Nonperturbative renormalization group approach to nonequilibrium systems

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These lecture notes are the basis of the presentation with the same title that will take place in the Summer School 'Nonequilibrium Statistical Mechanics: Fundamental Problems and Applications' at Boulder, USA, in July 2009. The reader will not find new results here and I am in debt with a set of authors who have contributed to the subjects presented here in the last years. I am particularly in debt with the PhD thesis of Leonie Canet (downloadable from http://lpm2c.grenoble.cnrs.fr /spip.php?article353) and with the lecture notes of B. Delamotte 'An introduction to the nonperturbative renormalization group' [1]. The textbook of M. Le Bellac [2] was always around the table while I was writing these notes as can be seen in the first chapter. I have learnt most of what I know in field theoretical treatment of reaction-diffusion systems from many discussions with L. Canet, H. Chaté and B. Delamotte. The lectures of U. Täuber (that can be downloaded from http://www.phys.vt.edu/ tauber/) have also been very useful. In summary: these notes are not original, almost all the good things the reader can find here have been explained in a better way elsewhere, and I be mainly the author of many mistakes that will certainly appear. However, I hope that in some parts, as in the motivation of the Derivative Expansion, a pedagogical progress to most part of the literature has been achieved. Moreover, in the third chapter I have tried to explain carefully some calculations made some time ago and never written in detail in the literature. I must acknowledge F. Benitez and L. Canet for reading a previous version of this manuscript and suggesting many improvements. L. Quintana and A. Rubini helped a lot to make this text closer to real English. I must acknowledge also the support of the uruguayan program PEDECIBA.

Contents

Chapter 3

Simple NPRG applications in reaction-diffusion systems

We have finally arrived to the applications of the NPRG to out-of-equilibrium phenomena. The first application of NPRG to this kind of phenomena was done in the Directed-Percolation (DP) universality class [54]. In this reference, critical exponents were calculated but also non-universal properties in out-of-equilibrium problems were analyzed also. In particular, reaction-diffusion processes called Branching and Annihilating Random Walks (BARW) were studied. In its odd case can be taken with the form (1.1) in the particular case of $\mu = 0$. For this particular process in the DP universality class, the structure of the phase diagram was studied [54, 55]. NPRG results for the even-BARW, that go beyond the scope of the perturbative analysis, can be found in [56]. The model-A has also been analyzed within this methods [57]. Finally, the analysis of the KPZ-equation has been performed within the NPRG under various approximation schemes [32, 33]. A brief review of the many of these results can be found in [30].

In the present chapter a single example will be presented: the application of the NPRG equation in the LPA approximation to DP universality class and, in particular, to the odd-BARW process realization.

3.1 LPA approximation for reaction-diffusion processes

In this section, the LPA approximation for any reaction-diffusion process of a single molecule species is presented. For all these types of reactions, the Doi-Peliti [10, 11] and the J.D.M.S.R. [7, 8, 9] procedures give actions of the

form:

$$\mathcal{S}[\varphi,\bar{\varphi}] = \int d^dxdt\Big\{\bar{\varphi}(t,\vec{x})\big(\partial_t - D\nabla^2\big)\varphi(t,\vec{x}) + V(\varphi,\bar{\varphi})\Big\} \tag{3.1}$$

As explained in the first chapter, for such an action, mean values can be taken with an expression similar to the one of equilibrium statistical mechanics (1.29). Correspondingly, one can define the equivalent of a partition function in presence of 'sources':

$$\mathcal{Z}[J,\bar{J}] = \int \mathcal{D}\varphi(t,\vec{x})\mathcal{D}\bar{\varphi}(t,\vec{x})\exp\Big(-\mathcal{S}[\varphi,\bar{\varphi}] + \int d^dxdt\{J\varphi + \bar{J}\bar{\varphi}\}\Big) \tag{3.2}$$

One can also consider, the equivalent of a Helmholtz 'free-energy', $\mathcal{W}[J,\bar{J}] = \log\mathcal{Z}[J,\bar{J}]$. From it, one can extract $\phi(t,\vec{x}) = \langle\varphi(t,\vec{x})\rangle\big|_{J,\bar{J}}$ and $\bar{\phi}(t,\vec{x}) = \langle\bar{\varphi}(t,\vec{x})\rangle\big|_{J,\bar{J}}$ where the mean values are taken in presence of sources $J(t,\vec{x})$ and $\bar{J}(t,\vec{x})$. Consequently, one can perform a Legendre transform in order to define the equivalent of a Gibbs 'free-energy':

$$\Gamma[\phi,\bar{\phi}] = \int d^dxdt\big\{J(t,\vec{x})\phi(t,\vec{x}) + \bar{J}(t,\vec{x})\bar{\phi}(t,\vec{x})\big\} - \mathcal{W}[J,\bar{J}]. \tag{3.3}$$

The NPRG procedure can be performed in the same way as in equilibrium phenomena. One modifies the action by a quadratic term:

$$\Delta\mathcal{S}_k[\varphi,\bar{\varphi}] = \frac{1}{2}\int\frac{d\omega}{2\pi}\frac{d^dq}{(2\pi)^d}\Phi(-\vec{q},-\omega)\hat{R}_k(\vec{q},\omega)\Phi^t(\vec{q},\omega) \tag{3.4}$$

where Φ is a vector with components $(\varphi,\bar{\varphi})$ and $\hat{R}_k$ is a 2×2 symmetric matrix. Correspondingly one can define the regularized $\mathcal{W}_k[J,\bar{J}]$ and $\Gamma_k[\phi,\bar{\phi}]$ as in previous chapters. If the matrix $\hat{R}_k$ is properly chosen, the behavior of Γ_k is, again,

$$\Gamma_k[\phi,\bar{\phi}] \sim \begin{cases} \mathcal{S}[\phi,\bar{\phi}] & \text{if } k\to\infty \\ \Gamma[\phi,\bar{\phi}] & \text{if } k\to 0 \end{cases} \tag{3.5}$$

The corresponding NPRG equation can be deduced in the same way than for equilibrium systems and takes the form:

$$\partial_k\Gamma_k[\phi,\bar{\phi}] = \frac{1}{2}\mathrm{Tr}\Big(\partial_k\hat{R}_k(\Gamma_k^{(2)} + \hat{R}_k)^{-1}[\phi,\bar{\phi}]\Big) \tag{3.6}$$

It is important to realize that now $\Gamma^{(2)}$ is a matrix not only in space and time indices but also in the 'internal' space $(\phi,\bar{\phi})$. Correspondingly, the 'trace' is now over space-time indices and also over internal indices.

Before going to specific applications, it is important to mention two technicalities:

- The form of various actions $\mathcal{S}$ has been deduced by considering a Itô prescription. This is automatic in the Doi-Peliti procedure and is part of a precise definition of the Langevin equation in the J.D.M.S.R procedure. When using the Itô prescription a very careful treatment of Fourier transforms is necessary. In these notes we will proceed in a naïve way, ignoring this problem. One can prove, however, that at the level of the LPA this problem has no consequences [58]. When considering more sophisticated approximations it is important to go back to a precise discretized version of the model before taking Fourier transforms in order to take into account the Itô prescription. At the end of the day the procedure is not very involved, but goes beyond the scope of the present lectures.

- The fields φ and $\bar{\varphi}$ are not real scalar fields. Depending on the specific procedure employed (Doi-Peliti or J.D.M.S.R) they will have different 'complexifications'. For the Doi-Peliti procedure they must be complex conjugated fields. In that case, the integration (1.29) must be done, in fact, over the real and imaginary parts of the field φ. For the case of the J.D.M.S.R, φ is real and $\bar{\varphi}$ purely imaginary. In that case, the integration must be done over φ and the imaginary part of $\bar{\varphi}$. In these notes we will ignore this point by supposing that some sort of analytical continuation is possible which allow us to treat φ and $\bar{\varphi}$ as independent real fields. At a perturbative level it does not seem to have any consequences but this is a delicate point for a non-perturbative analysis. This technical point is presently under active investigation [58].

To perform concrete calculations, a regulator profile matrix must be chosen. As mentioned before, it is important to chose a regulator that respect the known symmetries of the model. For reaction diffusion systems, it is important to respect space and time translation invariance, space rotations invariance and also other internal symmetries of specific reaction-diffusion actions.

For the DP action (1.34), the simplest regulator profile matrix $\hat{R}_k$ corresponds to chose a matrix *independent* of frequencies and only with off-diagonal components (as typical quadratic terms in the action):

$$\hat{R}_k(\vec{q},\omega) = \begin{pmatrix} 0 & R_k(q^2) \\ R_k(q^2) & 0 \end{pmatrix} \tag{3.7}$$

In particular, for the DP case it has the virtue of respecting the rapidity symmetry

$$\varphi(t,\vec{x}) \to -\bar{\varphi}(-t,\vec{x}), \qquad \bar{\varphi}(t,\vec{x}) \to -\varphi(-t,\vec{x}). \tag{3.8}$$

The function $R_k(q^2)$ should have the same properties than for the equilibrium case. The only difference is that the normalization at $q = 0$ should be taken $R_k(q^2 = 0) = Z_k D_k k^2$.

One difficulty that must be mentioned with such a regulator is that is has good properties for high wave-numbers but not for high frequencies. In principle one should ask the same kind of requirement in frequencies than in wave-numbers in order to have a NPRG equation ultraviolet finite and also with good decoupling properties of modes with high frequencies. However, in practice it seem to be unnecessary. Even if there are not a frequency cut-off induced by the regulator, in practice the integrals fall at least as $1/\omega^2$ at high frequencies (in fact for most of physical quantities is fall much faster). Even if it does not induce a decoupling of modes with high frequencies as fast as the one in wave-numbers, it seem to be enough in practice.

In order to extract physical information from the solutions of the NPRG equation, one proceeds in a similar way than in equilibrium problems. However, there are some minor differences to be analyzed. As explained in the first chapter, the weight associated to the action (1.29) allows to compute the expected values of functionals of the field variable $\varphi(t, \vec{x})$, as for example $\langle \varphi(t, \vec{x}) \varphi(t', \vec{x}') \rangle$. As it was also mentioned in that chapter, for out-of-equilibrium phenomena it is important to be able to calculate also response functions. A very important property of the J.D.M.S.R. formalism (and of the Doi-Peliti formalism) is that it also allows to calculate the response functions. For the simplest response function,

$$\chi(t, \vec{x}; t', \vec{x}') = \frac{\delta \langle \varphi(t, \vec{x}) \rangle}{\delta \bar{J}(t', \vec{x}')} = \langle \varphi(t, \vec{x}) \bar{\varphi}(t', \vec{x}') \rangle \tag{3.9}$$

as can be easily shown (see exercise 4). This is why the field $\bar{\varphi}$ is usually called *response field*. Given this fact, the response and correlation functions can be treated on an equal footing, and one can speak of a general 2-point correlation function $\langle \Phi_i(t, \vec{x}) \Phi_j(t', \vec{x}') \rangle$, where Φ_i may be φ or $\bar{\varphi}$. Consequently, correlation and response functions can be obtained from the matrix of second derivatives of Γ_k. As explained before, the matrix of second derivatives of $\Gamma_k + \Delta \mathcal{S}_k$ is the inverse (in space-time and internal indices) of the general 2-point function (including response fields) just defined.

A general property in out-of-equilibrium problems is that $\mathcal{S}[\varphi, \bar{\varphi} = 0] = 0$. This is manifest in the J.D.M.S.R. formalism presented before and can be shown to be true also in the Doi-Peliti formalism. If Itô prescription is taken into account properly it can be shown that this property is also true at the level of $\Gamma_k[\phi, \bar{\phi}]$. In particular, it implies that the second derivative of Γ_k with respect to ϕ at zero $\bar{\phi}$ is zero, giving a null entry in the matrix to be inverted. This is important, because as will be seen shortly, many times the physical

quantities that one wants to calculate in a 'symmetric phase' are evaluated at zero external sources. In that case, the various objects must be evaluated at zero external fields. In the particular case of the DP universality class, the 'symmetric phase' corresponds to the absorbing phase. In that case, the rapidity symmetry impose, moreover, that the second derivative with respect to $\bar{\phi}$ evaluated at zero external fields is also zero. In that case, the correlation function at zero external sources is simply zero and the response function is nothing but the operator inverse of $\frac{\delta^2\Gamma_k[\phi,\bar{\phi}]}{\delta\phi(t,x)\delta\bar{\phi}(t',\vec{x}')}\Big|_{\phi=\bar{\phi}=0}$. In the general case, where no rapidity symmetry is present (or it is broken spontaneously), the full matrix inversion is necessary and the correlation function is non-zero. Moreover, as explained in the previous chapter, in many cases it is useful to consider the system in an external field even if at the end of the day we will evaluate correlation function at zero fields. This is, in particular the case when considering the LPA approximation, where keeping the external field is associated to the encoding of all derivatives of Γ_k at zero wave-numbers (and frequencies).

In order to extract correlation length and critical exponents in out-of-equilibrium phenomena, one proceeds in a similar way as for the anomalous dimension in the equilibrium phenomena. One observes that, for low momenta and frequencies:

$$\begin{aligned}\chi^{-1}(\omega,\vec{p}) &= \int dtd^dx e^{i(\omega t-\vec{p}\cdot\vec{x})}\left.\frac{\delta^2\Gamma_k[\phi,\bar{\phi}]}{\delta\phi(t,x)\delta\bar{\phi}(0,0)}\right|_{\phi=\bar{\phi}=0}\\ &= \chi^{-1}(\omega=0,\vec{p}=\vec{0})+Z_k\big(i\omega+D_kp^2\big)+\mathcal{O}(\omega^2,p^4,p^2\omega)\end{aligned} \quad (3.10)$$

where $\chi^{-1}(\omega=0,\vec{p}=\vec{0})=\partial^2_{\phi\bar{\phi}}V_k(\phi=0,\bar{\phi}=0)$. Accordingly, the physical correlation length is

$$\xi_\perp^2=\chi_{k=0}(\omega=0,\vec{p}=\vec{0})Z_{k=0}D_{k=0}, \quad (3.11)$$

and the correlation time is

$$\xi_\parallel=\chi_{k=0}(\omega=0,\vec{p}=\vec{0})Z_{k=0}. \quad (3.12)$$

One observes also, that the product Z_kD_k plays an analogous role in out-of-equilibrium phenomena to the factor Z_k that appeared in the equilibrium case in the static limit $\omega\to 0$. Consequently, the extraction of η can be done like in that case:

$$\eta=-\lim_{k\to 0}k\frac{\partial\log(Z_kD_k)}{\partial k}. \quad (3.13)$$

In order to extract the scaling between correlation time and length, it is important to observe that the flow effectively stops when k becomes of the

order of $\xi_\perp^{-1}$. If we are close to the critical regime, for values of $k \gtrsim \xi_\perp^{-1}$, the evolution of the various parameters is as in the critical case. In that regime, the parameters $\chi_k(\omega = 0, \vec{p} = \vec{0})$, Z_k and D_k evolve in power laws if k is small enough. That is, in the regime $\xi_\perp^{-1} \lesssim k \ll \Lambda$,

$$\chi_k(\omega = 0, \vec{p} = \vec{0}) \sim k^{-l}, \qquad Z_k \sim k^{-m}, \qquad D_k \sim k^{-n}. \tag{3.14}$$

When k approaches $\xi_\perp^{-1}$, the flow of these quantities progressively stops and when k becomes smaller nothing new happens. So one concludes that

$$\chi_{k=0}(\omega = 0, \vec{p} = \vec{0}) \sim \xi_\perp^{l}, \qquad Z_{k=0} \sim \xi_\perp^{m}, \qquad D_{k=0} \sim \xi_\perp^{n}. \tag{3.15}$$

Comparing with (3.11) one deduces that $l+m+n = 2$ and replacing in (3.12) one concludes that

$$\xi_\parallel \sim \xi_\perp^{2-n}. \tag{3.16}$$

Given the definition of the exponent z, $\xi_\parallel \sim \xi_\perp^z$ one concludes that $z = 2-n$. Now, n can be extracted from the logarithmic derivative of D_k in the critical case:

$$2 - z = n = -\lim_{k\to 0} k \frac{\partial \log D_k}{\partial k}. \tag{3.17}$$

(here $k \to 0$ means when k is much smaller than the microscopic scale Λ and consequently D_k behaves as k^{-n}).

We consider now the LPA approximation for the particular form of the regulator (3.7), that can be used in most reaction-diffusion processes with only one molecule species. The LPA corresponds to neglect the flow of terms in Γ_k that includes derivatives (time derivatives and space derivatives) but to allow a free evolution of the effective potential (the part of Γ_k that do not includes derivatives). For reaction-diffusion problems, it corresponds to the Ansatz:

$$\Gamma_k[\phi, \bar{\phi}] = \int d^d x dt \Big\{ \bar{\phi}(t, \vec{x}) \big(\partial_t - D\nabla^2\big) \phi(t, \vec{x}) + V_k(\phi, \bar{\phi}) \Big\} \tag{3.18}$$

where now, the form of $V_k(\phi, \bar{\phi})$ evolves with k and it not fixed to its bare form. The bare form to be employed as initial condition at $k = \Lambda$ is the one that comes from the Doi-Peliti prescription:

$$V_\Lambda(\phi, \bar{\phi}) = \bar{\phi}\phi \Big(\mu - \sigma + \sqrt{2\sigma\lambda} \big(\phi - \bar{\phi}\big) + \lambda \bar{\phi}\phi \Big) \tag{3.19}$$

In order to deduce the LPA equation for $V_k(\phi, \bar{\phi})$ one must calculate the matrix of second derivatives of Γ_k, evaluate it in uniform fields and perform a Fourier transform. The calculation is straightforward and gives:

$$\Gamma_k^{(2)}(\vec{q},\omega) = \begin{pmatrix} \partial_\phi^2 V_k & i\omega + Dq^2 + \partial_{\phi\bar{\phi}}^2 V_k \\ -i\omega + Dq^2 + \partial_{\phi\bar{\phi}}^2 V_k & \partial_{\bar{\phi}}^2 V_k \end{pmatrix} \tag{3.20}$$

From it one can calculate the inverse matrix that appears in the NPRG equation:

$$\begin{aligned}\Big(\Gamma_k^{(2)}(\vec{q},\omega) + \hat{R}_k(\vec{q},\omega)\Big)^{-1} &= \frac{1}{\mathrm{Det}\big(\Gamma_k^{(2)}(\vec{q},\omega) + \hat{R}_k(\vec{q},\omega)\big)} \\ &\times \begin{pmatrix} \partial_{\bar{\phi}}^2 V_k & -(-i\omega + Dq^2 + \partial_{\phi\bar{\phi}}^2 V_k + R_k(q^2)) \\ -(i\omega + Dq^2 + \partial_{\phi\bar{\phi}}^2 V_k + R_k(q^2)) & \partial_{\bar{\phi}}^2 V_k \end{pmatrix}\end{aligned} \tag{3.21}$$

Performing the corresponding products of matrices and traces one arrives to the LPA equation for the effective potential:

$$\partial_k V_k(\phi,\bar{\phi}) = \int \frac{d\omega}{2\pi} \int \frac{d^dq}{(2\pi)^d} \frac{\partial_k R_k(q^2)(Dq^2 + R_k(q^2) + \partial_{\phi\bar{\phi}}^2 V_k)}{\omega^2 + (Dq^2 + R_k(q^2) + \partial_{\phi\bar{\phi}}^2 V_k)^2 - \partial_\phi^2 V_k \partial_{\bar{\phi}}^2 V_k} \tag{3.22}$$

The interest of taking a regulator that does not depend on ω becomes now evident: the integral over ω can now be done in analytical way, giving:

$$\partial_k V_k(\phi,\bar{\phi}) = \frac{1}{2} \int \frac{d^dq}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{\sqrt{1 - \partial_\phi^2 V_k \partial_{\bar{\phi}}^2 V_k / \big(Dq^2 + R_k(q^2) + \partial_{\phi\bar{\phi}}^2 V_k\big)^2}} \tag{3.23}$$

This is the general LPA equation for any reaction-diffusion process with a single molecule species or for any Langevin equation with a single scalar field if the regulator (3.7) is used. Other regulators may be necessary in cases (as for model-A and for the KPZ equation) where the regulator (3.7) does not satisfy all the symmetries of the model. In such cases, the procedure of deduction of the LPA is identical but the trace may include some other terms.

In the next section, the solutions of the LPA equation are analyzed for the DP universality class. Within this universality class, some non-universal properties associated to the odd-BARW model are also analyzed.

3.2 LPA for Directed Percolation

From the LPA for reaction-diffusion processes (3.23) one can analyze the DP exponents in any dimension of interest. In the particular case of the DP

universality class, the action must respect the rapidity symmetry (3.8). If the initial condition satisfies it, it is not hard to convince ourselves that the property is preserved by the LPA equation for any k (see exercise 3). In that case, the potential may be written in terms of the quantities $x = \bar{\phi}\phi$ and $y = \phi - \bar{\phi}$ that are manifestly invariant under the symmetry. A further property of the action is that $\mathcal{S}[\phi, \bar{\phi} = 0] = 0$. In particular, for DP, the potential is proportional to x. It is not difficult to show that if the initial condition satisfies such a property then the solution V_k of (3.23) satisfies it (up to an additive constant without physical consequences[1]).

The procedure mentioned in the previous section in order to extract exponents must be used here. Given the fact that LPA does not include renormalizations of terms in Γ_k with derivatives, the associated exponents (η and z) remain at their mean-field values $\eta = 0$ and $z = 2$. In the table 3.1 various exponents obtained from the LPA equation are compared to Monte-Carlo simulations. It is convenient to mention that, as for the Ginzburg-Landau

Table 3.1: LPA results for some critical exponents for the DP universality class compared to perturbative estimates [59] and Monte-Carlo results [60, 61, 62].

	LPA			Perturbative results			Monte-Carlo		
d	ν	β	z	ν	β	z	ν	β	z
1	1.056	0.528	2	0.877	0.398	1.49	1.096...	0.276...	1.580...
2	0.730	0.730	2	0.709	0.622	1.72	0.734(4)	0.584(4)	1.76(3)
3	0.584	0.872	2	0.584	0.822	1.89	0.581(5)	0.81(1)	1.90(1)

model, the critical exponents in the DP universality class become the mean-field ones above four dimensions. So, it is to be expected that the exponents are better when the dimension grows. It is interesting to observe, however, that the ν critical exponent prediction remains of excellent quality even in $d = 1$. The other exponents, are of much lower quality in low dimension, certainly because the mean field value for the anomalous dimension and z exponent are not good approximations in those dimensions.

It would be interesting to have at our disposal higher orders of the DE for this universality class. To be able to do so, it is necessary to properly take into account the technical issues mentioned in the previous section. This work is currently in progress [58].

[1] In fact, even such a constant remains equal to zero if the Itô prescription is implemented properly [58].

The most spectacular results of the LPA in this problems appear, in fact, when studying the phase diagram of the model, which is a non-universal property. Particularly important is the case of the odd-BARW that, as explained before, corresponds to the reaction-diffusion process (1.1) when no decaying probability is present $\mu \to 0$. In that case, mean-field approximation predicts the absence of phase transition (see subsection 1.1.1; if $\mu = 0$, one has that $\Delta \geq 0$) and only an active phase is present with stationary solution $n \equiv n_* = \Delta/(2\lambda)$. This is in contradiction to $d = 1$ and $d = 2$ simulations [63, 64] where a phase transition is observed in this model that belongs to the DP universality class.

This failure of mean-field was addressed by Cardy and Täuber [65, 66] in a seminal paper where they have shown that perturbative RG is able to correct the mean-field prediction in those dimensions. The concept of that work is that fluctuation can induce, at a coarse-grained level, an effective rate μ_k even if at the bare level this rate is absent. From the RG perspective it is natural to expect it, because the value $\mu = 0$ is not protected by any symmetry, so it is natural to expect that at the level of Γ_k, this parameter will become non-zero even if it is zero at the level of $\mathcal{S}$. One way to think about this effective rate is as the result of a *composite reaction*:

$$A \xrightarrow{\sigma} 2\,A \xrightarrow{\lambda} \text{nothing} \tag{3.24}$$

In the same way, the rate σ will have at the level of Γ_k a value σ_k that is different from its bare value. The non-trivial issue is the sign when $k \to 0$ of $\Delta_k = \sigma_k - \mu_k$. If the sign of this quantity keeps the positive sign of its bare expression, the system will be in an active phase, but if the sign changes along the evolution with k the system can go to an absorbing phase. The conclusion to which Cardy and Täuber arrived is that fluctuations are only strong enough in order to induce a non-trivial phase diagram for $d \leq 2$. They showed that the transition line is given by $\sigma_c = D(\lambda/(2D\pi\epsilon))^{2/\epsilon}$ for $d < 2$ (with $\epsilon = 2 - d$) and takes the form $\sigma_c = D\exp(-4\pi D/\lambda)$ in $d = 2$. For $d > 2$ they predicted that no transition could take place in such system, and such a conclusion was clearly under control in the perturbative domain.

This problem was re-analyzed within the NPRG [54, 55][2] and the result was compared with Monte-Carlo simulations in dimensions $1 \leq d \leq 6$. The corresponding results are shown in figure 3.1. It is important to stress that this line transition corresponds to a *non-universal* property of the model, usually claimed to be beyond the scope of field-theoretical treatments. The

[2]In fact, not only LPA but also a small variant of this approximation (sometimes called LPA') was analyzed in [54, 55]. This variant allows to estimate the exponents η and z beyond the mean-field approximation.

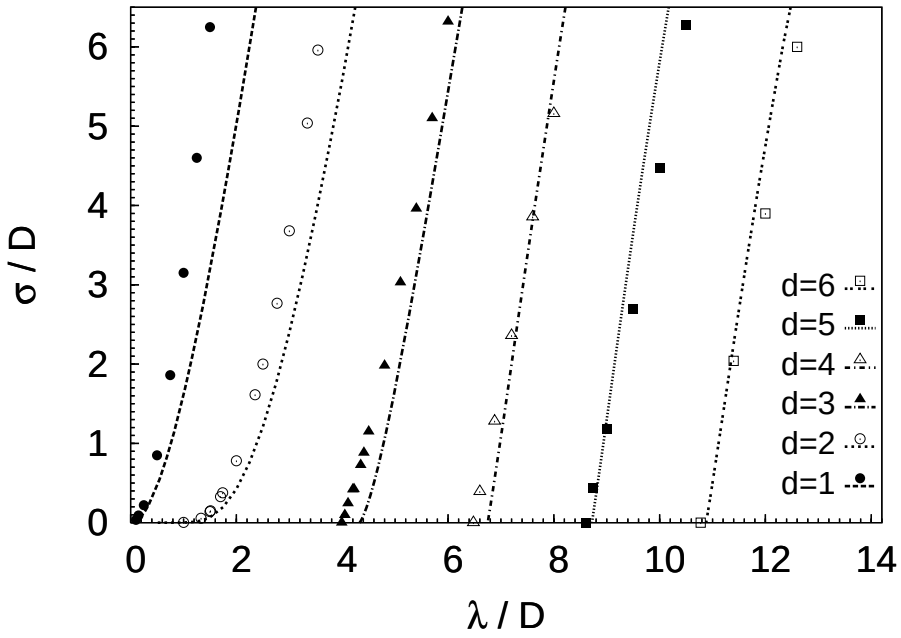

Figure 3.1: Phase diagram of the BARW $A \xrightarrow{\sigma} 2\,A$, $2\,A \xrightarrow{\lambda}$ nothing, in dimensions 1 to 6 (figure from [55]). Lines present NPRG results. Dots present Monte-Carlo numerical simulations. For each dimension the active phase lies on the left of the transition line and the absorbing phase on the right.

comparison between NPRG and Monte-Carlo results is surprisingly good and clearly shows phase transitions above $d = 2$. A technical point needed to be mentioned is that in order to compare Monte-Carlo and NPRG results the overall distance scale must be fixed. More precisely the relation of the lattice parameters and the continuum parameters at a microscopic scale Λ requires an adjustment of the points in the plane $(\lambda/D, \sigma/D)$ by an overall factor. It is fixed by dimensional analysis to take the form (C^{2-d}, C^2). Once the scaling factor C has been adjusted for a single point, it fixes the complete curve for any dimension.

Let's compare the results coming from perturbative RG and NPRG. In $d = 1$ and $d = 2$, perturbative results are reliable, because the transition curve includes the zero-coupling limit $\lambda = \sigma = 0$. In those cases, NPRG and perturbative results coincide: the curve in $d = 1$ is quadratic in both cases and the curve in $d = 2$ is exponential in D/λ (see parts (b) and (c) of figure 3.2). Moreover, the LPA pre-coefficient of the exponential is -11.8 that is very close to the perturbative RG one (-4π) [65, 66]. For higher dimensions, the transition curve does not approach the zero-coupling limit and it is beyond the scope of a perturbative analysis because it takes place only beyond a finite threshold λ_{th}/D. The value of this threshold coupling has also been investigated as a function for the dimension in [55] as shown in the part (a) of figure 3.2. It is seen in this curve that the threshold varies essentially linearly with the space dimension d. This suggests strongly that the mean-field prediction of non-existence of an absorbing phase is only recovered asymptotically for $d \to \infty$ (this results has been supported also by other techniques [67]). In this out-of-equilibrium phenomena, mean-field approximation or even perturbative corrections seem to be much less trustful than in the equilibrium case. Even characteristic dimensions where some phases exists or not, that normally are well under control in mean-field approximation for equilibrium phenomena, seem to require more sophisticated techniques in the out-of-equilibrium case. This seems to be a quite general phenomena [30]. In order to face this problems, the NPRG methods seem to be a power tool. Its use in this type of problem is almost all in the future.

3.3 Conclusions

These notes tried to present a pedagogical introduction to the application of the NPRG to out-of-equilibrium systems. In order to do so, it has been necessary to introduce before the physical motivations to use the NPRG and some basic examples. These examples, have been taken for simplicity from equilibrium problems, conceptually simpler than nonequilibrium ones. This

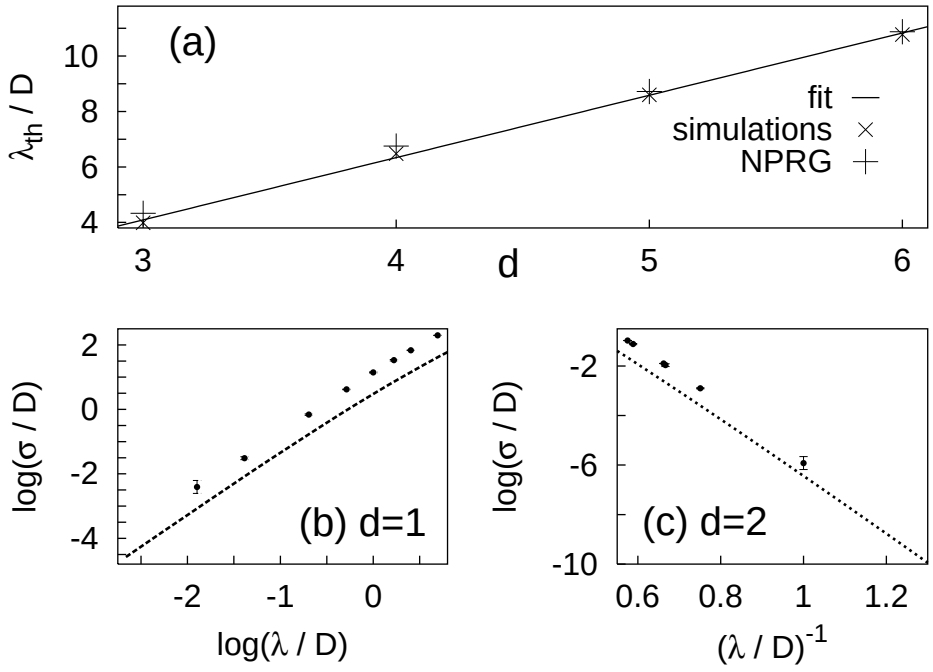

Figure 3.2: (a) Evolution of the thresholds λ_{th}/D with the dimension; (b) and (c) log plots of the transition curve in the vicinity of the origin in $d = 1$ and $d = 2$ respectively (figure from [55]).

took a large part of the notes. I tried to make it readable for students that did not know the basic perturbative results as the one-loop correction to the Landau approximation. Unfortunately it obliged me to shorten the space devoted to out-of-equilibrium problems that was the main objective.

I hope that the reader who arrived to this point will be able now to read the literature on the subject both in equilibrium and nonequilibrium problems. The notes of B. Delamotte [1] on NPRG and the short review of L. Canet on nonequilibrium applications of it [30] are certainly good complements in order to do so. I tried to present a relatively complete NPRG bibliography but the reading of some of the reviews mentioned will give some complements. I think that the notes are already too long, so it is impossible to analyze the case of 'parity conserving' transitions. However, I encourage the reader to try to reproduce the formulas of the article [56] as proposed in the last exercise of this chapter. If he or she manages to do so, the goal of these notes will be achieved.

3.4 Exercises

1) Calculate the matrix $\Gamma_k^{(2)}(q, \phi, \bar{\phi})$ from the LPA Ansatz (3.18).

2) Prove (3.23).

3) Prove that if the initial condition for V_k satisfy the rapidity symmetry (3.8) as in (3.19), then the solution V_k of the equation (3.23) has the symmetry for any k.

4) Prove that if the initial condition for V_k is proportional to $\bar{\phi}\phi$ as in (3.19), then the solution V_k of the equation (3.23) has also this property for any k (up to a constant with no physical consequences).

5) Prove (3.9).

6) Read and reproduce the various formulas in reference [56].

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