No such kink is observed in the experiments. In the following, we begin with the diffusion equation discussed in section IV, but pursue a direct solution, not dependent on the assumptions of standard reaction theory. No kink is found at $-H = H_k$.

V. 2. . Rotation in 2d. The procedure will first be demonstrated for a simpler model, in which the rotation of the magnetization vector is very nearly confined to a plane (as it would be in a thin, disc shaped sample). In that case, the diffusion equation may be written

$$\frac{\partial W}{D\partial t} = \frac{\partial}{\partial \theta}\left(\frac{\partial W}{\partial \theta} + \frac{v\partial E}{k_B T \partial \theta} W\right) \tag{83}$$

where E is the magnetic energy per unit volume, v the sample volume, and $D = \frac{\alpha\gamma k_B T}{vM}$ is the diffusion coefficient. In the present model

$$E = -HM\cos\theta - \frac{1}{4}H_k M \cos^2\theta$$

where the $\frac{1}{4}$ is just a convenience factor. We restate equation (83) in dimensionless form, measuring time in units of D^{-1}, and measuring magnetic energy in terms of thermal energy, so that $vHM/k_BT \to H$, $vH_kM/k_BT \to H_k$, and $v\partial E/k_BT \to E$. Then equation (83) reads

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial \theta}\left(\frac{\partial W}{\partial \theta} + \frac{\partial E}{\partial \theta} W\right). \tag{84}$$

As previously shown by Caroli, Caroli, and Roulet[19] in connection with diffusion in a bi-stable potential, it is convenient to transform this equation into a standard form familiar from quantum mechanics. In our particular case, the transformed equation relates to the quantum mechanics of a particle in a periodic potential, and allows us to bring to bear some of the associated folklore. The required transformation is

$$W(\theta, t) = w(\theta, t) e^{-\frac{1}{2}E(\theta)}, \tag{85}$$

which brings equation (2) into the form

$$\frac{\partial w}{\partial t} = \left(\frac{\partial^2 w_\lambda}{\partial \theta^2} + V(\theta) \right) w_\lambda \tag{86}$$

where

$$V(\theta) = \frac{1}{2}\frac{\partial^2 E}{\partial \theta^2} - \frac{1}{4}\left(\frac{\partial E}{\partial \theta}\right)^2 \tag{87}$$

Equation (86) resembles the Schrödinger equation, but for imaginary time. Written out in full, the potential is

$$V(\theta) = \frac{1}{2}(\ H\cos\theta + H_k \cos 2\theta) - \frac{1}{4}\ (H + H_k \cos\theta)^2 \sin^2\theta, \tag{88}$$

In a typical switching problem, for $t < 0, W$ may be the Boltzmann distribution centered on $\theta = 0$, and w will equal $\sqrt{W}$. At $t = 0$, a field $H = -|H|$ is applied. To simplify the argument, we shall assume that at $t = 0$, w is zero everywhere, except at $\theta = 0$. (The results are easily generalized to other initial values of θ, and the results for an arbitrary initial distribution can then be found by linear superposition.)

For orientation purposes we briefly discuss the case of free diffusion, $(V = 0)$, on the circumference of a circle, subsequent to $w(\theta, 0) = \delta(\theta)$ at time zero. If θ were not an angular position variable, but a coordinate on an infinite straight line, the well-known solution would be $w_\infty(\theta, t) = (2\pi t)^{-1/2} \exp(-\theta^2/4t)$. However, this function is not single valued on the circle. On the other hand, a single valued function can be constructed from it:

$$\begin{aligned} w(\theta, t) &= \sum_{-\infty}^{\infty} w_\infty(\theta + 2n\pi, t) \\ &= (4\pi t)^{-1/2} \sum_{-\infty}^{\infty} \exp\left(-(\theta + 2n\pi)^2/4t\right), \end{aligned}$$

a result which can be written in compact form as one of the ϑ functions. This sum is the response to the *periodic* delta function $\sum \delta(\theta + 2n\pi)$ that takes account of the fact that on the circle, there is nothing to distinguish θ from $\theta + 2n\pi$. A direct way of arriving at the same result is to note that the even eigenfunctions of the free diffusion equation are $\cos n\theta$, with eigenvalues n^2, so that an alternative form of the result is $\approx \sum_{-\infty}^{+\infty} e^{-n^2 t} \cos n\theta$, which is, in fact, the Fourier series for ϑ_3. For very short times, the series converges very slowly, and is a poor representation of the essential singularity at $t = 0$. For long times, the Fourier series is evidently better. The term of the series with $n = 0$ gives the equilibrium solution, which, in this field free case, is independent of θ. The 'switching time 'needed to very nearly reach its equilibrium value is just the lifetime of the initial state. Strictly, that time is infinite, but for practical purposes it is sufficient to seek the time needed to come to within a factor s of the final state, with s close to unity. That time is obviously dominated by the $n = 1$ term of the series solution. At $\theta = \pi$, and time t, the state is very nearly $1 - e^{-t}$, and this will equal $1 - s$ at a time $t = |\ln(1 - s)|$.

Analogous considerations apply when fields are present. The series solution of equation (86) following the initial periodic $\delta-$ function now has the form

$$w(\theta, t) = \sum_{\lambda} e^{-\lambda t} w_\lambda(0) w_\lambda(\theta)$$

where $w_\lambda(\theta)$ and λ are the eigenfunctions and eigenvalues that solve the equation

$$\frac{\partial^2 w_\lambda}{\partial \theta^2} + (V(\theta) + \lambda)\, w_\lambda = 0,$$

with only positive λ allowed, and only $2\pi-$ periodic eigenfunctions admitted. The smallest eigenvalue is $\lambda = 0$, and the corresponding eigenfunction is $w_0(\theta) = \exp(-\frac{1}{2}E(\theta))$, (the corresponding $W(\theta)$ is just the Boltzmann factor). Because $V(\theta) = V(-\theta)$, the w_λ must be either even or odd. Because the periodic delta function is even, only the even $w_\lambda(\theta)$ can be used here. Therefore $w'_\lambda(0) = w'_\lambda(2\pi) = 0$. Furthermore, $V(\theta)$ is symmetric about π, therefore the w_λ must be even or odd about π. Evidently, if they were odd about π, they could not satisfy $w_\lambda(0) = w_\lambda(2\pi)$. therefore they must be even about π, so that $w'_\lambda(\pi) = 0$. (Eigenfunctions odd about π must vanish at π. Therefore they are appropriate if there is a *sink* at π, and there is no sink in our present problem). Just as in the field-free case, the series solu-

tion is not a good description of the short time behavior of $w(\theta,t)$. At very short times, the diffusing particle cannot be aware of the presence of V; therefore a singularity very similar to that for the free particle must arise. To deal with it, we again resort to the solution $\bar{w}(\theta,t)$ with initial condition $\delta(\theta)$, with *no* periodicity restrictions, subsequently forming the desired periodic solution $w(\theta,t) = \sum \bar{w}(\theta + 2n\pi, t)$. (Clearly $\bar{w}(\theta,t)$ must reduce to the usual source solution in the field free case). At this point, the analogy with the quantum mechanics of a particle in a periodic potential becomes useful: $\bar{w}(\theta,t)$ describes the motion of a particle released at time zero at position zero in a periodic lattice, but in imaginary time. In the field free case, the source solution could have been found by superposing the infinite space eigenfunctions $e^{ik\theta}$ with a continuous spectrum of eigenvalues k, and corresponding time decay e^{-k^2t}. For the particle in a periodic lattice, the analogues of the plane waves are the Bloch-Floquet functions $e^{ik\theta}u(k,\theta)$, where $u(k,\theta)$ is periodic in θ. In this picture, although k still ranges from minus to plus infinity, the continuity of the eigenvalue (analogous to k^2), is interrupted by an infinity of gaps. As the result it is usually more convenient to restrict the range of k, and to introduce the 'band index' p, now writing the eigensolutions in the form $e^{ik\theta}u_p(k,\theta)$, decaying in time like $e^{-\lambda_p(k)t}$. The various $\lambda_p(k)$ are periodic functions of k, but for small k vary like $\mu_p k^2$. In an approximation in which this form is assumed to hold within each band, it then follows that the analog of the free field source solution is approximately equal to $\bar{w}(\theta,t) = \sum_p \frac{1}{\sqrt{4\pi\mu_p t}} e^{-\frac{\theta^2}{4\mu_p t}}$. This has the required singularity at $t = 0$. Within each band, the loss parameter α is effectively renormalized to $\mu_p\alpha$. The $\mu_p$can be calculated approximately by so called $k \cdot p$ perturbation theory[20]. Here we only note that the $\lambda_p(k)$ at $k = 0$ are the same as the eigenvalues λ used in the series representation.

A general feature of the eigenvalue spectrum should be noted: the eigenvalues depend on the magnitude, but not on the sign of H. This is a consequence of linearity of the diffusion equation, but may be checked by observing that the eigenvalue equation (4) is unchanged by the replacement $H \to -H, \theta \to \pi - \theta$. (Obviously the time to evolve in a field H from initial position around $\pi - \theta_1$ is the same as the time to evolve in a field $-H$ from initial position θ_1). The definition of switching time τ depends on practical considerations. The most exacting definition would require the final state to come within a factor $1-s$ of equilibrium in the switched field, with s extremely close to unity. The closer it is to unity, the longer the switching

time, so that only the term with the smallest λ need be retained in the series solution. Then τ is given by

$$w(\pi,\tau) = sw_0(\pi) = w_0(\pi) + w_1(0)w_1(\pi)\exp(-\lambda_1\tau)$$

, where $\lambda_1 is$ the smallest non-zero eigenvalue, and we have chosen a normalization of the eigenfunctions such that $w_0(0) = 1$. It follows that

$$\lambda_1\tau = \left|\ln\frac{(1-s)w_0(\pi)}{w_1(0)w_1(\pi)}\right|$$

and if s is close to one, the right hand side is very nearly equal to $|\ln(1-s)|$. Thus, for this exacting definition of switching time, only the smallest non-zero eigenvalue needs to be known, but not the eigenfunctions, . In practice, less stringent definitions must be adopted and are usually considered sufficient. In the case of magnetic recording, the switching field is applied for only a short period as the medium passes below the recording head. Therefore one may have to settle for values of s not much greater than one- half. Then higher terms in the series may have to be included, and the result will depend on several eigenfunctions and eigenvalues. A formal procedure of finding these, is to note that equation (4) has the form of Hill's equation and to evaluate an infinite determinant derived from that equation. Exact analytic results are probably out of reach (the Hill determinant is not sufficiently sparse, having four filled superdiagonals and four filled subdiagonals). Therefore, in Appendix A, we indicate a very rapid computational way of solving this problem. The results are shown in figures V. 1, 2, and 3. Figure 1 shows the first five eigenvalues as functions of H, for $H_k = 3$. Figure V 2 shows the switching field H as a function of τ for the case $1 - s = e^{-1}$, and various H_k, when only the lowest non-vanishing eigenvalue is considered. Figure V 3 shows the contours of equal τ in the $(H, H_k)-$ plane for $1 - s = e^{-1}$, again using λ_1 only.

To end this section, we briefly discuss the case of anisotropy axis inclined to the direction of H by a finite angle. (A spread in such angles is normally found in media particles) . Initially, the magnetization is aligned close to one of the two anisotropy directions, along angle θ_a, say. The field is then applied to try to force the magnetization into the direction π. It will succeed only if H is large enough. The final angle θ_f will be less than π as long as the equation $-|H|\sin\theta_f + \frac{1}{2}H_k \sin 2(\theta_f - \theta_a) = 0$ has a solution. Figure V.3 shows θ_f as a function of θ_a for values of the parameter $r = |H|/H_k$

increasing from zero to 2 in steps of 0.125. Evidently, when r is very slightly greater than 1, the equation has a solution only if θ_a exceeds approximately 2 radians. When it is less than that, the final angle snaps to the value π. In practice, this is the important regime. the potential $V(\theta)$ is now

$$\frac{1}{2}\left(H\cos\theta + H_k\cos(2(\theta-\theta_a)\right) - \frac{1}{4}\left(H^2\sin^2\theta + \frac{H_k^2}{4}\sin^2 2(\theta-\theta_a) + HH_k\sin\theta\sin 2(\theta-\theta_a)\right)$$

and is of mixed parity. Therefore each eigenfunctions can be written as a linear combination of even and odd functions. Substituting that combination in the 'Schroedinger' equation then yields two coupled equations for the even and odd parts. Evidently the even part will have much the same properties as the even eigenfunction for $\theta_a = 0$ discussed above. In particular, its derivative at $\theta = \pi$ will be zero. The procedure explained in Appendix A can then be used. The results for a few values of θ_a are shown in figure V 4.

V. 2. Rotation in 3d. In that case also, the diffusion equation

$$\frac{\partial W}{\partial t} = \nabla\cdot(\nabla W + (\nabla E)W)$$

can be transformed into Schroedinger-like form

$$\frac{\partial w}{\partial t} = \nabla^2 w + \left(\frac{1}{2}\nabla^2 E - \frac{1}{4}(\nabla E)^2\right)w$$

with $W = e^{-\frac{1}{2}E}w$, and, in polar coordinates, $\nabla = (0,\ \frac{\partial}{\partial\theta}, \frac{\partial}{\sin\theta\partial\phi})$ for constant M. Here, E stands for $vE/(k_BT)$. The range of θ is (0,π), that of ϕ is $(0, 2\pi)$. for consistency with the notation of the previous section, assume the field H applied along the $z-$ axis, i.e.$\theta = 0$. The energy is then $-H\cos\theta + E_k(\theta,\phi)$, with anisotropy energy E_A. The effective potential is then

$$\begin{aligned} V(\theta,\phi) &= \frac{1}{2}\left(H\cos\theta + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial E_k}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 E_k}{\partial\phi^2}\right) - \\ &\quad -\frac{1}{4}\left(\left(H\sin\theta + \frac{\partial E_k}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial E_k}{\partial\phi}\right)^2\right) \end{aligned}$$

and the eigenvalue problem to be solved is

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial w}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial\phi^2} + (V(\theta,\phi)+\lambda)w = 0$$

For uniaxial crystalline anisotropy, with axis aligned along Oz, and for particles with rotational symmetry around that axis, E will be a function of θ only, as will V. In that case, even though the magnetization vector will precess in the $\phi-$ direction also, that motion cannot affect the switching process. Thus we can average over ϕ, in which case the second term on the left of the preceding equation drops out, since, for single-valuedness, w must be periodic in ϕ. The resulting w is a function of θ only. The further transformation $w = u/\sqrt{\sin\theta}$ then gives

$$\frac{\partial^2 u}{\partial\theta^2} + (\hat{V} + \lambda)u = 0,$$

where

$$\hat{V} = \frac{1}{2}\left(H\cos\theta + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial E_k}{\partial\theta}\right) - \frac{1}{4}\left(H\sin\theta + \frac{\partial E_k}{\partial\theta}\right)^2 + \frac{1}{2}\mathrm{cosec}^2\theta - \frac{1}{4}\cot^2\theta$$

For the case $E_k = -\frac{1}{2}H_k\cos^2\theta$, the first terms of $\hat{V}$ becomes $\frac{1}{2}\left(H\cos\theta + \frac{3}{2}H_k\cos(2\theta) + \frac{1}{2}H_k\right)$,while the second term remains unchanged at $-\frac{1}{4}\left(H + H_k\cos\theta\right)^2\sin^2\theta$ as in section 1. The singularities of the last two terms at $\theta = 0$ and $\theta = \pi$ do not impair the eigenvalue solutions of the equation for u. (For θ close to zero the third and fourth term dominate, and one solution there has the form $u \approx \sqrt{\theta}J_0(\sqrt{\lambda}\theta)$, where J_0 is a Bessel function, so the corresponding w is finite, and its derivative vanishes there, as it should. (The other solution goes to infinity). Similarly one solution remains finite and its derivative vanishes as θ approaches π. Thus nothing qualitatively different from the results of section 1 arises in this totally uniaxial case. Suppose, however, that the crystalline anisotropy is as before, but the particle is no longer rotationally symmetric around the direction of H and H_k. Then a demagnetizing energy $E_{dmg} = \frac{1}{2}N\sin^2\theta\,\sin^2\phi$ arises, and averaging over ϕ is no longer allowed. We have $\nabla E_{dmg} = \left(\frac{1}{2}N\sin 2\theta\sin^2\phi, \frac{1}{2}N\sin\theta\sin 2\phi\right)$ and $\nabla^2 E_{dmg} = N\left((2\cos^2\theta - \sin^2\theta)\sin^2\phi + \cos 2\phi\right)$, so V acquires an additional term

$$\frac{N}{2}\left((2\cos^2\theta - \sin^2\theta)\sin^2\phi + \cos 2\phi\right) - \frac{N^2}{16}\left(\sin^2 2\theta\right)\sin^4\phi + \sin^2\theta\sin^2 2\phi$$

Here N is in reduced units; in the original units it is $N_x v M^2/k_B T$. Hence N will be very large in the limit in which $M >> H, H_k$. In this case, some progress is possible. Clearly ϕ, which is a measure of how far the

magnetization sticks into the energetically unfavorable direction, will be very small. Then the foregoing expression may be truncated at $\phi\hat{}2$. The eigenvalue equation for u then becomes

$$\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\sin^2\theta\partial\phi^2} \; - \; \frac{N}{2}\left(3+\frac{N}{2}\right)\phi^2\sin^2\theta \; u + \hat{V}(\theta)u + \lambda' u = 0$$

where $\lambda' = \lambda + N/2$. An exact separation of variables in this equation is not possible. However, note that the N-dependent term is small near $\theta = 0$ or π, and is largest at $\theta = \pi/2$. A reasonable approximation is therefore to replace $\sin\theta$ by some constant, c, between zero and one, in the second and third term. Then the equation is separable. The eigenfunctions are products of the functions without demagnetizing, times harmonic oscillator eigenfunctions, and the eigenvalues are the corresponding sums of eigenvalues. These functions all have a Gaussian decay factor that concentrates w around $\phi = 0$, as expected. An estimate of the constant c may be made by considering the exact ground state, which, to order ϕ^2 is $w_g = \exp -\frac{1}{2}(E + N\phi^2\sin^2\theta)$. A reasonable value of c is then obtained by minimizing the mean deviation

$$\frac{1}{4\pi}\int_0^\pi d\phi\int_0^\pi d\theta\sin\theta\left|\exp -\frac{1}{2}(E+\frac{1}{2}N\phi^2\sin^2\theta) - \exp -\frac{1}{2}(E+\frac{1}{2}N\phi^2c^2 \;\;)\right|$$

from the true ground state numerically. It is found that for the case $|H| = 10$, $H_k = 3$ the smallest fractional deviation is obtained for $c \cong .45$ over the range of $N-$ values from zero to five, and increases linearly from just below one percent at $N = 1$ to just over five percent at $N = 5$. Assuming that not only this picture, but also the value of c, does not change much in the higher states, each eigenstates of u will be products of the form $u_{\lambda_n}(\theta)$ times a harmonic oscillator wave function of ϕ, with corresponding $\lambda'_{n,m} = \lambda_n + \omega(m+\kappa_m)$, where $\omega = c\sqrt{N(3+N/2)}$. It would be too much to expect the 'zero point value' κ_m to be independent of m (all we know is that its value is zero for $m = 0$).

V.3. A note on the conventional treatment. In spite of some pitfalls already mentioned in the introduction of this chapter, it is worthwhile to explore the conventional and more intuitive approach with its much simpler algebra. Equation 1 happens to have the form treated by Kramers in the so-called high friction limit for particle motion over a potential barrier.

Kramers defines the escape rate as the steady state current divided by the number of particles in the originating potential well. In the steady state, equation 1 (in reduced units) has a first integral: the current

$$-J = \frac{\partial W}{\partial \theta} + \frac{\partial E}{\partial \theta} W$$

which may be written

$$-J = e^{-E(\theta)} \frac{\partial}{\partial \theta} (e^{E(\theta)} W)$$

whence

$$-J \int_{\theta_0}^{\theta_1} e^{E(\theta')} d\theta' = e^{E(\theta_1)} W(\theta_1) - e^{E(\theta_0)_{H=0}} W(\theta_0)_{H=0}$$

θ_0 is identified with the location of the originating well, in our case $\theta_0 = 0$. Kramers neglects back flow from the final well at θ_1; therefore discards the first term on the right. He also takes $W(\theta_0)$ to be the Boltzmann distribution; in our case it would be $n_0 e^{-E(\theta_0)} \; / \int d\theta_0 e^{-E(\theta_0)}$ (The integration is to extent only through the quadratic portion of the initial well). Kramers defines the escape rate

$$r = \frac{J}{n_0} = \frac{1}{\int d\theta' e^{-E(\theta')} \int_{\theta_0}^{\theta_1} e^{E(\theta')} d\theta'}$$

Most of the second integral in the denominator on the right comes from the vicinity of the barrier $\triangle$ (in our case $\triangle = \frac{1}{2}(H_k - |H|)^2$for negative H.) In that vicinity, $E(\theta) = \triangle - \frac{1}{2}(\theta - \theta_B)^2 |E''(\theta_B)|$, where $\theta_B = \arccos(|H|/H_k)$ and the integration may then be extended from $-\infty$ to ∞. Also

$$\int d\theta' e^{-E(\theta')} \approx e^{\frac{1}{2}H_k - |H|} \int d\theta e^{-(H_k - |H|)\theta^2/2} = \sqrt{\frac{2\pi}{H_k - |H|}} e^{\frac{1}{2}H_k - |H|}$$

and $E''(\theta_B) = (|H|^2 - H_k^2\}/H_k$, so that, finally

$$\begin{aligned} r &= \frac{1}{2\pi} \sqrt{(H_k - |H|)^2 (H + H_k)/H_k} e^{-\triangle - \frac{1}{2}H_k + |H|} \\ &= (H_k - |H|) \sqrt{\frac{H_k + |H|}{H_k}} e^{\frac{1}{2}H} \; e^{-\frac{1}{2}[(H_k - |H|)^2 + H_k(H_k - |H|)]/H_k} \end{aligned}$$

This works as long as $|H|$ is reasonably smaller than H_k. The procedure breaks down completely when $|H| = H_k$. At that point not only is $\triangle = 0$, but also $E''(\theta_B) = 0 = E'''(\theta_B)$. The lowest non-vanishing derivative is $E^{(iv)}(\theta_B)$. That derivative is negative, so the required integral converges. Exactly at $|H| = H_k$ it can be carried out by setting $(\theta - \theta_B) = x/|H|^{1/4}$. The integral then becomes $|H|^{-1/4} \int_{-\infty}^{\infty} e^{-x^4/6} dx = 2.837 |H|^{-1/4}$. Unfortunately no such simple result can be found for $|H|$ close to H_k, because the quadratic and higher parts of the exponent scale differently.

However, there is a more serious difficulty. A steady state requires the existence of a sink somewhere in the system. Then the initial well cannot be an equilibrium distribution as is assumed here. However, if the barrier is sufficiently high, the current due to escape will be small, and n_0 will decline very slowly, so the rate formula holds at least initially. Then back-flow may be introduced by considering also the time variation of n_{θ_1} in the final well, and using a master equation to link the two numbers:

$$\begin{aligned} dn_0/dt &= -r_{01} n_0 + r_{10} n_{\theta_1} \\ dn_{\theta_1}/dt &= -r_{10} n_{\theta_1} + r_{01} n_0 \end{aligned}$$

In this 'adiabatic' picture, $r_{01}$is the Kramers rate , and then r_{10} may be determined by detailed balance in the usual way. As $|H|$ approaches $H_k$from below, the barrier moves toward zero angle; finally, at $|H|$ just above H_k, the initial well at $\theta = 0$ is converted into a maximum, and this adiabatic picture fails. Then one must resort to the treatment in sections V. 1 through 3, which does not appeal to adiabaticity . But it is possible to compare the Kramers method with the method of sections 1 through 3 in the range $|H| < H_k$. Solving the master equation for n_0 gives

$$n_{\theta_1}(t) = n_0(0) \frac{r_{10} - r_{01} \exp -(r_{01} + r_{10})t}{(r_{01} + r_{10})} \tag{89}$$

Comparison with the earlier results then shows that λ_1 is essentially equivalent to $(r_{01} + r_{10})$. Furthermore it implies that $w_{\lambda_1}(\pi)$ is closely related $r_{01}/(r_{01} + r_{10})$. This immediately suggests that one can take a purely phenomenological approach even in the range $|H| > H_k$, since the form of the exact result for $w(\theta, t)$ at long times has the same form as the result of the diffusion equation for large times. One only has to abandon the usual derivation of r. One may even be tempted to regard the entire series for $w(\theta, t)$ as the solution of a master equation of a many-barrier problem with

specified transition probabilities between any two minima i, j. However, we do not know at this point whether the pre-exponential factors as well as the λ 's of the exact treatment can be made to match the corresponding quantities in the solution of the master equation for some particular choice of all the r_{ij}.

VI.Magnetization Reversal in Large Specimen.

VI. Overview. The simplest and oldest treatment of this subject, due to Stoner and Wohlfarth[21], envisages a square hysteresis loop. It is assumed that the magnetization switches uniformly as soon as the applied field exceeds a shape and/or crystalline anisotropy field H_k that opposes reversal. For reasons that will presently become clear, the reversal condition is written as $VM_s|H|_{crit} = VM_sH_k$. However, experiments[22] gave a value of $|H|$ substantially less than this relation indicates. The results are frequently stated in volume terms: observed $|H|_{crit} = \frac{V'}{V}H_k$.where $V' < V$ came to be known as the magnetic volume. The reason appears to be that, for samples larger than a typical domain wall width, spatial variation of $\vec{M}$ cannot be ignored, and, in fact, facilitates switching. This problem has two aspects: a. Initiation of the reversal as the result of an applied field larger than a critical value, and b.the subsequent course of the reversal towards its completion.

a. Evidently, a. is within the province of linearized theory, and therefore much easier to deal with than b, which strictly speaking, is a problem of diffusion in function space. We begin (and almost, but not quite, end) with a discussion of a. The first task is to establish the stationary configurations in the space of $\vec{M}(\vec{r})$. As discussed in section I., these satisfy the equation

$$\frac{\vec{M}(\vec{r})}{M_s} = \frac{\tilde{d}\mathrm{E}}{\tilde{d}\vec{M}(\vec{r})} \tag{90}$$

which states that in a stationary configuration, the magnetization must be everywhere along the total effective field. From hereon, we shall refer to such a configuration as a 'point'in $\vec{M}$ space, even though, in configuration space, it is, of course, an extended feature(for example, a domain configuration). To be a stable minimum, a further condition must be satisfied: the second functional derivative of E, evaluated at a point satisfying eqn(1),

must be positive definite. It ceases to be positive definite for the same parameter values (e.g. the applied field), for which the natural frequency of at least one type of deviation from the solution of 1 goes to zero. To prove this, write the equation of motion, for the increment in $\vec{M}$:

$$\delta\dot{M}_i(\vec{r}) = i\omega\delta M_i(\vec{r}) = \epsilon_{ijk}M_j^0(\vec{r})\int dv' \frac{\tilde{d}^2\mathrm{E}}{\tilde{d}M_k^0(\vec{r})\tilde{d}M_l^0(\vec{r}')}\delta M_l(\vec{r}') \qquad (91)$$

where the M^0 components are the solution of equation 18. One of the eigenvalues ω will be zero if the functional determinant with elements $\frac{^2\mathrm{E}}{M_k^0(\vec{r})M_l^0(\vec{r}')}$ is zero. But this is also the condition for the second functional derivative of E to cease being positive negative.

The energy landscape of E consists of 'maxima', saddle 'points', and 'minima'. In zero applied field, the system sits at the lowest, (the absolute) minimum. By and large, in an applied reversal field less than a certain critical value, the landscape changes a bit, acquiring a 'tilt', but the $\vec{M}(\vec{r})$ field still starts out at the same minimum, which is now less deep, (and will no longer be the global minimum). It can escape from its minimum by diffusion over the nearest and lowest saddle point, only to fall into the next minimum, from which it again escapes. Eventually, the 'tilt' provided by the reversal field makes it reach the new absolute minimum, at which point reversal is complete. All this looks like process b., until the reversal field reaches a critical value . At that point, the initial minimum is swallowed up by the adjacent saddle point and becomes a new saddle point (a very flat one at critical, but a regular quadratic saddle point beyond critical). The energy in the critical field needed to bring about this merger is approximately equal to the height of the original nearby saddle prior to reversal of the field. An approximate idea of the scenes near the original minimum before and after the merger can be obtained by expanding the total energy E up to quartic terms in the deviation of $\vec{M}$ from its value at the minimum. Beyond the critical field, the descent of $\vec{M}$ into the nearest minimum is almost ballistic, but assisted by diffusion enough to take off from the saddle point. The neighborhood of the critical condition is conveniently studied by linearization. This yields the initial exponential growth rate of a small disturbance, but says nothing about its 'post-exponential' development, (although fourth order development of E may shed some light on it). A classic calculation of a critical field in an extended sample is that of Frei, Shtrickman, and Treves[23], who evaluated $|H|_{crit}$ needed to start reversal of the magnetization

in an axially magnetized circular cylinder, and they indeed found $V'/V < 1$. Their procedure is as follows: From equation 18, it is clear that for small deviations $\delta\vec{M}(\vec{r})$ from the initial minimum $\vec{M}_0(\vec{r})$ should be at right angles to $\vec{M}_0(\vec{r})$; deviations along $\vec{M}_0(\vec{r})$ must be higher order in the $\delta\vec{M}(\vec{r})$. Their $\vec{M}_0(\vec{r})$ is along the cylinder axis, therefore $\delta\vec{M}(\vec{r})$ must lie in the transverse plane. To allow it to have a radial component at the surface would cost demagnetizing energy and raise the critical field. Therefore they take the lines of $\delta\vec{M}(\vec{r})$ to be purely circular, with only a component $\delta M_\theta(r)$ around the cylinder axis. To avoid a singularity at the center, they take $\delta M_\theta(r)$ to be one of the Bessel functions $J_n(\lambda r)$. Finally, they determine the possible values of λ by imposing the boundary condition that has been used traditionally[24]: derivative of total $\vec{M}$ in the direction of the surface normal should equal zero. So the λ 's are the roots of $J_n'(\lambda a) = 0$. This mode is called a curling mode, because the lines of total $\vec{M}$ are evidently spirals.

However, for all these modes the magnetic volume factor V'/V, though less than the Stoner-Wohfarth value 1, is still much bigger than the one observed. The reason is that the energies of these states are still of order of, though less than, the total volume, since the entire sample is required to act as nucleation center. In fact, even if the sample is perfect, nucleation will occur in a thin surface region. The reason is that the boundary condition is inhomogeneous, in contrast to the homogeneous boundary condition $dM/dn = 0$ assumed in the traditional treatments. In the case of surface pinning, this is quite obvious. But even for a pure Heisenberg nearest neighbor exchange model, at least one of the nearest neighbors of a surface spin is missing. Therefore that surface spin has to be treated differently from the bulk spins, and in the continuum mode, this results in the normal gradient of $\vec{M}$ to be of order of the applied and anisotropy fields. In consequence of an inhomogeneous boundary condition a localized state arises, with magnetization rapidly decreasing into the interior of the sample. As the result, the V'/V ratio becomes at most an A/V ratio, where A is the surface area. Furthermore, if the inhomogeneity is confined to a small region in the interior, the ratio will become much smaller still.

A rough, qualitative way of discussing this for plane waves is in terms of the spin wave spectrum, with dispersion relation

$$\omega^2 = (H + H_k + J\ell^2k^2)(H + H_k + J\ell^2k^2 + 4\pi k_T^2/k^2)$$

where $\vec{k}$ is the wavenumber of a plane wave disturbance. For instability,

one or the other of the two factors must go to zero. Consider the first factor. Checking first for the Stoner-Wohlfarth condition, for which the reversal mode is uniform, so that $k = 0$, we see that $H = -H_k$. If we are looking for an instability for $H > -H_k$ (i.e. $|H| < H_k$), we would need a negative value of k^2, which means that k must be imaginary, so that this spin wave becomes a localized disturbance (rejecting the spatially growing possibility). As shown in reference 25, homogeneous boundary conditions can never give such a mode. For a more rigorous discussion for planar geometry, including the effects of the dipolar field inside and outside the sample, see referenc[25] . It requires the use of the Wiener-Hopf method[26].

VI. 2. The reversal process itself. Diffusion in the function space of $\vec{M}$.

As in the section on reversal in small particles, we begin by considering the case of a very thin film in which rotation of the $\vec{M}$ vector is confined to the plane because of very strong demagnetizing inhibiting rotation out of the plane. Further, for orientation purposes, it is useful to consider the case of free diffusion, but this time on a hypercircle, rather than on the circumference of a circle, as previously, for a small particle. To avoid notational challanges, continuous two-dimensional $\vec{r}$ space will be replaced by a two dimensional lattice at discrete points $\vec{n}$, with spacing allowed to go to zero at the end of the calculation. Each $\vec{M}(\vec{n})$ is specified by an angle θ_n, with domain (0,2π). At time $t = 0$, each $\theta_n = 0$. the diffusion equation in the field-free case is

$$\frac{\partial W}{\partial t} = \sum_n \frac{\partial^2 W}{\partial \theta_n^2} \tag{92}$$

with initial condition $W(t = 0) = \Pi_n \delta(\theta_n)$. If we rule out a sink at $\theta = \pi$, the eigenfunctions are $\Pi_n \cos m_n \theta_n$ with eigenvalues $-\sum_n m_n$, where the m_n are zero or positive integers. The solution corresponding to the $\delta-$ function at $t = 0$ is then

$$W = \sum_{\{m_n\}} e^{-t\sum_n m_n} \Pi_n \cos m_n \theta_n \tag{93}$$

As $t \to \infty$, only the term with all $m_n = 0$ survives, with $W \to 1$, uniformly spread on the hypercircle. The longest lived term that still has some time-dependence has just a single one of the $m_n = 1$, and all the others equal to zero. Thus after a long, but not infinite, time,

$$W = 1 + e^{-t} \sum \cos \theta_n \tag{94}$$

For very short times, as in the case of small particles, the eigenfunction expansion converges too slowly to conveniently represent the singular behavior at $t = 0$. There, the solution is a product of ϑ_3 - functions. When there is a magnetic energy E, measured as before in units of k_BT, the diffusion equation is

$$\frac{\partial W}{\partial t} = \sum_n \frac{\partial}{\partial \theta_n}\left(\frac{\partial W}{\partial \theta_n} + \frac{\partial(\mathrm{E}/k_BT)}{\partial \theta_n} W\right) \tag{95}$$

and can be transformed into Schrödinger form

$$\frac{\partial w}{\partial t} = \sum_n \frac{\partial^2 w}{\partial \theta_n^2} + \sum_n \left(\frac{1}{2}\frac{\partial^2(\mathrm{E}/k_BT)}{\partial \theta_n^2} - \frac{1}{4}\left(\frac{\partial(\mathrm{E}/k_BT)}{\partial \theta_n}\right)^2\right) w \tag{96}$$

by the substitution $W = e^{=E/2}w$. Suppose that E consists of exchange energy only , with $(\mathrm{E}/k_BT) = \int dv\ E(\vec{r}) = = (J/k_BT)\frac{1}{2}\int dv\ell^2(\nabla\theta)^2 \rightarrow (Jv/k_BT)\frac{1}{2}\sum_{nm}(\theta_n - \theta_m)^2 = J'\ \frac{1}{2}\sum_{nm}(\theta_n - \theta_m)^2$, where $J' = (Jv/k_BT)$. If $\sum_{nm}$ denotes summation over nearest neighbors, equation 97 becomes

$$\frac{\partial w}{\partial t} = \sum_n \frac{\partial^2 w}{\partial \theta_n^2} + \sum_n \left(\frac{1}{2}J'Z - J'^2\left(\sum_{\ell=1}^{Z}(\theta_n - \theta_{n+\ell})^2\right)^2\right) w \tag{97}$$

where the ℓ are the vector distances to the Z nearest neighbors. With $w = e^{\frac{1}{2}NJ'Zt}u$, where N is the total number of lattice points, equation (97) can be written

$$-\frac{\partial u}{\partial t} = -\sum_n \frac{\partial^2 u}{\partial \theta_n^2} + J'^2\left(\sum_{\ell=1}^{Z}(\theta_n - \theta_{n+\ell})^2\right)^2 u \tag{98}$$

With $u \propto e^{-\lambda t,}$, this is Schrödinger's equation for a set of harmonically interacting particles centered on sites $\vec{n}$ of a planar lattice, and energy eigenvalue λ. Let $\theta_n = \frac{1}{\sqrt{2N}}\sum_k \varsigma_k e^{i\vec{k}\cdot\vec{n}}$, $p_n = i\frac{\partial}{\partial\theta_n} = \frac{1}{\sqrt{2N}}\sum_k \pi_k e^{i\vec{k}\cdot\vec{n}}$. Then the equation becomes

$$\lambda u = \frac{1}{2}\sum_k \left(\pi_k^2 + \omega_k^2\zeta_k^2\right) u \tag{99}$$

where

$$\begin{aligned} \omega_k &= J' \sum_{\ell} (1 - e^{i\vec{k}\cdot\vec{\ell}}) \\ &= J'\ Z \left(1 - \frac{1}{Z} \sum_{\ell} \cos \vec{k} \cdot \vec{\ell} \right) \end{aligned} \tag{100}$$

Equation 99 describes harmonic oscillators. With the ground state and eigenvalue given by

$$\begin{aligned} u_g &= \prod_k \frac{\exp(-\frac{1}{2}\omega_k \theta_k^2)}{\sqrt{\pi \omega_k}} \\ \lambda_g &= \prod_k \exp(-\frac{1}{2}\omega_k) \end{aligned}$$

it follows that

$$\begin{aligned} w_g &= e^{\frac{1}{2}NJ'Zt} u_g \\ &= \prod_k \frac{\exp(-\frac{1}{2}\omega_k \theta_k^2)}{\sqrt{\pi \omega_k}} \end{aligned} \tag{101}$$

with the time factor cancelling out because of zero point motion. Thus, $\lambda = 0$ for the ground state. This agrees with the fact that (101), translated back into θ_n space, solves (97), with the time variation equal to zero, as it should for thermal equilibrium. For the excited states, the eigenvalues are of the form

$$\lambda_{\{m\}} = \sum_k m_k \omega_k,$$

where the m_k are positive integers or zero. The decaying state with the lowest nonzero decay constant has the lowest permissible value of $\vec{k}$, and the corresponting $m_k = 1$, all other m equal to zero. Disregarding boundary conditions, the smallest k value is zero. This is, of course unrealistic. The plane wave modes employed here are appropriate to the infinite medium. For the bounded medium, the smallest possible k will be of order of the reciprocal sample dimensions, even if the boundary conditions are homogeneous. The reason for these very low eigenvalues is simple: In the absence of any energy other than exchange, making very long wavelength domain walls

requires very little energy. This immediately explains why nucleation in a very small portion of the sample can lead to very rapid reversal. Consider first an abrupt domain wall in the yz plane, spins up for $x \leq 0$ and down for $x > 0$. Now turn the up spin layer at $x = 0$ through some angle ξ towards the down direction. This makes the layer's initially negative interaction energy with the left nearest neighbor upspins less negative by an amount $J - J\cos\xi$. At the same time it makes its interaction energy with the nearest neighbor downspins less negative by an amount $-J\cos(\pi - \xi) = J\cos\xi$. The net change of energy is zero. This is why there is no energy cost involved in moving that domain wall. The same is true for a Bloch wall of finite width of order $\sqrt{\frac{J}{K}}\ell$: There is a cost in establishing the domain wall (because of the anisotropy energy K), but once it is established, there is no barrier to its movement. This is really a large amplitude version of the Goldstone mode: because the wall is equally happy wherever it sits, no energy is required to move it. But the cost in nucleating the domain wall is proportional to the area S of the wall. So even if the wall runs through the whole sample, the magnetic volume is only $S\delta$, where δ is of order of the wall width. If nucleation occurs near a small imperfection, the problem almost reduces to the switching of a small particle discussed previously. Once the highly localized reversal has occured, the remaining impediment to complete the process in an otherwise pure sample is the frictional retardation of the wall motion. That motion may be calculated purely ballistically; diffusion is negligible at that stage. Of course, if the specimen has imperfections at which the wall can get hung up, the magnetic volume will increase in proportion to the imperfection density. Dipolar interaction could conceivably change this picture . That interaction can create domain walls even in the absence of exchange coupling[28], and the long range character of dipolar coupling might increase the magnetic volume. But there is probably no serious barrier to domain wall motion in that case either.

To summarize, it appears that because of domain wall motion, the formidable concept of diffusion in function space is largely circumvented. The $effective$ part of the undoubtedly very complex function space of $\vec{M}$ seems to reduce to a much simpler space of small nucleation centers, or, at worst, surface configurations of $\vec{M}$.

Appendix A: The method required to find the eigenfunctions w from the differential equation is the so-called 'shooting method'. Normally such

a second order equation is solved by specifying the values of the unknown function and its first derivative at the initial point, $\theta = 0$ in this case. But here, the requirement is different: we must specify the value of w at $\theta = 0$ and at $\theta = 2\pi$. Deferring normalization of w to the end, we may assign an arbitrary initial value to w at $\theta = 0$, but then require $w(2\pi) = w(0)$. A 'quick and dirty' method of finding the eigenvalues and functions on a desktop computer is to make the assignment $w(0) = 1$, $w'(0) = 0$, and then to run the equation, assigning a large number (400 seems to be good) of trial eigenvalues. Make a table of these trial eigenvalues λ and the corresponding value of $[dw(\theta, \lambda)/d\theta]_{\theta=\pi}$. We know from the symmetry of the problem that for the correct eigenvalues, this derivative must vanish. Because we are making a table of discrete values, we will not hit this value exactly. So we select from that table successive values of λ between which $[dw(\theta, \lambda)/d\theta]_{\theta=\pi}$ changes sign. The correct value must between these two successive λ 's. Evidently, the larger the number of table entries, the more closely one will come to the exact eigenvalues. A very sensitive test of the quality of this procedure is to evaluate $w(2\pi, \lambda)$ for the quasi-exact eigenvalue found. The better that eigenvalue, the closer will $w(2\pi, \lambda)$ equal 1.

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III. 4. Intrinsic damping due to valence fluctuations.

In the previous two examples, the non-local loss torque each had an asymptotic expansions, of which the leading term was of Gilbert form. We now show that valence fluctuations in a ferromagnetic compound result in a loss torque of totally different structure. In the limit of motions small enough to permit linearization, the result can always be *interpreted* as arising from LL-G damping , but this does not help in the case of large motions. The first suggestions of a relation between valence fluctuations and magnetic losses were made almost fifty years ago by Galt[15] , by Wijn and van der Heide [16] , and by Clogston[17]. The process was analyzed in detail by Clogston, and his procedure is utilized here. Holes or electrons are imagined to hop by thermal activation between ions introduced substitutionally or interstitially in the host lattice. In particular, the authors of refs. 15 to 17 considered the case of nickel ferrite with small amounts of divalent iron replacing divalent nickel on octahedral sites. Their magnetic energies on these sites, and hence their thermal distribution, will depend on the prevailing direction of $\vec{M}$ (considering that Nickel ferrite is *ferrimagnetic*, not *ferro*magnetic). . If the electrons cannot instantly come to equilibrium as $\vec{M}$ changes, the delay provides a drag on the motion of $\vec{M}$, and therefore a magnetic loss. Clogston postulates a set of master equations for the occupation numbers N_i of the various sites, and for simplicity replaces the set by a single one,

$$\dot{N}_i = (N_i \quad \square N_i) / \square$$

where $N_i \quad \exp(\square \square_i(\vec{M}) / kT)$, with $\square_i(\vec{M})$ the energy of the *i*th site. Consider the free energy of the mobile carriers $F = \sum_i N_i \square_i(\vec{M})$. Its rate of change may be divided into a heating rate $\sum_i \dot{N}_i \square(\vec{M})$ and a rate of ‘

work done on the system $\sum_i N_i d\varepsilon_i(\vec{M})/dt$. We shall be interested only in the latter. If n_i is the deviation of N_i from its equilibrium value, the solution of the simple master equation is

$$n_i = N_i(t) - N_i\ (\varepsilon_i(\vec{M}(t))) = \int_{-\infty}^{t} dt' e^{-(t-t')/\tau} \frac{dN_i}{dt'}$$

Hence, and since we are interested only in the rate of work done on the system, the non-equilibrium part of the torque is

$$\vec{Q} = -\vec{M} \times \sum_i n_i \partial\varepsilon_i / \partial\vec{M}$$

$$= \frac{1}{kT} \vec{M} \times \int_{-\infty}^{t} dt' e^{-(t-t')/\tau} \sum_i \vec{h}_i(t) \left(\vec{h}_i(t') \cdot \dot{\vec{M}}(t') \right) N_i\ (\varepsilon_i(\vec{M}))$$

where, for small concentrations of the mobile carriers, we have set $\partial N_i / \partial\varepsilon_i = N_i / kT$, and where $\vec{h}_i(t) \equiv \vec{h}_i(\vec{M}(t)) = \partial\varepsilon_i / \partial \vec{M}(t)$ are the local anisotropy fields. For processes in which the magnetization changes slowly on the scale of the valence equilibration time, a series expansion is justified, with the result

$$\vec{Q} = \frac{\tau}{kT} \vec{M} \times \sum_i \vec{h}_i(t) [1 - \sum_{n=1} \tau^n \frac{d^n}{dt^n}] \left(\vec{h}_i(t) \cdot \dot{\vec{M}}(t) \right) N_i\ (\varepsilon_i(\vec{M}))$$

The first term in the series slightly resembles the Gilbert form, but only slightly, because the anisotropy fields strongly depend on the magnetization. For example, if there are three inequivalent sites, for which $\varepsilon_i = \frac{1}{2}\varepsilon M_i^2 / M^2$, i=1,2,3, where ε is a constant, and the damping torque is considered small with the non-dissipative torque $-\vec{M} \times \vec{H}$, then the equations of motion for the direction cosines $\alpha_i = M_i / M$ are

$$\dot{\Omega}_1 = \Omega_2 H - g\Omega^2 H \Omega_2 \Omega_3 \int_{-\infty}^{t} dt' e^{-(t-t')/\tau} \Omega_1(t) \Omega_2(t)$$

$$\dot{\Omega}_2 = -\Omega_1 H - g\Omega^2 H \Omega_3 \Omega_1 \int_{-\infty}^{t} dt' e^{-(t-t')/\tau} \Omega_1(t) \Omega_2(t)$$

$$\dot{\Omega}_3 = 2g\Omega^2 H \Omega_1 \Omega_2 \int_{-\infty}^{t} dt' e^{-(t-t')/\tau} \Omega_1(t) \Omega_2(t)$$

where $g = \langle N_i \rangle / kT$, with all occupation numbers approximated by the same average $\langle N_i \rangle$.

It turns out that reversal of magnetization using the last three equations is much more abrupt than reversal calculated with LL-G damping. Presumably the reason is that the damping term in the last three equations does not immediately 'get off the ground' when the field is switched, because of the strong dependence of the damping term on the direction cosines . For detailed pictures, see reference 10.

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