Nonperturbative renormalization group approach to nonequilibrium systems

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These lecture notes are the basis of the presentation with the same title that will take place in the Summer School 'Nonequilibrium Statistical Mechanics: Fundamental Problems and Applications' at Boulder, USA, in July 2009. The reader will not find new results here and I am in debt with a set of authors who have contributed to the subjects presented here in the last years. I am particularly in debt with the PhD thesis of Leonie Canet (downloadable from http://lpm2c.grenoble.cnrs.fr /spip.php?article353) and with the lecture notes of B. Delamotte 'An introduction to the nonperturbative renormalization group' [1]. The textbook of M. Le Bellac [2] was always around the table while I was writing these notes as can be seen in the first chapter. I have learnt most of what I know in field theoretical treatment of reaction-diffusion systems from many discussions with L. Canet, H. Chaté and B. Delamotte. The lectures of U. Täuber (that can be downloaded from http://www.phys.vt.edu/ tauber/) have also been very useful. In summary: these notes are not original, almost all the good things the reader can find here have been explained in a better way elsewhere, and I be mainly the author of many mistakes that will certainly appear. However, I hope that in some parts, as in the motivation of the Derivative Expansion, a pedagogical progress to most part of the literature has been achieved. Moreover, in the third chapter I have tried to explain carefully some calculations made some time ago and never written in detail in the literature. I must acknowledge F. Benitez and L. Canet for reading a previous version of this manuscript and suggesting many improvements. L. Quintana and A. Rubini helped a lot to make this text closer to real English. I must acknowledge also the support of the uruguayan program PEDECIBA.

Contents

Chapter 2

A simple NPRG application at equilibrium

In this chapter we will see a concrete application of the NPRG. The Ginzburg-Landau model with a single scalar field is used as the simplest possible example. The generalization to the slightly more involved model with N scalar fields and $O(N)$ symmetry is also mentioned. Along the presentation we will introduce a widely used approximation scheme in the NPRG context: The Derivative Expansion (DE). A detailed presentation of the leading order (called Local Potential Approximation, LPA) is done and the corresponding results shown. The presentation will be mainly focused on critical properties of the model but will also contain some comments on non-critical results. We will also present limiting behavior of the LPA in the perturbative and large N limits as well as the limit corresponding to the Non-Linear σ model around $d = 2$. Finally, we will see the critical exponents obtained from higher orders of the Derivative Expansion and compare them to the best estimates in the literature. Some comments are added for other approximation schemes explored in the literature in the NPRG context.

2.1 Approximating the NPRG

The equation (1.78) looks simple because of its similarity to the one-loop expression (1.77). However, it is extremely involved. In the perturbative expression (1.77) all the r.h.s. elements are known a priori. In the NPRG equation (1.78) they are part of the problem. In fact, the NPRG equation is a functional partial differential equation that can not be solved without performing approximations. As explained in the previous chapter, the equation is exact, but the first thing we will have to do is to replace it with an

approximate but manageable equation.

As explained before, the main advantage of the NPRG equation is that its r.h.s. is dominated by a small shell of wave-numbers. This will allow us to formulate approximation schemes otherwise impossible to implement. Of course, one could just employ the perturbation theory. The equations could be organized according to the powers of coupling constant appearing in each term. By doing so, one recovers the full perturbative expansion [26], and can use this calculation in order to justify the perturbative RG improvement to direct perturbative calculations. We will not follow this strategy here and, in fact, there are much simpler and convenient ways to implement the perturbative RG analysis both at equilibrium and out-of-equilibrium. The reader is encouraged to look at the notes of B. P. Vollmayr-Lee for a presentation of perturbative techniques. The strength of the NPRG is that it is extremely adapted to the formulation of other approximation schemes that do not rely on perturbation theory.

Here we will present an approximation scheme called Derivative Expansion [27, 28, 25]. This approximation scheme is well suited *only* in order to study the large distance (and long time in the out-of-equilibrium case) regime of a model. More precisely, if the correlation length of a model is ξ, the DE makes sense only for quantities with typical wave-numbers p verifying $p\xi \ll 1$. We will be interested mainly in the critical regime of various models where $\xi \to \infty$. In that regime, the DE is only able to calculate correlation functions and its derivatives at *zero* wave-numbers. It includes, however, a lot of very useful physical information, including the phase diagram of a model, the thermodynamical equation of state, the existence and value of the mean value of a field, the values of all critical exponents, etc. In particular, the DE will allow us to calculate all quantities that can be extracted from the effective potential (the Gibbs free-energy per unit of volume in a constant background field) whose typical wave-number is zero in the thermodynamical limit.

So, for the sake of concreteness, let's calculate the exact equation for the effective potential $V_k(\phi)$ by evaluating the NPRG equation (1.78) in a constant background field ϕ. In that case, as mentioned before $\Gamma_k[\phi] = \mathcal{V}V_k(\phi)$, where $\mathcal{V}$ is the volume of the space. Also, in such case, the system is translationally invariant. This implies that the (connected) correlation function of two fields only depends on *differences* of positions:

$$G_c(\vec{x},\vec{y}) = \langle\varphi(\vec{x})\varphi(\vec{y})\rangle - \langle\varphi(\vec{x})\rangle\langle\varphi(\vec{y})\rangle = G_c(\vec{x}-\vec{y}). \tag{2.1}$$

By performing a Fourier transform, this implies that

$$\tilde{G}_c(\vec{p},\vec{p}') = \int d^dx \int d^dy e^{i(\vec{x}\cdot\vec{p}+\vec{y}\cdot\vec{p}')} G_c(\vec{x},\vec{y}) = (2\pi)^d\delta^{(d)}(\vec{p}+\vec{p}')G_c(\vec{p}). \tag{2.2}$$

The last equality can be seen as a convenient definition of the Fourier transform of the correlation function in the transitionally invariant case, which is a function of a single wave-vector.

We have seen that G_c can be seen as a matrix which is the inverse of $\Gamma_k^{(2)}+R_k$. For these quantities, translational invariance also makes convenient to define the Fourier transform of $\Gamma_k^{(2)}(\vec{x},\vec{y})$ by factorizing the delta function associated with conservation of wave-vectors:

$$\tilde{\Gamma}_k^{(2)}(\vec{p},\vec{p}') = \int d^dx \int d^dy e^{i(\vec{x}\cdot\vec{p}+\vec{y}\cdot\vec{p}')}\Gamma_k^{(2)}(\vec{x},\vec{y}) = (2\pi)^d\delta^{(d)}(\vec{p}+\vec{p}')\Gamma_k^{(2)}(\vec{p}). \quad (2.3)$$

In fact, because of rotational invariance, the functions $\Gamma_k^{(2)}(\vec{p})$, $R_k(\vec{p})$ and $G_c(\vec{p})$ are functions of p^2. Moreover, the functions $\Gamma_k^{(2)}(p^2)+R_k(p^2)$ and $G_c(p^2)$ are algebraic inverses:

$$G_c(p^2) = \frac{1}{\Gamma_k^{(2)}(p^2)+R_k(p^2)}. \quad (2.4)$$

For the particular wave-vectors appearing in the NPRG equation, one concludes that in a constant background field

$$\Big(\Gamma_k^{(2)}+R_k\Big)^{-1}_{\vec{p},-\vec{p}} = (2\pi)^d\delta^{(d)}(0)\Big(\Gamma_k^{(2)}(p^2)+R_k(p^2)\Big)^{-1} = \mathcal{V}\Big(\Gamma_k^{(2)}(p^2)+R_k(p^2)\Big)^{-1}. \quad (2.5)$$

One concludes that

$$\partial_k V_k(\phi) = \frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\frac{\partial_k R_k(q)}{\Gamma_k^{(2)}(q^2,\phi)+R_k(q^2)}. \quad (2.6)$$

This is the exact NPRG equation for the effective potential. Its difficulty is clear: it is not a closed equation because it depends on the, also unknown, 2-point function $\Gamma_k^{(2)}(q^2,\phi)$. One could deduce from (1.78) an equation for $\Gamma_k^{(2)}(q^2,\phi)$, by taking two functional derivatives and performing a Fourier transform. However, the corresponding exact equation would depend on $\Gamma^{(3)}$ and $\Gamma^{(4)}$. In general, the equation for $\Gamma^{(n)}$ depends on all $\Gamma^{(m)}$ with $m \leq n+2$. This is sometimes called the *closure problem* of the $N-$body problem.

In order to face this problem, one must perform *approximations*. It is interesting to note, however, that we actually know one part of $\Gamma^{(2)}(q^2,\phi)$ (and, in fact, of any $\Gamma^{(n)}$) if we know the effective potential. In order to express it, let's define the corresponding Fourier transforms as

$$\int d^dx_1 \ldots \int d^dx_n e^{i(\vec{x}_1\cdot\vec{p}_1+\cdots+\vec{x}_n\cdot\vec{p}_n)}\ \Gamma_k^{(n)}(\vec{x}_1,\ldots,\vec{x}_n,\phi)\Big|_{\phi(\vec{x})\equiv\phi} = (2\pi)^d\delta^{(d)}(\vec{p}_1+\cdots+\vec{p}_n)\Gamma_k^{(n)}(\vec{p}_1,\ldots,\vec{p}_n,\phi). \quad (2.7)$$

where, because of the delta-function, the last argument $\vec{p}_n$ is not an independent variable but is fixed by $\vec{p}_n = -(\vec{p}_1 + \cdots + \vec{p}_{n-1})$. If so defined, then

$$\Gamma_k^{(n)}(\vec{p}_1 = \cdots = \vec{p}_n = \vec{0}, \phi) = \frac{\partial^n V_k(\phi)}{\partial \phi^n}. \tag{2.8}$$

This is intuitive: to take the derivative with respect to a constant background field of the Gibbs free-energy is the same than to take the derivative with respect to the zero wave-number component of the field. The precise proof is left for exercise.

In particular, concerning the equation for the potential (2.6) this implies that the part of $\Gamma_k^{(2)}(q^2, \phi)$ that can be expressed in terms of the potential is $\Gamma_k^{(2)}(q^2 = 0, \phi) = \partial^2 V_k(\phi)/\partial \phi^2$.

The properties of the NPRG must now be exploited. First of all, given the fact that the NPRG equation is both ultraviolet and infrared finite, the functional $\Gamma_k[\phi]$ is regular in the field and in its derivatives. This means that its derivatives $\Gamma_k^{(n)}$ are regular functions of the wave-vectors and of the external background field. Second, the integral appearing in (2.6) is dominated by wave-vectors with $q^2 \lesssim k^2$. This motivates the Derivative Expansion approximation scheme: one expands, in the r.h.s. of NPRG equation the differences between $\Gamma_k^{(n)}$ and $\mathcal{H}^{(n)}$ functions to a given power of wave-number and neglect higher powers. At leading order, called Local Potential Approximation (LPA), one performs the replacement:

$$\Gamma_k^{(n)}(\vec{p}_1, \ldots, \vec{p}_n, \phi) \overset{LPA}{\to} \Gamma_k^{(n)}(\vec{p}_1 = \cdots = \vec{p}_n = \vec{0}, \phi) = \partial_\phi^n V_k(\phi), \tag{2.9}$$

except for the 2-point function that has a *bare* wave-number dependence:

$$\Gamma_k^{(2)}(p^2, \phi) \overset{LPA}{\to} p^2 + \Gamma_k^{(2)}(p^2 = 0, \phi) = p^2 + \partial_\phi^2 V_k(\phi). \tag{2.10}$$

This information can be summarized by performing this approximation in the effective potential equation

$$\partial_k V_k(\phi) \overset{LPA}{\sim} \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q)}{q^2 + \partial_\phi^2 V_k(\phi) + R_k(q^2)}. \tag{2.11}$$

This gives the LPA equation for the effective potential. We now have an approximated but *closed* equation. One can extract from it a lot of physical information as will be analyzed in detail in the next section.

We must now explain how to formulate higher orders of the DE. This approximation scheme corresponds to expanding the Fourier-transform of the derivatives of Γ_k in wave-vectors. Wave-vectors correspond in direct space

to derivatives of the field. Then, the DE can be seen as an expansion of Γ_k (or more precisely, an expansion of $\Gamma_k - \mathcal{H}$) in the number of derivatives of the fields. Then, the leading order (LPA) will correspond to take an Ansatz for the averaged Gibbs free-energy of the form:

$$\Gamma_k[\phi] \overset{LPA}{\sim} \int d^d x \Big\{ \frac{1}{2} \big(\nabla\phi(\vec{x})\big)^2 + V_k(\phi) \Big\}. \tag{2.12}$$

One sees in this ansatz that the part containing derivatives is left as in $\mathcal{H}$ and the part without derivatives is allowed to flow with k. From this ansatz one can calculate derivatives of Γ_k, substitute them in the NPRG equation (1.78) and obtain the LPA equation for the potential (2.11).

In the same way, the next-to-leading order of the DE, called $\mathcal{O}(\partial^2)$ corresponds to take the ansatz:

$$\Gamma_k[\phi] \overset{\mathcal{O}(\partial^2)}{\sim} \int d^d x \Big\{ \frac{Z_k(\phi)}{2} \big(\nabla\phi(\vec{x})\big)^2 + V_k(\phi) \Big\}. \tag{2.13}$$

Now both the term without derivatives of the fields and the term with the minimum number of derivatives (two) are allowed to flow, but the terms with more derivatives are maintained at zero. The next-to-next-to-leading order, called $\mathcal{O}(\partial^4)$ includes three further terms, that are all possible independent terms with four derivatives of the field, etc. Plugging these ansatz in the NPRG equation (1.78) one deduces closed flow equations for the various coefficient functions $V_k(\phi)$, $Z_k(\phi)$, etc.

In the next sections concrete results obtained from this approximation scheme will be presented. Before doing so, it is important to discuss why such an approximation scheme has a chance to correctly describe the long distance regime of several models including their strongly coupled regime.

In order to analyze this it is important to ask: what is the expansion parameter of the DE? As stated before, it is an expansion of the derivatives of Γ_k in the norm of wave-vectors. However, wave-vectors are dimensionful, so when expanding in a typical wave-vector scale p, one is expanding, in fact, in p divided by another wave-vector or inverse distance scale M_k. In fact, in a typical problem there are many of such possible scales, and the DE is an expansion in all these p/M_k parameters. In order to gauge the quality of the approximation, what matters is the *smallest* scale M_k that corresponds to the inverse of the correlation length ξ_k *in presence of the regulator*. For non zero k, the regulator term $\Delta\mathcal{H}_k$ will give a contribution to M_k of order k as explained before. This contribution comes on the top of genuinely physical contributions that would be present in the systems in absence of this artificial regulator term. In general, the most unfavorable situation is

the critical regime where the physical correlation length is infinite (and then, gives no contribution to M_k). In that case, only the contribution from the regulator is present and consequently M_k is of the order of k.

Let's concentrate on the critical case for the moment where the expansion parameter is of the order of p/k. The physical limit of the model corresponds to $k \to 0$, so in that limit, the best one can hope to calculate with this approximation scheme are quantities with zero typical wave-number. The delicate point is the following: even if we are calculating quantities with zero typical wave-numbers (because of interactions between different modes) in the equations, correlation functions with arbitrary wave-number will appear anyway. So, normally, we are not able to focus only on the small wave-number regime because it is coupled to the rest of the theory. Here the form of the NPRG comes to the rescue as explained before, because internal wave-numbers q that contributes significantly to NPRG equations are bounded by k. So, in practice, correlation functions with all wave-numbers smaller than k are not coupled via the NPRG equation to other sectors of correlation functions. So, the good news is that we are expanding in a parameter that is bounded. The bad news is that the DE is an expansion in a parameter q/k that includes values of order 1.

Why should an expansion in a parameter of order 1 work? In fact, in critical phenomena it is usual to expand in parameter which is *a priori* of order 1. For example, one usual approximation is to expand in $\epsilon = 4 - d$ to obtain critical exponents in $d = 3$! The usual justification for such a procedure is the following: the mean-field approximation gives estimates for critical exponents in the Ginzburg-Landau model that are not exact but that are reasonably close to those of $d = 3$. One can formulate an expansion in ϵ that is well justified in 3.99 dimensions and that gives corrections to mean-field critical exponents that come from perturbative loop corrections. If one extrapolates to $\epsilon = 1$, the typical size of the corrections will be the difference between real critical exponents and mean-field critical exponent (that is small). So one can reasonably guess that the ϵ is accompanied by a small pre-factor that makes $\epsilon = 1$ still in the domain of validity of the expansion. And if one looks at orders ϵ and ϵ^2 this reasoning works! The difficulty comes from the fact that the perturbative expansion does not converge, but we will leave the task of explaining Borel resummation and other sophisticated techniques to the perturbative experts.

The justification of the Derivative Expansion for Ginzburg-Landau models is similar, but it is in fact better grounded for many reasons. First of all, the leading order of the DE already includes all one-loop corrections to the effective potential. So for properties with zero typical wave-numbers, the LPA is one-loop exact [25, 1]. Then, for the small momentum regime, the DE

is as good as the perturbative expansion at one-loop when $\epsilon = 4 - d$ is small. In fact, in the perturbative regime, the situation is even better because at zero external field $\phi = 0$, the corrections to the wave-number dependence of the 2-points correlation function is of order ϵ^2. So, if we are interested not only in small wave-numbers but also in small external fields, the LPA approximation is something of intermediate precision between $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$. Second, the DE expansion is not an extrapolation but an *interpolation*. It is not only a good approximation near $d = 4$ but also reproduces the leading order of the Non-linear σ model for $O(N)$ models around $d = 2$ (at order $\mathcal{O}(\partial^2)$) [25, 1]. Third, in any dimension, the LPA reproduces exactly the limit $N \to \infty$ for the effective potential of $O(N)$ models [25, 1]. So, the DE is justified as the perturbative RG expansion but in a triple way.

The same kind of reasoning can be made for other models, including out-of-equilibrium cases. Typically, the Derivative Expansion will work (in the sense of giving good results in the leading orders) when anomalous dimensions are small. In fact, as will be explained in the next section, the LPA approximation gives an anomalous dimension equal to zero and so one expects DE to work better when anomalous dimensions are small. Fortunately, anomalous dimensions are frequently small.

The DE has been applied to a large family of problems (many of them quoted in the reviews [25, 1]). Empirically the DE works surprisingly well in any dimension and for almost any tested model. Semi-quantitative results are obtained already at order LPA and they seem to improve in almost any tested model at next-to-leading order [25, 1]. The next-to-next-to-leading order has only been tested in the Ginzburg-Landau model in the Ising universality class and the corresponding results for critical exponents have the same quality level than 7-loops with Borel resummation [29]. There are only two known exceptions to these results: the applications to Quantum Chromodynamics and to the KPZ Langevin equation. In these problems, the DE has shown comparatively poor results [31, 32, 33]. These models, with physically very different motivations, have two properties in common: both models have very involved symmetries and they posses derivative interactions. It is possible that DE is not well suited for models where symmetries impose that interactions are dominated by derivative terms.

In what refers to the non-critical regime where there is a finite correlation length ξ, we are better positioned if the critical regime is under control. In that case the expansion parameter will be p/k in the part of the flow where $k \gg \xi^{-1}$ and $p\xi$ in the opposite case (here ξ is the physical correlation length). If the critical regime is under control, the validity of the results for any model will be for $p\xi \lesssim 1$.

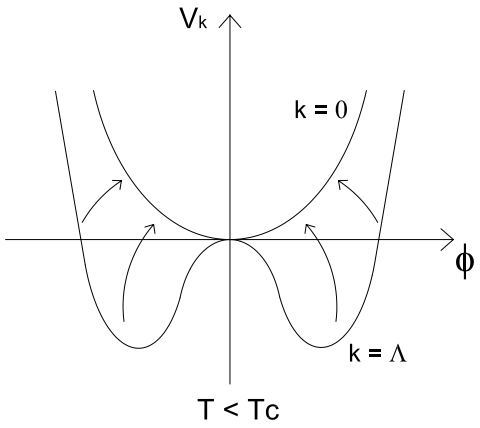

Figure 2.1: Flow with k of the effective potential V_k for $T > T_c$.

2.2 Results of LPA

Before presenting results obtained from the solution of the LPA equation, let's explain how physical results can be obtained in the DE from the behavior of the different functions $V_k(\phi)$, $Z(\phi)$, etc. In order to do so, one must solve the corresponding equations and for almost any application this must be done numerically. The set of initial conditions at $k = \Lambda$ is fixed by imposing that $\Gamma_\Lambda[\phi] = \mathcal{H}[\phi]$. The NPRG equations in a given approximation is a set of differential equations of first order in k. So, given the mentioned initial conditions, one can follow the solution from $k = \Lambda$ to the physical limit $k \to 0$.

The solution for the potential can have two typical behaviors in the case of the Ginzburg-Landau model. For example, by keeping u_0 fixed in (1.35) and varying $r_0(T)$ the system can go when $k \to 0$ in a symmetric/paramagnetic phase or in broken/ferromagnetic phase. A schematic evolution of the potential in the symmetric phase is presented in the figure 2.1.

One initially observes the effective potential favors the formation of a spontaneous magnetization. However, fluctuations tend to restore the symmetry and at a given value of $k = \bar{k}_{sym}$, the two minimums of the potential reach $\phi = 0$. The potential then continues to evolve but rapidly stabilizes. From the corresponding potential, one can obtain the equation of state:

$$\frac{\partial V_{k\to 0}(\phi)}{\partial \phi} = J. \tag{2.14}$$

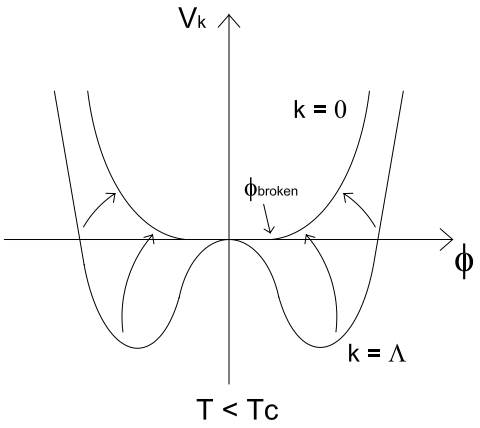

Figure 2.2: Flow with k of the effective potential V_k for $T < T_c$.

Because there is a single minimum of the potential, the only solution when $J \to 0$ is $\phi = 0$ and the system has no spontaneous magnetization.

If the temperature is low enough the evolution is quite different as shown in figure 2.2.

The minimums of the potential evolve, but stabilize at a certain values $\phi = \pm\phi_{broken}$, with $\phi_{broken} > 0$. All the potential for $|\phi| > \phi_{broken}$ stabilizes rapidly, but the part with $|\phi| < \phi_{broken}$ continues to evolve and slowly approach a flat behavior, that gives a convex physical effective potential when $k \to 0$. In this situation, the solution of the equation of state (2.14) for $J \to 0^+$ is $\phi = \phi_{broken}$ corresponding to the spontaneous magnetization.

Varying the microscopic parameters that intervene in the solution via the 'initial conditions' for the flow at $k = \Lambda$, one can also study this way the phase diagram of a given model. In particular, one observes that there exists a value of $r_0(T)$ in the edge of these two phases where $\partial^2_\phi V_{k\to 0}(\phi = 0) = 0$. This will correspond to the critical temperature T_c. Moreover, $r_0(T)$ is expected to be a regular function of the temperature. So around T_c, $r_0(T) \sim \bar{r}_0(T - T_c)$. Varying r_0 we are, in fact, varying the temperature of the model.

From the solution of the DE, one can also calculate all critical exponents. For example, the exponent ν can be obtained from the behavior of the physical correlation length as a function of the temperature. There are many different definitions of the correlation length that agree in mean-field approximation but that are slightly different in its exact form. One frequently used definition of the correlation length (that is not exactly equivalent to the

one presented in the previous chapter) is obtained from the low wave-number behavior of the physical 2-point function. If one considers this function at zero magnetization ϕ and at low wave-number:

$$\Gamma^{(2)}_{k\to 0}(p^2) = \partial^2_\phi V_{k\to 0}(\phi = 0) + Z_{k\to 0}(\phi = 0)p^2 + \mathcal{O}(p^4), \tag{2.15}$$

a usual definition of the correlation length (sometimes called the 'gap definition' of the correlation length) is

$$\xi^2 = \frac{Z_{k\to 0}(\phi = 0)}{\partial^2_\phi V_{k\to 0}(\phi = 0)}. \tag{2.16}$$

If the terms of order p^4 and higher are neglected, the 2-point function has essentially the same form than in mean-field approximation. By identifying the correlation length with the mean-field approximation, one obtains this 'gap definition' of the correlation length (2.16). It is important to mention that $\xi \to \infty$ when $T \to T_c$ because, at that temperature, the denominator of the r.h.s. of (2.16) goes to zero as explained before.

Now, by tuning the initial condition, the solution at $k \to 0$ can be varied. One can then represent ξ as a function of the temperature and from the relation

$$\xi \propto |T - T_c|^{-\nu} \tag{2.17}$$

extract the exponent ν. It is worth noticing that this procedure is not numerically very precise in practice. There are other procedures that are conceptually equivalent but numerically more efficient that are normally used in the literature.

In the same way, one can prove (see exercise 2) that at $T = T_c$ one can obtain the critical exponent η from the flow of $Z_k(\phi)$ by

$$\eta = -\lim_{k\to 0} \frac{\partial \log Z_k(\phi = 0)}{\partial \log k}. \tag{2.18}$$

Here, we can point out that the function Z_k appearing in the behavior of the regulator profile $R_k(q^2)$ (1.74) may be chosen as $Z_k(\phi = 0)$. This allows for the regulator term to be normalized in the same way that $\Gamma^{(2)}$. With this definition, k^2 can be interpreted as a contribution to the inverse running correlation length squared, M_k^2 (compare the equations (2.15) and (1.74)).

Having explained how to extract several physical quantities from the various functions that appear in the DE, let's explain in more detail the results that come from the LPA approximation.

First, comparing the LPA approximation and the next-to-leading order (2.13), one observes that LPA approximation corresponds to take $Z_k(\phi) \equiv 1$.

As can be seen from equation (2.18) this gives:

$$\eta \overset{LPA}{\sim} 0. \tag{2.19}$$

That is, concerning the wave-number dependence of the 2-point function, the LPA gives no improvement with respect to mean-field. In ϵ expansion a similar result is found: η has no correction of order ϵ, being of order ϵ^2. In fact, this is not far from true in $d = 3$: the correct value of η for the Ginzburg-Landau model (1.35) is quite small (near 0.036 [34]).

Second, for almost any application, the LPA equation needs to be solved numerically. It can be solved in any dimension with the same numerical difficulty. In fact, after analytically performing the angular integration in (2.11) the dimension d becomes an arbitrary real parameter that can be chosen at will:

$$\partial_k V_k(\phi) = \frac{1}{2^d \pi^{d/2} \Gamma(d/2)} \int_0^\infty dq q^{d-1} \frac{\partial_k R_k(q)}{q^2 + \partial_\phi^2 V_k(\phi) + R_k(q^2)}. \tag{2.20}$$

In order to obtain a concrete solution of the LPA equation one must chose a regulator profile $R_k(q^2)$. Physical results can not depend on the choice of regulator function we make. In fact, in the physical limit $k \to 0$, $\Delta\mathcal{H}_k$ is eliminated and correspondingly all dependence on $R_k(q^2)$ should disappear on observable quantities. However, once we have done approximations (as the LPA), one introduces a (hopefully small) dependence on the choice of regulator we make. On one hand this is bad news because it introduces a spurious dependence on physical predictions coming from a non-physical object. On the other hand, varying $R_k(q^2)$ and studying the dependence of physical results on this variation allow us to gauge the quality of the approximation. A good criterion to chose a regulator is the stability of its results: if when slightly changing the shape of the regulator the results change in an important way one commonly says that the regulator is not *optimized* [35, 36]. The variations of physical quantities around an optimized regulator allows for the estimation of the order or magnitude of the error bars for the results.

Anyway, in order to concretely solve the LPA equation one needs to make a choice of $R_k(q^2)$. In the last few years mainly two regulators have been largely used, each of them having their points in favor and their drawbacks. In order to be able to perform the radial integral in (2.20) analytically, the theta-regulator [35] has been widely used. It has the shape:

$$R_k(q^2) = Z_k(k^2 - q^2)\theta\big(1 - q^2/k^2\big) \tag{2.21}$$

If one chooses it, the LPA equation becomes a nonlinear but numerically relatively simple partial differential equation:

$$\partial_k V_k(\phi) = \frac{1}{d 2^{d-1} \pi^{d/2} \Gamma(d/2)} \frac{k^{d+1}}{k^2 + \partial_\phi^2 V_k(\phi)}. \tag{2.22}$$

There are two advantages in the theta-regulator. First of all, its simplicity: in order to handle the LPA approximation or the next-to-leading order of the DE, it gives equations which are extremely simple to solve numerically. Second, there are other regulators that give simple NPRG equations (for example, the sharp cut-off [16, 24, 37] and the power-law cut-off [28]), but none of them gives good results in the sense that they seem to be very far from an optimal regulator. The theta-regulator seems to be optimal among *all* regulators at the level of LPA [35], and modified by a pre-factor it can be optimized at next-to-leading order [36]. Given these good properties and its simplicity, the theta-regulator is normally the first choice when exploring a new problem.

However, it also has an important drawback: this regulator is not analytical in wave-numbers. This implies that the DE can not be implemented with it beyond next-to-leading order because the non-analyticities of the regulator give a non-analytic behavior of the Gibbs free-energy [29]. This problem becomes even more severe, for example, in finite temperature field theory [38] where even the LPA can be pathological with this regulator.

Another very used regulator is the exponential regulator [22]. Its shape is

$$R_k(q^2) = Z_k \frac{q^2}{\exp(q^2/k^2) - 1}. \tag{2.23}$$

Its drawback is evident: it does not allow the calculation of integrals analytically. However this regulator has proven to be extremely stable in many situations. In general, it is convenient to multiply it by a constant and vary this prefactor in order to optimize the results (the optimum pre-factor is frequently around 2 [36]). In general, the optimized exponential regulator is a good alternative to the theta-regulator. Because of its analyticity, the DE can be formulated at any order for this regulator and the concrete calculation at order $\mathcal{O}(\partial^4)$ has been formulated with it [29].

All critical exponents can be calculated in LPA in any dimension. They can be related to η and ν and so we will concentrate on the exponent ν here. Corrections of scaling and in particular the sub-critical exponent ω can also been calculated by these methods [39]. At the level of precision of LPA, one can extract ν numerically as explained before. For higher precision, more elaborated methods are required. Some results for ν and ω of the LPA for

Table 2.1: LPA results for some critical exponents for the $O(N)$ model for $d = 3$ and theta-regulator.[39]

N	0	1	2	3	4	5	10	∞
ν	0.59	0.65	0.71	0.76	0.80	0.84	0.92	1
ω	0.66	0.66	0.67	0.70	0.73	0.77	0.87	1

the $O(N)$ models (see exercise 3) are summarized in the table 2.1. They have been calculated with the theta-regulator and they come from [39]. If we compare them to the best estimates in the literature (see table 2.3), we can observe that the quality is quite good, given the tremendous simplicity of the method and the fact that it is only the leading order of the DE. In the next section, the quality of the results for various orders of the DE is analyzed.

It is important to mention that the LPA equation for the potential becomes *exact* when $N \to \infty$ and so do the various critical exponents [40]. The LPA has been used for the calculation of critical exponents in a large variety of models. To give just another example, it is interesting to mention that this approximation is able to identify the infinite set of multi-critical points in $d = 2$ for a single scalar model [41] and calculate the corresponding critical exponents with a relatively good precision. This is quite impressive for an approximation which neglects the anomalous dimensions, that tend to be large in $d = 2$.

Before considering higher orders of the derivative expansion, it is important to mention that the NPRG is able to analyze not only critical and universal properties but it is also able to calculate non-universal and/or non-critical properties. In particular, the LPA approximation is able to calculate delicate non-universal properties as the critical temperature of vapor-liquid transition of a given gas [25]. Moreover, the LPA has shown to be compatible with the general properties of convexity of the effective potential [21]. These is very important for practical applications in studying various broken phases, because it is a property very difficult to satisfy in standard mean-field or perturbative approximations.

2.3 Higher orders of Derivative Expansion

Here we present results for higher orders of the DE, focusing on the Ginzburg-Landau model (1.35) and its generalization with $O(N)$ symmetry (1.87). In the model with a single scalar, the next-to-leading order of the DE corre-

sponds to the Ansatz (2.13). When considering the model with N scalars and with $O(N)$ symmetry, the $\mathcal{O}(\partial^2)$ order corresponds to the most general expression with $O(N)$ symmetry and with zero or two derivatives:

$$\Gamma_k[\phi] \overset{\mathcal{O}(\partial^2)}{\sim} \int d^dx \Big\{ V_k(\rho) + \frac{Z_k(\rho)}{2} \sum_i \big(\nabla\phi_i(\vec{x})\big)^2 + \frac{Y_k(\rho)}{4} \big(\sum_i \phi_i \nabla\phi_i\big)^2 \Big\}, \tag{2.24}$$

where $\rho = \frac{1}{2}\sum_i \phi_i^2$.

When including these gradient terms, the DE gives non-trivial values for the anomalous dimensions. This is important in $d = 3$ in order to improve the precision of the results, but around $d = 2$ this is important even at the qualitative level. Once these terms are included, the DE correctly reproduces (with this linear realization of $O(N)$ models) the correct one-loop result of Non-linear σ model around $d = 2$. It is important to stress that no perturbative approach is known allowing to reproduce both the $4 - \epsilon$ and the $2 + \epsilon$ expansions. As mentioned before, this is one of the reasons for the precision achieved in the DE even at low order [25, 42]. This not only allows to a correct treatment of the non-abelian case ($N > 3$) but also for a quite impressive treatment of the Kosterlitz-Thouless transition, corresponding to $d = 2$ and $N = 2$ [25, 43]. The treatment of the $d = 2$ Ising universality class becomes semi-quantitatively correct also [43]. The corresponding results are presented in table 2.2.

Table 2.2: $\mathcal{O}(\partial^2)$ results for some critical exponents for the $O(N)$ model for $d = 2$ [43] compared with the exact result.

N	ν		η	
0	0.70	0.75	0.222	0.2083...
1	0.92	1	0.295	0.25
2	–	–	0.287	0.25

For $d = 3$ and at order $\mathcal{O}(\partial^2)$, the results achieve a relatively good quantitative level, as can be seen in the table 2.3. It is important to mention that results from [43] have been done with the exponential regulator without improving it by including a constant pre-factor.

In $d = 3$, for the particular case of $N = 1$, corresponding to the Ising universality class, the results of better quality are known for two reasons. First, the results have been optimized as explained before [36]. Second the DE has been pushed up to the $\mathcal{O}(\partial^4)$ order [29]. The corresponding results are shown in table 2.4. As it is visible in this table, the results from $\mathcal{O}(\partial^4)$ are

Table 2.3: $\mathcal{O}(\partial^2)$ results for some critical exponents for the $O(N)$ model for $d = 3$ [43] compared with best estimates in the literature.

N	ν		η	
0	0.590	0.5882(11) [44]	0.039	0.0284(25) [44]
1	0.6307	0.6302(1) [45]	0.0467	0.0368(2) [45]
2	0.666	0.6717(1) [46]	0.049	0.0381(2) [46]
4	0.739	0.741(6) [44]	0.047	0.035(4) [44]
10	0.881	0.859 [47]	0.028	0.024 [47]

of the same level of precision than those coming from 7-loops perturbative RG analysis.

Table 2.4: Results from Derivative Expansion a different orders [36, 29] for $N = 1$ and $d = 3$, compared to best estimates in the literature [45] and best perturbative estimates [44].

order	ν	η
∂^0 =LPA	0.651	0
∂^2	0.628	0.044
∂^4	0.632	0.033
7-loops [44]	0.6304(13)	0.0335(25)
best estimate [45]	0.6302(1)	0.0368(2)

Before finishing this chapter, it is important to quickly mention that in the context of the NPRG there are other approximation schemes that have been implemented and that are not based on perturbation theory. It is impossible to present here any of them in detail, but the interested reader is advised to look at the corresponding references that can be found in [25, 42]. Just for the sake of completeness, most of the other schemes are based on Weinberg's proposal of an approximation scheme [48]. It consists in expanding Γ_k in the fields and truncating this expansion to a given power of the fields, but including an arbitrary wave-number dependence. This kind of approximation has been used in other contexts as in truncated Schwinger-Dyson equations. It is impossible to resume here the literature on it. In the context of the NPRG it has also been largely used together or not with the Derivative Expansion approximation [25, 42]. For models such as Ginzburg-Landau, where the interactions are dominated by potential terms and not by derivative terms, it

seems that DE is more powerful in order to extract low wave-number quantities [49]. When interactions including derivatives are present as in QCD and in the solution of KPZ equation, it seems that the inclusion of some sort of full momentum dependence is convenient [50, 51, 33]. Recently an approximation scheme that includes the advantages of both DE and Weinberg's approximation, has been proposed with very promising results [52, 53].

2.4 Exercises

1) Prove the equation (2.8).

2) At the critical temperature the systems shows the property of *scaling invariance*. In particular, the quotient of two correlation functions is a function of dimensionless variables *without* any dependence on microscopic parameters of the model:

$$\frac{\Gamma_k^{(2)}(p')}{\Gamma_k^{(2)}(p)} = f\Big(\frac{p'}{p}, \frac{k}{p}\Big). \tag{2.25}$$

The function $\Gamma_k^{(2)}(p)$ has a limit when $k \to 0$. By admitting this, show that $\Gamma_{k\to 0}^{(2)}(p)$ is a power law when $T = T_c$ and comparing to (1.18) show that the exponent is

$$\Gamma_{k\to 0}^{(2)}(p) \propto p^{2-\eta}. \tag{2.26}$$

Show also that the same η appear in the formula (2.18):

$$Z_k(\phi = 0) \sim k^{-\eta} \tag{2.27}$$

Hint: consider (2.25) for $p = k$.

3) Show that the LPA equation for the $O(N)$ model (1.87) is:

$$\partial_k V_k(\rho) = \frac{1}{2^d \pi^{d/2} \Gamma(d/2)} \int_0^\infty dq q^{d-1} \partial_k R_k(q) \times \Big\{ \frac{N-1}{q^2 + \partial_\rho V_k(\rho) + R_k(q^2)} + \frac{1}{q^2 + \partial_\rho V_k(\rho) + 2\rho \partial_\rho^2 V_k(\rho) + R_k(q^2)} \Big\}, \tag{2.28}$$

where $\rho = \frac{1}{2}\sum_{i=1}^N \phi_i \phi_i$

4) Consider the ϕ^4 approximation to the $O(N)$ form of $\Gamma_k[\phi]$ (see [25] or [1]):

$$\Gamma_k[\phi] = \int d^d x \Big\{ \frac{Z_k}{2} \sum_i \nabla\phi_i(\vec{x})\cdot\nabla\phi_i(\vec{x}) + \frac{r_k}{2} \sum_i \phi_i^2(\vec{x}) + \frac{u_k}{4!} \Big(\sum_i \phi_i^2(\vec{x}) \Big)^2 \Big\} \tag{2.29}$$

Deduce the NPRG equations for the three constants r_k, u_k and Z_k in that approximation. Prove that the systems go to a critical regime when $k \to 0$ if the renormalized and dimensionless constants $\tilde{r}_k$, $\tilde{u}_k$ defined by

$$\tilde{r}_k = \frac{r_k}{Z_k k^2} \qquad\qquad \tilde{u}_k = \frac{u_k}{Z_k^2 k^{4-d}} \tag{2.30}$$

approach a *fixed point* solution $k\partial_k \tilde{r}_k = k\partial_k \tilde{u}_k = 0$.

Deduce from the NPRG equation for the dimensionless constants and from the running anomalous dimension $\eta_k = -k\partial_k \log Z_k$, the values of η and ν in $4-\epsilon$ dimensions and in $2+\epsilon$ dimensions.