Topological phases

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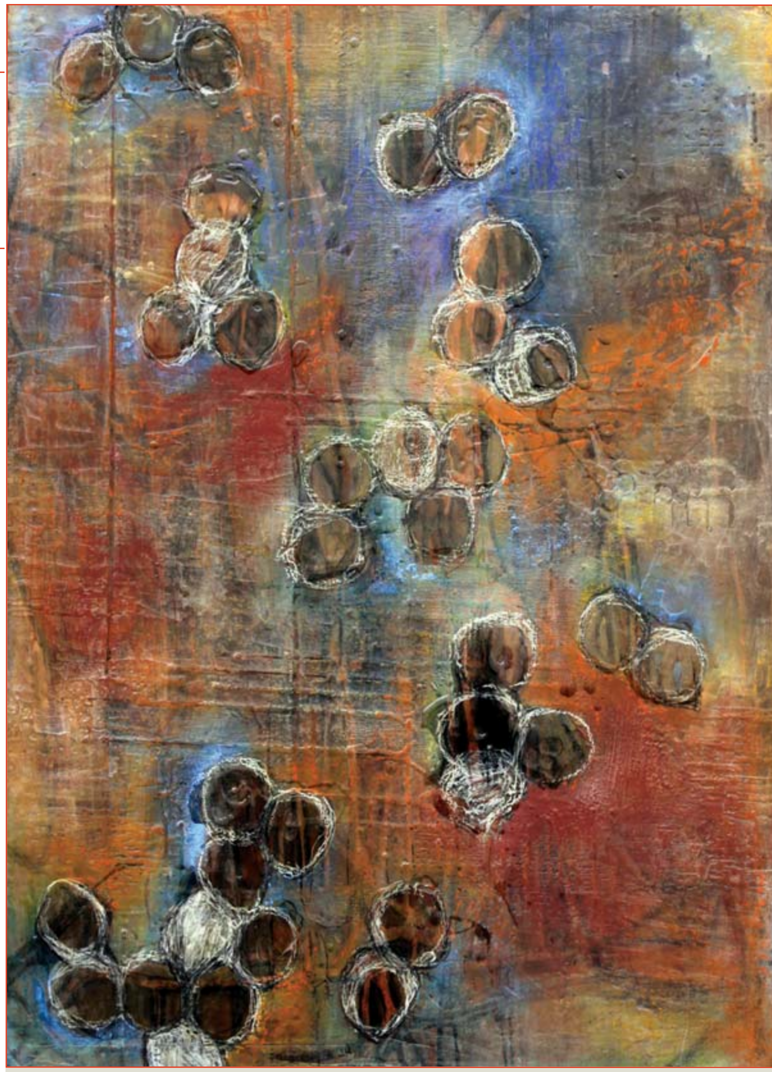
Phases of matter distinguished by topological properties sometimes possess quasiparticles with bizarre statistics. Those that do may see applications in quantum information processing.



Topology #7046 by Renee DeCarlo Johnson, 2010.

You were probably told in high school that there are three phases of matter: solid, liquid, and gas. In reality, many more phases exist. But what exactly is a phase of matter? How can one phase be distinguished from another?

Basically, for a large collection of similar particles, a phase is a region in some parameter space in which the thermal equilibrium states possess some properties in common that can be distinguished from those in other phases. That means that some discontinuity of the properties occurs at the boundary between the phases.

Traditionally, the properties used to distinguish between phases of a system are expectation values of local observable quantities. By "local quantity," I mean a function that depends only on the coordinates and momenta of the particles that are within some finite distance of a particular point in space. The density at a point is a local quantity; so is the number of particles in a small region close to some point.

For example, as water transitions from liquid to gas by passing through the boiling point at atmospheric pressure, the particle density jumps from high to low. For water, however, other paths in parameter space go continuously from liquid to gas and evade the discontinuity in the density. When two phases cannot be smoothly connected by *any* path in parameter space, the traditional view is that there must be a difference in their symmetry properties. For example, liquid water is translationally invariant on average, whereas the crystal lattice of solid ice breaks the translation symmetry—its particle density has a periodic modulation. That distinction is sharp, and it has physical consequences: There must be a phase transition at the boundary between zero and nonzero modulation.

The same ideas can be applied to distinguish phases at very low or even zero temperature. In that regime, quantum mechanical effects are usually important; one then speaks of quantum phases. As the temperature is lowered, the particles find their way into the lowest-energy state. That ground state is specified by a wavefunction that depends on the positions of all the particles. Changes in Hamiltonian parameters, for example the magnetic field, can cause a quantum phase transition, driving the system into one with different symmetry properties (see the article by Subir Sachdev and Bernhard Keimer, PHYSICS TODAY, February 2011, page 29).

In quantum mechanics, quantities in general and observables in particular are described in terms of operators. A local observable is described by a local operator, which, as illustrated in figure 1, affects the system only within a finite distance of a given point, the position of the operator. For two local operators whose positions are widely separated, the order in which they are applied makes no difference; they commute.

Mind the gap

During the past 30 years, theorists have devised a paradigm for a different kind of quantum phase—

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and quasiparticle braiding

a topological phase.[1–3] Unlike in the traditional telling, the notion of a topological phase does not depend on the use of symmetries. Moreover, it involves operations that are not local.

In describing the phases, I will think in terms of a system of structureless point particles that could be moving in free space or, as with electrons in a crystalline medium, be in the presence of a periodic background potential. The Hamiltonian determines both the time evolution of the system and the energy of its states. It acts locally, meaning that it may be expressed as a sum of local operators over their positions; the locality implies that particles have more or less short-range interactions. Associated with the Hamiltonian is its energy spectrum, the set of the energies of all the stationary states of the system.

Application of a local operator to the ground state typically changes it to some higher-energy excited state. Physically, applying the operator means nudging the system as sketched in figure 1, allowing a photon to be absorbed, or something along those lines—all of which could be done locally. The resulting state contains collective excitations of the particles, and those are local; at least initially, the state remains unchanged at positions far from the location of the operator, though the excitations may eventually propagate away from their point of origin. The possibility of creating excitations with a local operator reflects the existence of the system's microscopic local degrees of freedom.

The minimum energy of a local excitation relative to the ground-state energy is called the energy gap. If that gap persists as the size of the system goes to infinity—say, as the number of particles increases without bound as the density remains fixed—then the Hamiltonian, or the system it describes, is said to lie in a topological phase. The energy gap will depend continuously on the parameters in the Hamiltonian. So if a system has an energy gap, that gap will remain if the Hamiltonian is slightly altered. According to the definition just given, an ordinary insulator is a topological phase, but we will see that not all are so trivial.

Beyond local action

The above discussion leaves unaddressed the issue of states of the infinite system that cannot be created by any local operator. Consider the possibility that such states could have zero excitation energy and so be degenerate with the ground state. One way that might occur is if some symmetry of the Hamiltonian is broken in the ground state. For example, an infinite crystal is a periodic array of particles. Displacing that array slightly, as in figure 2a, yields a distinct ground state with the same energy as the original. But because the translation involves displacing all the particles, no local operator can effect it.

On the other hand, the two ground states can be distinguished by the different positions of the

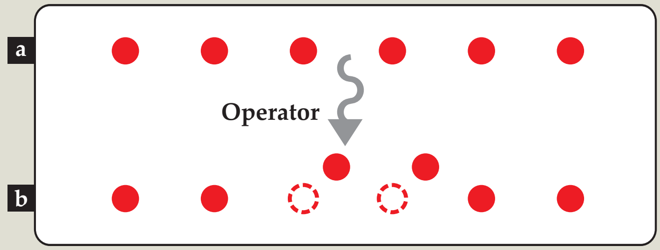

Figure 1. A local operator changes the state of a system only locally—for example, by moving the positions of a few particles.

peaks and troughs in their particle densities. Because the density is a local operator, we can add to the Hamiltonian a term consisting of a periodic potential, as illustrated in figure 2b, that acts on each particle. Degeneracies that can be split in that way are a part of the traditional story, and I will exclude them from further consideration.

There remains the possibility of degenerate ground states that cannot be mapped onto one another by application of any local operator and that cannot be distinguished by the expectation value of any local operator. In that case a small change in the Hamiltonian cannot split the degeneracy of the states.[1] In practice, the multiplicity of the degenerate ground states in a topological phase depends on both the phase and the boundary conditions, and there exists a boundary condition for which the ground state is unique.

Properties that are unchanged throughout a topological phase are called topological properties. I have presented two examples so far: the existence, though not the magnitude, of an energy gap and the multiplicity of ground states for a given boundary condition. The multiplicity can differ in distinct topological phases; if so, that difference provides a means for distinguishing between the phases. By

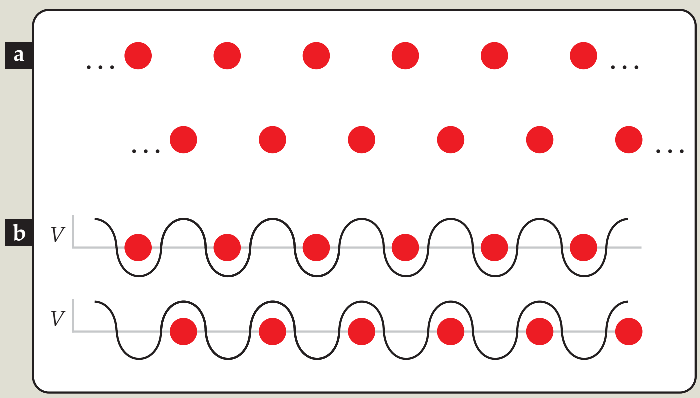

Figure 2. (a) A one-dimensional crystal (top) and the same crystal displaced by less than a period (bottom) are distinct states with the same energy. **(b)** If, however, the crystals in panel a are situated in a potential *V* with the same period as the crystal, the two states will differ in energy.

definition, topological properties don't change if the system Hamiltonian is perturbed. Indeed, the term "topological" suggests an analogy with the field of topology in mathematics, which studies geometric properties of an object that are unchanged even as the numerical parameters that fully characterize the object's shape are varied slightly.

If, as I have assumed, the ground-state degeneracy cannot be lifted by adding a small local operator to the Hamiltonian, it follows that all the degenerate ground states look the same locally. The states are different over very large scales, but any attempt to capture that difference locally will fail. Like Macavity the Mystery Cat of T. S. Eliot's poem, the difference is always somewhere else at the time. So topological properties are very much global, collective effects, but of a much more subtle nature than the collective effects in traditional phases. Think of these states as fluids, not crystals, and think of the wavefunction as a quantum superposition of many different configurations of the particles, not as dominated by a single one. Their topological properties rely on some long-range quantum entanglement, not on conventional correlations. They are like a secretive underground movement whose name is known to all, whose members are known to none, and whose influence is long range.

Unerasable excitations

Let us move on to excitations of the topological phase that cannot be created or destroyed by local operators and that are not degenerate in energy with the ground state. Somewhere in space, those states can be distinguished from the ground state by means of local operators; otherwise, they would not have a higher energy. I will assume those regions, often called defects or cores, are pointlike—as opposed to strings, for example—and that none of the defects can be erased using a local operator. In that case I'll call them quasiparticles; the nominal position of each quasiparticle is that of the center of its core.

Outside each quasiparticle core, the state is locally indistinguishable from the ground state. Quasiparticle positions can be discerned or changed via local operators. It follows that the quasiparticles can move around like particles and, in some cases at least, have a kinetic energy and an effective mass. They can be attracted to or repelled from any particular point by including a suitable local term in the Hamiltonian. They really do behave like particles.

Just as the ground state can be degenerate, so, too, can be the states for a set of quasiparticles with fixed, well-separated positions.[2] That degeneracy cannot be split by any local term in the Hamiltonian, and in particular not by the terms used to control or change the quasiparticle positions: The distinct quasiparticle states cannot be distinguished locally. Consider the case of a system of n quasiparticles, all of the same type, and a boundary condition that yields a nondegenerate ground state. Then, as n becomes very large, the multiplicity of the degenerate quasiparticle states for fixed positions increases exponentially as d^n for some constant $d \geq 1$. The value of d depends on the phase and on the quasiparticle type. For example, in one simple case to which I will return, the degeneracy is $2^{n/2}$ (whence $d = \sqrt{2}$) for all even values of n. When d is not an integer, as in that case, the multiplicity clearly cannot be shared equally as integral numbers of states per quasiparticle. Thus the degeneracy cannot be attributed to local degrees of freedom associated with each quasiparticle, a result consistent with the nonlocal nature of the degenerate states.

The nature of the degenerate states has implications for quantum information science.[3] Quantum information is stored as a state of a quantum system; if it is stored in the space of degenerate quasiparticle states, then it is stored nonlocally. The virtue of that

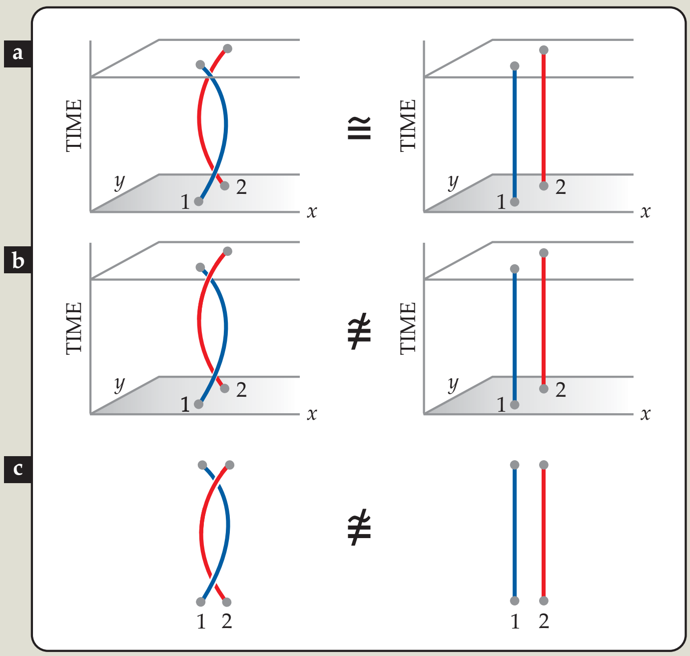

Figure 3. Isotopic equivalence and inequivalence. The exchange of identical quasiparticles in a topological phase can be represented by worldlines on a diagram with two spatial axes and one time direction. **(a)** In the diagram to the left, particles at positions 1 and 2 are exchanged, with the particle at position 1 passing over the one at position 2. Then they are exchanged again, with the particle now at position 2 passing over the particle at position 1. The two worldlines, differently colored for ease of visualization, can be separated without passing through each other to yield the pair of worldlines in the right diagram. Thus the pair of exchanges to the left is isotopically equivalent to no exchange at all. **(b)** In this variation, both exchanges involve the particle at position 1 passing over the particle at position 2. The entangled worldlines cannot be separated, and so the pair of exchanges is not isotopically equivalent to no exchange. Evidently, in two dimensions, the exchange in which the particle at position 1 passes over the one at position 2 (τ_{12}) is not equivalent to the inverse exchange in which 2 passes over 1. In three dimensions, the extra wiggle room afforded by the z dimension makes the exchange and its inverse equivalent. **(c)** This abstract rendering represents the inequivalence shown in panel b.

nonlocality becomes apparent when you consider that the system couples to its environment—for example, via the electromagnetic field. The interaction terms in the Hamiltonian that enforce that coupling are local, and so a given initial storage state cannot decohere within the subspace of degenerate states. In other words, the information stored in the system by initializing it in one state survives for a very long time. That long lifetime is an attractive feature, inasmuch as errors due to decoherence are a persistent problem in other systems.

Exchanges, permutations, and braids

Quantum statistics asks how the wavefunction of a state changes upon an exchange of identical particles (or quasiparticles). In older textbook discussions, an exchange is simply a permutation of the particle positions in the wavefunction, and only two possibilities are admitted: A transposition (an exchange of just two particles) either leaves the wavefunction unchanged, in which case one speaks of Bose statistics and particles called bosons, or it returns the same wavefunction multiplied by −1 and one speaks of Fermi statistics and fermions. The effect of an arbitrary permutation of the particles can be found by compounding transpositions. The net result is a factor of +1 for bosons or ±1 for fermions, depending on whether the number of transpositions is even or odd.

The quantum statistics of quasiparticles in a topological phase can be significantly more complicated and, in fact, is a topological property of the phase.[2] An exploration of the statistics will help answer the question of how one can perform operations, or quantum computation, on the information stored in degenerate quasiparticle states.

A sufficiently careful and flexible way to get at the statistics is to move the quasiparticles adiabatically (that is, very slowly) until they have returned to their original positions, up to a possible permutation.[4] Performing the exchange adiabatically ensures that the very act of transporting the particles doesn't excite the state. The end result of moving the quasiparticles is to multiply the state by a phase factor, or Berry phase, if it is nondegenerate, or otherwise to transform it to another state in the same subspace of degenerate quasiparticle states. In the degenerate case, the transformation is described by a unitary matrix operation that is independent of the specific initial state.

It is helpful to picture an arbitrary exchange process as a diagram showing the paths, or worldlines, taken by the quasiparticles in spacetime, not only in space. (Figure 3, which illustrates the concept of isotopy developed in this paragraph, gives a few examples.) The particles must be exchanged in such a way that their separations remain large at all times; in particular, the worldlines cannot intersect. If the effect of an adiabatic exchange is to be a meaningful notion of quasiparticle statistics, the effect must be independent of the precise paths. That is, the resulting phase factor or matrix needs to be the same if the worldlines are deformed slightly, provided that the initial and final positions remain fixed and that the quasiparticles remain well separated. Two diagrams of worldlines that can be continuously deformed from one to the other in that way are said to be isotopic, or topologically equivalent, and all diagrams that are isotopic are said to be in the same equivalence class. In sum, the effect of exchanges carried out by two sets of worldlines in the same equivalence class should be the same.

To understand how isotopic invariance comes about in topological phases, first consider adiabatically moving one quasiparticle around a small closed curve far from the other quasiparticles, which remain at fixed positions. Such an operation is tantamount to applying a local operator to the initial state. The expectation value of that operator should not suffice to detect the presence of the other quasiparticles or to distinguish among any degenerate states. Thus the only possible result of the circular excursion is to multiply the state by a phase factor that is independent of the other quasiparticles and of the specific degenerate state of the system. The same is true of the additional effect due to one quasiparticle circuiting a small curve, instead of being stationary, as part of a process in which the other quasiparticles exchange adiabatically. Building on that kind of argument, one can show that all isotopic exchanges have the same effect except for those uninformative phase factors.

In three or more space dimensions, all exchange processes that effect a given permutation of identical quasiparticles are isotopically equivalent. In one space dimension, no exchanges are possible because the worldlines would necessarily cross. But in two space dimensions, as figure 3 shows, two worldlines that make a circuit around one another cannot be

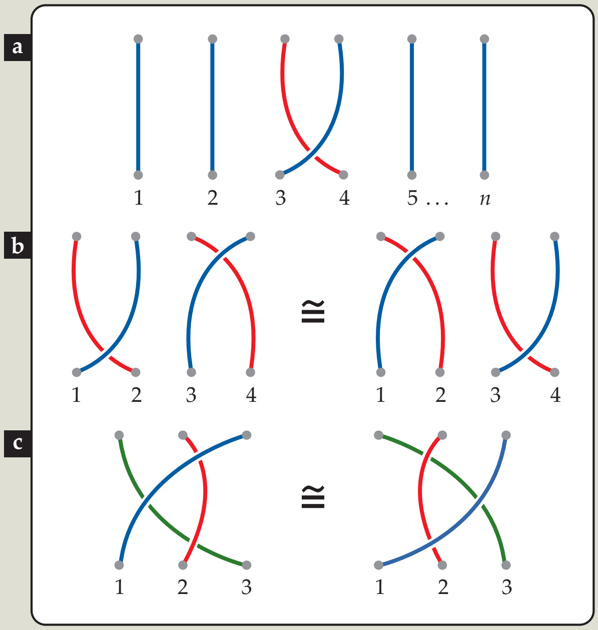

Figure 4. The braid group. (a) One of the elementary exchanges, τ_{34}, that, along with their inverses, generate all braids. **(b)** Diagram representing the braid-group relation $\tau_{12}\tau_{34} = \tau_{34}\tau_{12}$. **(c)** Diagram representing the braid-group relation $\tau_{23}\tau_{12}\tau_{23} = \tau_{12}\tau_{23}\tau_{12}$. Relations like this and the commutation property of panel b define the braid group.

Box 1. Topological quantum computation

In a classical computer, information is stored as a string of bits, each being either a 0 or a 1. Physically, each bit is stored in the state of a bistable classical system. In a computation, a set of bits is sent into a logic circuit built from a sequence of gates. Each gate acts on either one or two bits and performs one of the elementary Boolean operations AND, OR, or NOT. The output of the logic circuit is one or more bits that are easily read. Errors, such as the accidental flipping of a 1 to a 0, are manageable for macroscopic bistable systems and can be corrected with additional bits.

One model for a quantum computer uses a quantum version of a bit, called a qubit. A qubit is a quantum two-state system, such as a spin-½ particle. A computation is a unitary operation that acts on an initial state of some qubits. Some set of unitary operations acting on either one or two qubits at a time forms the analogue of the classical gates. With them one can construct circuits that perform arbitrary computations to within any desired accuracy. The output is a quantum state of the qubits, read by performing measurements. In practice, ordinary quantum systems have a strong propensity to decohere during the course of the computation. Thus conventional quantum computation will require many additional qubits to compensate for uncontrollable errors.

Topological quantum computation replaces the qubits with a space of degenerate states of non-abelian quasiparticles in a topological phase. Computations are performed by braiding, with exchange of two neighbors as the elementary gates. Due to the intrinsic topological protection of the states from decoherence, the need for error correction is minimal. In a quantum Hall system, to give one example, the output can be read via measurements of interference properties of edge-state quasiparticles.[3]

deformed to two that do not. Consequently, an infinite number of different equivalence classes exists in two dimensions; those classes are called braids.

Non-abelian statistics

Just as all permutations can be built up from transpositions of neighboring particles, all braids can be built up from elementary exchanges of neighboring quasiparticles via paths that do not enclose any other quasiparticle—braids, like permutations, form a group (see figure 4). Thus the statistics for a given type of quasiparticle can be specified by the matrices giving the effects of those elementary exchanges.

In three or more spatial dimensions, the only possibilities allowed by the assumptions I have specified are that the multiplicity of an n-quasiparticle state is 1 for all n and that the effect of any transposition is to multiply the state by ±1. Thus the quasiparticles are bosons or fermions. But in two dimensions, the possibilities are much richer. Even if the multiplicities are 1 for all n, the effect of an elementary exchange can be any phase factor $e^{i\theta}$, where θ is the phase angle. Under those circumstances, the quasiparticles are called anyons or are said to possess fractional statistics; bosons and fermions are special cases. Because the phase factors associated with fractional statistics are ordinary, commuting numbers, the statistics of anyons is called abelian.

The effect of an elementary exchange in two dimensions could also be a unitary matrix acting in the space of degenerate quasiparticle states. In that case the quasiparticle statistics is termed non-abelian because the matrices for different exchanges do not generally commute. On the mathematical side, non-abelian statistics is connected with the topologically invariant knot polynomials of Vaughan Jones (University of California, Berkeley, and Vanderbilt University) and others, with their interpretation using Chern–Simons gauge theories,[5] and with conformal field theories in two dimensions.[6]

Because quasiparticles that have degenerate states are governed by non-abelian statistics, information stored in those states can be manipulated. In other words, the degenerate states can be used for topological quantum computation, as proposed by Alexei Kitaev, now at Caltech.[3,7] Box 1 sketches the basic idea (see also the article by Sankar Das Sarma, Michael Freedman, and Chetan Nayak, PHYSICS TODAY, July 2006, page 32). The topological model of quantum computation is exactly as powerful as conventional models.[7,8] As such, suitable topological phases can serve for a universal quantum computer.

My idealized presentation in this article has taken the system size to be infinite and the quasiparticles to be always well separated. In the real world of finite-sized systems, the degeneracy of quasiparticle states—a cornerstone of my discussion—generally is not exact. But because of the nonzero gap for local excitations, the energy splittings between quasiparticle states approach zero exponentially with increasing quasiparticle separation in units of a length that is usually microscopically small. The assumption of degenerate quasiparticle states is actually rather reasonable in practice.

Real-world examples

The earliest phase of matter in which topological properties were recognized occurs with the integer quantum Hall effect (IQHE), which is observed for a gas of electrons confined to move in two dimensions perpendicular to a magnetic field (reference 9; see also the article by Joseph Avron, Daniel Osadchy, and Ruedi Seiler, PHYSICS TODAY, August 2003, page 38). In that case the energy gap for local excitations is due to the quantization of cyclotron motion. There exists a distinct IQHE phase for every integer $\nu \neq 0$. Those phases are distinguished from the trivial one, $\nu = 0$, by topological properties I do not have space to discuss in detail here: a quantized Hall conductivity proportional to ν and the presence of chiral gapless edge states in the finite-sized, real-world system. (In the bulk material far from the edge, the IQHE phases have an energy gap.)

The fractional quantum Hall effect (FQHE) was discovered soon after the IQHE, under similar experimental conditions.[10] In those phases, ν is a rational fraction, and the energy gap is due to interactions between the electrons.[11] In addition to quantized Hall conductivity and gapless edge excitations, those phases possess quasiparticles with fractional electric charge and admit degenerate ground states for some boundary conditions. In many cases those are abelian phases—that is, the quasiparticles are anyons with abelian statistics.[12]

In 1991 Gregory Moore (now at Rutgers University) and I described an FQHE state having quasiparticles with non-abelian statistics as well as others with abelian statistics.[2] For the non-abelian variety, the quasiparticle-state multiplicity is $2^{n/2}$ for n well-separated quasiparticles. The phase can occur with $\nu = 5/2$, and a state with that ν has been observed.[13] Numerical investigations suggest that the observed state is either the Moore–Read phase or its particle–hole conjugate, which is a distinct but similar phase. The nature of the $\nu = 5/2$ state is the subject of intense current experimental interest (see, for instance, PHYSICS TODAY, June 2008, page 14). It may be the first real-world system to exhibit non-abelian quasiparticles, but the issue has not yet been definitively settled.

Topological phases can also be found in systems that don't display the quantum Hall effect. One such system is the two-dimensional $p + ip$ superconductor, so called because its Cooper pairs have angular momentum $\hbar$. At weak coupling, the superconductor has a topological phase with gapless chiral edge states and also has two types of quasiparticles: Majorana fermions, which are their own antiparticles, and vortices with a quantum of magnetic flux equal to $hc/2e$. At each vortex core is a localized self-adjoint fermionic zero-mode operator, described in box 2. The existence of the operators implies that the topological phase has $2^{n/2}$ degenerate states for n well-separated vortices, and that the vortices have non-abelian statistics.[14] Using the concept of composite fermions (see the article by Jainendra Jain, PHYSICS TODAY, April 2000, page 39), the non-abelian quasiparticles of the Moore–Read quantum Hall phase can be understood as being analogous to those vortices.

Other topological phases with properties somewhat like the $p + ip$ superconductor phase occur in one, two, or three dimensions in so-called topological insulators (reference 15; see also the article by Xiao-Liang Qi and Shou-Cheng Zhang, PHYSICS TODAY, January 2010, page 33). Topological insulators differ from ordinary ones, which have fermionic quasiparticles and trivial topological properties. The search for evidence of Majorana zero modes in a system comprising a topological insulator coupled to a superconductor is currently an enormous experimental effort; indeed, evidence may have recently been obtained (reference 16; see also PHYSICS TODAY, June 2012, page 14). The $p + ip$ superconductors and topological insulators show that strong correlations, as in FQHE, are not necessary for a topological phase. What is necessary is some long-range entanglement in the ground state. Also fascinating are so-called fractional topological insulators, which are analogs of the FQHE systems but without the applied magnetic field, and topological phases in lattice systems of quantum spins.

By any accounting, ice and water are different phases. In particular, the solid and liquid have different symmetries. Moreover, the degenerate ground states of the ice crystal are related by symmetries. But the traditional telling that those symmetry characteristics are universal is now understood to be incorrect—phases may differ only in their topological properties, among them ground-state degeneracy and quasiparticle statistics. When the quasiparticle states are degenerate, the quasiparticles are governed by non-abelian statistics, one of the most bizarre concepts to crop up in studies of quantum matter, but one that just might change our approach to quantum computation.

Box 2. Fermion algebra and zero modes

Quasiparticles that are fermions, including the electrons that underlie many topological phases, can be created or destroyed at some point by an operator located at that point. Fermionic quasiparticles are not, strictly speaking, local excitations, because the operators $\psi^\dagger(x)$ and $\psi(x)$ that create or destroy them do not commute when applied at different locations. Instead, they anticommute; the two different orders differ by a minus sign as, for example, in the relation $\psi(x)\psi(y) = -\psi(y)\psi(x)$. Thus, for a term in the Hamiltonian to be local, it must contain a product of an even number of fermion operators.

Fermionic quasiparticles in a superconductor may be their own antiparticles. That is, two of them may be able to annihilate each other, in which case they are called Majorana fermions. In some situations, when one decomposes the fermion operators into parts that change the energy of the system by different amounts, those parts include operators that leave the energy unchanged. Those operators, γ_j, for $j = 1, 2, \ldots, n$, are called fermion zero-mode operators. When the quasiparticles are Majorana fermions, the zero-mode operators are Hermitian (self-adjoint): $\gamma_j^\dagger = \gamma_j$ for all j. The zero-mode operators with different indices anticommute, $\gamma_j\gamma_k = -\gamma_k\gamma_j$, and for any fixed index j, $\gamma_j^2 = I$ (the identity).

If the zero-mode operators are to be written as matrices, they need to act on more than one state.[14] In fact, if n is even, the minimum number of states required is $2^{n/2}$. That state counting applies in particular to the zero-mode operators supported on n vortices of a $p + ip$ superconductor and to similar objects in other situations. It is wrong to call the vortices themselves Majorana fermions; fermions are particles with Fermi statistics, but the vortices are non-abelian quasiparticles. Rather than creating a quasiparticle from the ground state, the role of the Majorana zero-mode operators is to change the state in the degenerate subspace of $2^{n/2}$ states shared among the already-existing vortices. The zero-mode operators also provide a representation of the non-abelian statistics, because the effect on the space of degenerate states of an elementary exchange can be written in terms of the zero-mode operators of the exchanged vortices.[17]

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