Lecture II

- Reentrance problems in perturbation theory: need to sum all orders.

- Resummation technique:
  - solution by maximum section
  - solution by matrix diagonalization

- Tuning $T_c$ suppression with other parameters
  - magnetic impurities $\frac{1}{15}$ ✓
  - magnetic field $H$ ✓
  - width in wires $a$ ✓
  - electromagnetic environment
  - spin-orbit scattering
Quick Summary of Lecture I

- Disorder in metals causes electrons to move diffusively, so Coulomb interaction less effectively screened.
- In equation for $T_c$,

$$T_c = 1.13 \omega_0 \exp[-\frac{1}{N(0) L_x - \mu^*}]$$

find that DoS $N(0) \downarrow$, pseudopotential $\mu^* \uparrow$.
- Simple-minded predictions for 2D films

$$\ln \left( \frac{T_c}{T_{c0}} \right) \sim \begin{cases} - \frac{R_o}{R_o} \left\{ \ln^3 \left( \frac{1}{2\pi T_{cT}} \right) + \ln^2 \left( \frac{1}{2\pi T_{cT}} \right) \right\} & \text{(featureless potential)} \\ - \frac{R_o}{R_o} \left\{ \ln^2 \left( \frac{1}{2\pi T_{cT}} \right) \ln(Dx^2T) + \ln^3 \left( \frac{1}{2\pi T_{cT}} \right) \right\} & \text{(screened Coulomb)} \end{cases}$$

- Need honest calculation to check if this is true.
Reentrance Problems in PT

Basic PT predicts exponential decay of $T_c$ with $R_0$, so $T_c$ never suppressed to zero:

$$\ln \left[ \frac{T_c(R_0)}{T_{co}} \right] = -\frac{1}{3} \frac{R_a}{R_0} \ln^3 \left( \frac{1}{2\pi T_{co} \tau} \right)$$

"Self-consistent" version involves solution of cubic eqn $x = \frac{1}{3} b (x+b)^3$

$$x = -b + \frac{2}{\sqrt{3}} \sin \left[ \frac{1}{3} \sin^{-1} \left( \frac{3 \sqrt{3} b}{2} \right) + \frac{2\pi n}{3} \right] \quad n = 0, 1, 2$$

There are 3 real solutions, but only $n = 0$ version gives $T_c(R_0 = 0) = T_{co}$, and so this is physical correct soln $[ n = 1 \Rightarrow T_c(R_0 = 0) = 0 \quad \text{and} \quad n = 2 \Rightarrow T_c(R_0 = 0) = \infty ]$

The $n = 0$ and $n = 1$ solutions meet at the reentrance point $\tau = \frac{4}{9} b^2 \quad x = \frac{b}{2}$ beyond which no physical soln exists.

For $b = 6$ this is $R_0 \approx 2 k \beta L$, $T_c/T_{co} \approx 0.05$

Reentrance means still can't get to zero $T_c$. 
FIG. 1. Superconducting transition temperature $T_c$ plotted as a function of the inverse of the thickness for a variety of alloy compositions. The solid line is a best fit to the data. Data replotted as a function of sheet resistance $R_\square$. Theory curve is that of Maekawa or Fukuyama (Ref. 5).
FIG. 2. \( \frac{T_c}{T_{c0}} \) vs \( R_0 \) for various compositions, with fit to theory. Values of \( T_{c0} \) are 7.6, 7.2, 6.8, and 5.8 K for increasing Ge content.
Beyond Perturbation Theory

- Problem with PT is that phonon and Coulomb interactions treated differently.

- Cancellation of low-q singularity in Coulomb potential means we can fix this:

\[
\Gamma(\varepsilon_n, \varepsilon_e) = \left[ -\lambda + t\Lambda(\varepsilon_n, \varepsilon_e) \right] - 2\pi T \sum_{m=0}^{M} \left[ -\lambda + t\Lambda(\varepsilon_n, \varepsilon_m) \right] \frac{\Pi(\varepsilon_m, \varepsilon_e)}{|\varepsilon_m|}
\]

\[
\Lambda(\varepsilon_n, \varepsilon_e) = \ln\frac{1}{(\varepsilon_n + \varepsilon_e)^{\frac{d}{2}}}
\]

[Oreg + Finkelstein PRL 83 191 (1999)]
Solution By Maximum Section Method

Continuous version of \( \Gamma \) eqn is:

\[
\Gamma(\varepsilon, \varepsilon') = \int_{T}^{u} \frac{d\varepsilon_1}{\varepsilon_1} f(\varepsilon, \varepsilon_1) \Gamma(\varepsilon_1, \varepsilon')
\]

\[
f(\varepsilon, \varepsilon') = -\lambda + e \ln \left( \frac{1}{(\varepsilon + \varepsilon')^T} \right)
\]

Define \( \Gamma'(\omega) = \Gamma(\omega, \omega) \), the effective magnitude of \( \Gamma \) at energy scale \( \omega \). If we consider \( \omega = T \) then \( \Gamma'(\omega) \) is given by:

\[
\Gamma'(\omega) = f(\omega) - \int_{\omega}^{u} \frac{d\varepsilon_1}{\varepsilon_1} f(\omega, \varepsilon_1) \frac{1}{\varepsilon_1} f(\varepsilon_1, \omega)
\]

\[
+ \int_{\omega}^{u} \frac{d\varepsilon_1}{\varepsilon_1} \int_{\omega}^{u} \frac{d\varepsilon_2}{\varepsilon_2} f(\omega, \varepsilon_1) \frac{1}{\varepsilon_1} f(\varepsilon_1, \varepsilon_2) \frac{1}{\varepsilon_2} f(\varepsilon_2, \omega) + \ldots
\]

\[
= f(\omega) - Q(\omega)
\]

Consider \( N \)-th term in \( Q(\omega) \):

\[
Q_N(\omega) = (-1)^{N+1} \int_{\omega}^{u} \frac{d\varepsilon_1}{\varepsilon_1} \ldots \frac{d\varepsilon_N}{\varepsilon_N} f(\omega, \varepsilon_1) \frac{1}{\varepsilon_1} f(\varepsilon_1, \varepsilon_2) \frac{1}{\varepsilon_2} \ldots \frac{1}{\varepsilon_N} f(\varepsilon_N, \omega)
\]

Split integral into \( N \) regions with \( j \)-th region

\[ \omega < \varepsilon_j < \varepsilon_1, \varepsilon_2 \ldots \varepsilon_{j-1}, \varepsilon_{j+1} \ldots \varepsilon_N \]

[Sudakov, Doklady 1 662 (1956)]
Consider terms to left and right of $d\varepsilon$ and use approximation $f(\varepsilon, \varepsilon') = f(\max[\varepsilon, \varepsilon'])$:

\[
Q_{N}^{Lj}(\omega) = (-1)^{j} \int_{\varepsilon}^{\varepsilon'} d\varepsilon_{j} \cdots d\varepsilon_{1} f(\varepsilon_{j}, \varepsilon_{j}) \frac{1}{\varepsilon_{j}} \cdots \frac{1}{\varepsilon_{1}} f(\varepsilon_{0}, \varepsilon_{0})
\]

\[
Q_{N}^{Ri}(\omega) = (-1)^{N-j+i} \int_{\varepsilon}^{\varepsilon'} d\varepsilon_{j+1} \cdots d\varepsilon_{0} f(\varepsilon_{j+1}, \varepsilon_{j+1}) \frac{1}{\varepsilon_{j+1}} \cdots \frac{1}{\varepsilon_{0}} f(\varepsilon_{0}, \varepsilon_{0})
\]

where we noted $f(\omega, \varepsilon_{0}) = f(\varepsilon_{0}, \varepsilon_{0})$

\[
f(\varepsilon_{0}, \omega) = f(\varepsilon_{0}, \varepsilon_{0})
\]

The $N$ different $j$ values just give the $N$ different terms with $N+1$ $f$s in integral

\[
\int_{\varepsilon}^{\varepsilon'} \frac{d\varepsilon}{\varepsilon} \Gamma(\varepsilon)^{2}
\]

\[
\Rightarrow \quad \Gamma'(\omega) = \frac{d\Gamma}{d\varepsilon} = \int_{\varepsilon}^{\varepsilon'} \frac{d\varepsilon}{\varepsilon} \Gamma(\varepsilon)^{2}
\]

\[
\Rightarrow \quad \Gamma(\omega) = -\lambda + 6\ln(\frac{1}{\varepsilon'}) - \int_{\varepsilon}^{\varepsilon'} \frac{d\varepsilon}{\varepsilon} \Gamma(\varepsilon)^{2}
\]

Define variable $z = \ln(\frac{1}{\varepsilon'})$ to get

\[
\frac{d\Gamma}{dz} = t - \Gamma^{2}
\]
Integrating this gives

\[ \ln \left( \frac{1}{\omega t} \right) = \frac{1}{2\sqrt{\nu E}} \ln \left| \frac{\nu + \sqrt{\nu E}}{\nu - \sqrt{\nu E}} \right| + \text{const} \]

We know limits:
\( \omega = \frac{1}{t} \Rightarrow \Gamma = -\lambda \)
\( \omega = T_c \Rightarrow \Gamma = -\infty \)

\[ \Rightarrow \ln \left( \frac{1}{tc} \right) = \frac{1}{2\sqrt{\nu E}} \ln \left( \frac{\nu + \sqrt{\nu E}}{\nu - \sqrt{\nu E}} \right) \]

When \( t = 0, T_c = T_{co} \Rightarrow \ln \left( \frac{1}{tc} \right) = \frac{1}{\lambda} \)

\[ \ln \left( \frac{T_c}{T_{co}} \right) = \beta - \frac{1}{2\sqrt{\nu E}} \ln \left( \frac{1 + \beta \sqrt{\nu E}}{1 - \beta \sqrt{\nu E}} \right) \]

\[ \beta = \ln \left( \frac{1}{T_{co}c} \right) \]

Expanding as a power series in \( t \), the leading term \(-\frac{1}{3} t \beta^3 \) reproduces P.T.

\( T_c \) goes smoothly to zero at \( t = \frac{1}{\beta^2} \).

For \( \beta = 6 \Rightarrow R_0 \approx 4.5 \times R \).
Why We Need Non-Perturbative Approach

- First-order PT
- Ad hoc extended PT
- RG Solution

never goes to zero

reentrance
Fig. 14. Suppression of superconductivity in amorphous Mo$_{79}$Ge$_{21}$ [42]. The solid line is a theoretical fit with Eq. (13).
Aside: "RG" View of Superconductivity

- Take pair amplitude $\Gamma(\omega)$ at large energy scales and integrate out high energies until $\Gamma$ diverges at $T_c$:

$$\frac{d\Gamma}{d\omega} = -\frac{\Gamma^2}{\omega} \Rightarrow \Gamma^{-1} = A + \ln \omega$$

- First start with Coulomb $\Gamma(\varepsilon_F) = \mu$ and integrate down to $\omega = \omega_0$

$$\Gamma'(\omega_0) = \mu^* = \frac{\mu}{1 + \mu \ln (\varepsilon_F/\omega_0)}$$

- At this energy add in phonons so total $\Gamma'(\omega_0) = \mu^* - \lambda = -\lambda^*$. Integrating gives

$$\Gamma'(\omega) = -\frac{\lambda^*}{1 - \lambda^* \ln (\omega_0/\omega)}$$

$$\Rightarrow T_c = \omega_0 e^{-\frac{1}{\lambda^*}}$$

- Magnetic impurities give eqn

$$\frac{d\Gamma}{d\omega} = -\frac{\Gamma^2}{\omega + \frac{1}{\tau_3}} \Rightarrow T_c(\frac{1}{\tau_3}) - T_{c0} = -\frac{1}{\tau_3}$$
Solution by Matrix Diagonalization

- Eqn for $\hat{\Gamma}$ is a matrix eqn with rows and columns labelled by Matsubara frequency:

$$\hat{\Gamma} = \left[ -\hat{\chi} \hat{1} + \hat{t} \hat{\Lambda} \right] - \left[ -\hat{\chi} \hat{1} + \hat{t} \hat{\Lambda} \right] \hat{\varepsilon}^{-1} \hat{\Gamma}$$

where $\Gamma_{nm} = \Gamma(E_n, E_m)$; $\Lambda_{nm} = \Lambda(E_n, E_m)$; $1_{nm} = 1$ and $\varepsilon_{nm} = (\eta + \frac{\nu}{2}) S_{nm}$. Solving for $\hat{\Gamma}$ with $1_{nm} = S_{nm}$

$$\hat{\Gamma} = \left\{ \hat{1} + \left[ -\hat{\chi} \hat{1} + \hat{t} \hat{\Lambda} \right] \hat{\varepsilon}^{-1} \right\}^{-1} \left[ -\hat{\chi} \hat{1} + \hat{t} \hat{\Lambda} \right]$$

$$= \hat{\varepsilon}^{-1} \left\{ \hat{1} + \hat{\varepsilon}^{-1} \left[ -\hat{\chi} \hat{1} + \hat{t} \hat{\Lambda} \right] \hat{\varepsilon}^{-1} \right\} \hat{\varepsilon}^{-1} \left[ -\hat{\chi} \hat{1} + \hat{t} \hat{\Lambda} \right]$$

- When the matrix in brackets has a zero eigenvalue, $\hat{\Gamma}$ is singular and we are at $T_c$.

- Rank of matrix $M = \frac{1}{2\pi T}$ depends upon $T$. Start at $M_0 = \frac{1}{2\pi T_{co}}$, find lowest eigenvalue and increase $M$ by one until eigenvalue changes sign. Then $T_c/T_{co} = M_0/M$.

- Fast + adaptable using Lanczos on Pentium PC.
Effect of Magnetic Impurities

- Extend \( \Gamma \) eqn to include magnetic impurities:
  \[
  \Gamma(\varepsilon_n, \varepsilon_e) = \left[-\lambda + \lambda \Lambda(\varepsilon_n, \varepsilon_e)\right] - \pi T \sum_{m}^{M} \left[-\lambda + \lambda \Lambda(\varepsilon_n, \varepsilon_m)\right] \frac{\Gamma(\varepsilon_n, \varepsilon_e)}{\varepsilon_{m1} + \sqrt{\varepsilon_{e3}}}
  \]

\[
\Lambda(\varepsilon_n, \varepsilon_m) = \begin{cases}
  \ln\left[\frac{1}{(1\varepsilon_n + 1\varepsilon_m)\tau}\right] & \varepsilon_n \varepsilon_m < 0 \\
  \ln\left[\frac{1}{(1\varepsilon_n + 1\varepsilon_m)\left(2/3\right)\tau}\right] & \varepsilon_n \varepsilon_m > 0
\end{cases}
\]

- Matrices double in size to rank 2M due to breaking of time-reversal invariance.

- Magnetic impurities affect both mean field and perturbative terms. Hard to approximate by differential eqn unless we ignore 2nd effect
  \[
  \frac{d\Gamma}{d\omega} = \frac{\Gamma^2}{\omega + \frac{1}{\tau_3}} - \frac{\Gamma}{\omega}
  \]

- Experimentally measure pair-breaking rate per impurity:
  \[
  \alpha(R_0) = \frac{T_c(R_0, 0) - T_c(R_0, \sqrt{\tau_3})}{\sqrt{\tau_3}}
  \]

Theory predicts \( \alpha(R_0) \) roughly constant.
MAGNETIC IMPURITIES EXPT

[CHERVENAK + VALLES PRB51 11977 (1995)]

FIG. 1. A 10-Å, 800 – Ω Pb0.9Bi0.1/Ge film in which $T_c$ is suppressed by incrementing the thickness of the Gd top layer. The dashed line suggests the regime of linear suppression of $T_c$ with $n_P$.

LINEAR SUPPRESSION OF $T_c$ BY $\frac{1}{T_s}$

FIG. 2. Inverse $T_c$ shift vs film thickness of PbBi. The deviation from the dashed line indicates reduced pair-breaking strength in the thickest films. Inset: Data from three experimental runs showing $T_c \propto \frac{1}{d}$ in PbBi films.

LINEAR SUPPRESSION OF $T_c$ BY $Vd$
$T_c$ (K)

$R_{sq}$ (kΩ)

MAGNETIC IMPURITIES EXPT

SUPPRESSION OF $T_c$ BY RA
$R_{sg}(U)$ vs $R_{sg}$ of film
Pair-Breathing Rate Per Impurity
Magnetic Impurities Expt
First Order P.T.: Magnetic Field


\[
\ln \left( \frac{T_c}{T_{co}} \right) = \psi(\frac{1}{2}) - \psi(\frac{1}{2} + \frac{1}{2\pi T_c t_{th}}) + R_{HF} + R_v
\]

\[
R_{HF} = -\frac{1}{2} \ln^2 \left( \frac{1}{2\pi T_c t_{th}} \right) - t \ln \left( \frac{1}{2\pi T_c t_{th}} \right) \left[ \psi(\frac{1}{2}) - \psi(\frac{1}{2} + \frac{1}{2\pi T_c t_{th}}) \right]
\]

\[
R_v = -\frac{1}{3} \ln^3 \left( \frac{1}{2\pi T_c t_{th}} \right) - t \ln^2 \left( \frac{1}{2\pi T_c t_{th}} \right) \left[ \psi(\frac{1}{2}) - \psi(\frac{1}{2} + \frac{1}{2\pi T_c t_{th}}) \right]
\]

- Basic idea: magnetic fields scramble the phase coherence needed for superconductivity and the localization processes that suppress superconductivity: WHO WINS?

- Above formula predicts upturn in $H_{c2}(T)$ at low $T$ as $R_0$ increased i.e. localization loses.

- We find effect disappears if sums in perturbation theory performed carefully.

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**NO UPTURN IN $H_{c2}(T)$ PREDICTED BY LOCALIZATION THEORY**
Effect of Magnetic Fields

- We extend eqn to include magnetic fields by calculating $\Gamma$ at finite external momentum and setting $Dq^2 = 2\gamma T_H = 2DqH$: 

$$\Gamma(\varepsilon_n, \varepsilon_e) = \left[-\lambda + \kappa \Lambda(\varepsilon_n, \varepsilon_e)\right] - \pi T \sum_{m=-(M+1)}^{M} \left[-\lambda + \kappa \Lambda(\varepsilon_n, \varepsilon_m)\right] \frac{\Gamma(\varepsilon_m, \varepsilon_e)}{1|\varepsilon_m|}$$

$$\Lambda(\varepsilon_n, \varepsilon_m) = \begin{cases} \ln \left[\frac{1}{(1|\varepsilon_n| + 1|\varepsilon_m|)^\tau}\right] & \varepsilon_n \varepsilon_m > 0 \\ \ln \left[\frac{1}{(1|\varepsilon_n| + 1|\varepsilon_m|)^\tau}\right] + \frac{2}{(1|\varepsilon_n| + 1|\varepsilon_m|)^\tau} & \varepsilon_n \varepsilon_m < 0 \end{cases}$$

- Matrices double in size to rank $2M$ due to breaking of time-reversal invariance.

- Find NO POSITIVE CURVATURE IN $H_{\mathbb{C}_2}(T)$. 
FIG. 3. Upper critical-field data, with GLAG fits (dashed lines). Inset shows MEF theory for same samples.
Fit of Graybeal and Beasley $H_{c2}$ Data to MEF Theory

$T/T_{\phi 0}$

$(0'(0)^{T2}H/(T')^{T2}H)$
Fit of $H_{c2}$ Data to Exact PT and MEF Theory

[Expt: Graybeal + Beasley PR8294,167 (1984)]
Fit of $H_2$ Data to Exact PT and MEF Theory


In/InO$_x$ films

$\alpha/\alpha_0$

$T/T_0$

33000 K

29000 K

22500 K
Fit of $H_{c2}$ Data to Exact PT and MEF Theory


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The graph shows the fit of $H_{c2}$ data to Exact PT and MEF Theory. The graph plots $\alpha/\alpha_0$ against $T/T_{c0}$ for Zn films. Two lines are represented: the dashed line for MEF Theory and the solid line for Exact PT. The graph includes data points at 400 $\Omega$ and 600 $\Omega$, indicating measurements at different resistances.
$T_c$ Suppression in Quasi-1D Wires

- Pb wires of length $L \sim 2 \mu m$ and widths $a \sim 1000 - 150 \AA$. Find decreasing width further suppresses $T_c$.

- Theoretically new feature is that transverse momentum sum cannot be replaced by an integral:

$$\Lambda(\varepsilon_n, \varepsilon_m) = \frac{4\pi D}{a L} \sum \frac{1}{D q^2 + 1\varepsilon_n + 1\varepsilon_m}$$

$$= \frac{4\pi D}{a} \sum \int_0^\infty \frac{2d2q}{2\pi} \frac{1}{D q^2 + D q^2 + 2\pi T (n+m+1)}$$

$$= \frac{4}{a} \sum \frac{1}{\sqrt{q^2 + \frac{2\pi T}{D} (n+m+1)}} \tan^{-1} \left[ \frac{1/\sqrt{D T}}{\sqrt{q^2 + \frac{2\pi T}{D} (n+m+1)}} \right]$$

- Find that result is very sensitive to boundary conditions:

  $q_a = \frac{2\pi p}{a}$

  $p = 0, \pm 1, \pm 2 \ldots$ (periodic)

  $q_a = \frac{2\pi p}{a}$

  $p = \pm 0, \pm 1, \pm 2 \ldots$ (Oreg + Finkel'stein)

  $q_a = \frac{\pi p}{a}$

  $p = 0, 1, 2 \ldots$ (zero gradient) ✓
Zero gradient boundary conditions are correct since these correspond to zero supercurrent flowing out of sides of wire.

An additional problem is that $T_c(R_0)$ data on 2D films doesn't fit well to theory; quantity to consider is $T_c$ suppression of wires relative to films:

$$\left[ \frac{T_c(a, R_0)}{T_c(a, 0)} \right] / \left[ \frac{T_c(2D, R_0)}{T_c(2D, 0)} \right]$$

2D $\rightarrow$ 1D crossover occurs when width $a$ is of same size as thermal length $L_T = [2\pi D/T]^{1/2} \sim 300A_0$.

Get qualitative but not quantitative agreement between theory + expt.
$T_c$ vs Normal State $R_n$ for
2D Pb Films + Wires

[Sharifi et al, PRL 71, 428 (1993)]
$T_c$ vs Normal State $R_n$ for 2D Pb Films + Wires [Xiong et al. PRL 78 927 (1997)]
Results

To obtain 2D Data by Fitting 2D Data

Perfect Diagonalization Results
Xiong Data: Fractional $T_c$ Suppression in Wires Relative to Films
Theory vs Expt

zero gradient b.c.: \( q_a = 0, \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4} \ldots \)

![Graph showing data points and curves for different values of Tc(2D)/Tc(2D) vs. Resistance R/\Omega.](image)

- \( \nabla 1000A \)
- \( \Delta 580A \)
- \( \square 350A \)
- \( \bullet 250A \)
Theory vs Expt

periodic b.c. : \( g_a = 0, \pm \frac{2\pi}{a}, \pm \frac{4\pi}{a} \ldots \)

[Graph showing the relationship between \( \frac{T_c(a)/T_{c0}(2D)}{T_c(2D)/T_{c0}(2D)} \) and Resistance \( R/\Omega \). The graph includes data points for different resistances labeled as 1000A°, 580A°, 350A°, and 250A°.]
Theory vs Expt

Oreg's b.c.: \( q_a = \pm 0, \pm \frac{2\pi}{a}, \pm \frac{4\pi}{a} \ldots \)

![Graph showing resistance vs. normalized temperature ratio for different currents.](image)
Turning Off Coulomb Interaction

- Start with superconducting Ta film of $R_0 \approx 100 \Omega$ which shows $T_c$ suppression.

- Add thin oxide layer and thick Au layer: image charges induce in Au weaken effective Coulomb interaction.

- $T_c$ increases showing suppression due to Coulomb interaction.

Spin-Orbit Scattering

- Spin-orbit scattering leads to an antilocalization effect: $\sigma(T)$ increases as $T$ decreases.

- Would this lead to turning off $T_c$ suppression effects: i.e., $T_c \uparrow$ as $1/T_{so} \uparrow$?

- Seen in expt with granular Al + Bi.
Turning Off Tc Suppression in Ta Films by Turning off Coulomb Interaction

[Astrakharchik + Advans PRB50 13622(1994)]

![Graph showing the effect of Au layer on R/R_N vs Temperature(K)]

- **no Au layer**
- **with Au layer**

- **Au (2700A°)**
- **TaO_2 (50-100A°)**
- **Ta Film (~100L)**
Effect of Spin-Orbit Scattering on $T_c$ Suppression in Granular Al

[Miller et al. PRL 61 2717 (1989)]

![Graph showing the relationship between $T_c$ (K) and $\rho_{rt}$ (Ωcm)].

- Granular Al
- Granular Al + 2% Bi