

# Nonperturbative renormalization group approach to nonequilibrium systems

N. Wschebor  
Instituto de Física, Facultad de Ingeniería,  
Universidad de la República,  
Montevideo, Uruguay

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These lecture notes are the basis of the presentation with the same title that will take place in the Summer School 'Nonequilibrium Statistical Mechanics: Fundamental Problems and Applications' at Boulder, USA, in July 2009. The reader will not find new results here and I am in debt with a set of authors who have contributed to the subjects presented here in the last years. I am particularly in debt with the PhD thesis of Leonie Canet (downloadable from <http://lpm2c.grenoble.cnrs.fr/spip.php?article353>) and with the lecture notes of B. Delamotte 'An introduction to the nonperturbative renormalization group' [1]. The textbook of M. Le Bellac [2] was always around the table while I was writing these notes as can be seen in the first chapter. I have learnt most of what I know in field theoretical treatment of reaction-diffusion systems from many discussions with L. Canet, H. Chaté and B. Delamotte. The lectures of U. Täuber (that can be downloaded from <http://www.phys.vt.edu/tauber/>) have also been very useful. In summary: these notes are not original, almost all the good things the reader can find here have been explained in a better way elsewhere, and I be mainly the author of many mistakes that will certainly appear. However, I hope that in some parts, as in the motivation of the Derivative Expansion, a pedagogical progress to most part of the literature has been achieved. Moreover, in the third chapter I have tried to explain carefully some calculations made some time ago and never written in detail in the literature. I must acknowledge F. Benitez and L. Canet for reading a previous version of this manuscript and suggesting many improvements. L. Quintana and A. Rubini helped a lot to make this text closer to real English. I must acknowledge also the support of the uruguayan program PEDECIBA.

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# Chapter 1

## Introduction to the Non-Perturbative Renormalization Group

This first chapter is a quick introduction to the Non-Perturbative Renormalization Group (NPRG). We will present the sort of problem this technique tries to address in and out of equilibrium. For the sake of simplicity, we will leave the out-of-equilibrium examples for the third chapter and focus on using the equilibrium Ginzburg-Landau model. In particular, the Ginzburg criterion for the validity of perturbative calculations is presented. Finally, the NPRG is introduced and its general properties shown.

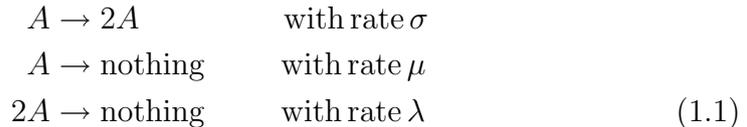
### 1.1 Some examples of physical problems out-of-equilibrium

The goal of these notes is to present the Non-perturbative Renormalization Group (NPRG) and its applications in nonequilibrium statistical mechanics. As in many areas of theoretical physics, the technique is involved. This makes very difficult to learn while trying to approach the different physical problems it would be applied to. This is the reason why a significant part of these lectures use examples in equilibrium phenomena. The technical aspects are simpler in this case and it is easier to acquire a physical intuition of the methods employed before addressing the nonequilibrium examples.

However, it is important to start by describing some examples of physical problems that one may be interested in analyzing and of the kind of quantities to be calculated.

### 1.1.1 Reaction-diffusion problems

Suppose we consider a reaction-diffusion problem where some molecules of type  $A$  can diffuse with a diffusion coefficient  $D$  and can experiment reactions of the type



This reaction-diffusion problem is included within a large family of phenomena included in the 'Directed Percolation' universality class [3, 4]. If you change the values of the reaction rates this system experiments a second order phase transition between two very different scenarios.

In order to have a rough estimate to what happens in such a phenomena one can consider a *mean field approximation* and describe it in terms of a uniform average concentration of particles  $n(t)$  whose evolution in time is the mean field evolution equation:

$$\partial_t n(t) = (\sigma - \mu)n(t) - 2\lambda n(t)^2 \tag{1.2}$$

If the initial value of  $n(t)$  is  $n_0$ , the solution takes the form

$$n(t) = \frac{n_0 n_*}{n_* - n_0(1 - e^{\Delta t})} e^{\Delta t}. \tag{1.3}$$

where  $n_* = \Delta/(2\lambda)$  and  $\Delta = \sigma - \mu$ . The behavior of the solution depends on the sign of  $\Delta$ :

- If  $\Delta > 0$ , there are two stationary solutions  $n(t) \equiv 0$  and  $n(t) \equiv n_*$ . For any non-zero initial condition, the concentration tends to  $n_*$  exponentially when  $t \gg \Delta^{-1}$ .
- If  $\Delta < 0$ , there is only one stationary solution  $n(t) \equiv 0$  and the concentration tends to it exponentially when  $t \gg \Delta^{-1}$ .
- At  $\Delta = 0$  a second order phase transition takes place and  $n(t)$  tends to zero at large times but only algebraically:

$$n(t) = \frac{n_0}{1 + 2\lambda n_0 t} \tag{1.4}$$

In fact, Monte Carlo simulations of the full reaction-diffusion problem, *without mean field approximation*, show that this second-order phase transition takes place [3]. Typical evolutions in a  $d = 1$  system are shown in the

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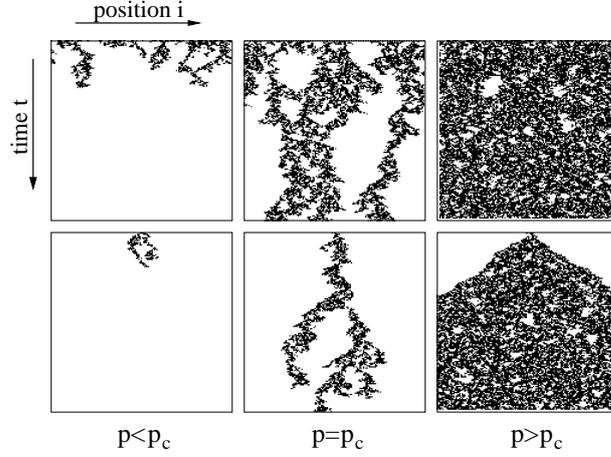


Figure 1.1: In this simulation [3] all the rates  $D$ ,  $\mu$ ,  $\sigma$  and  $\lambda$  are fixed in terms of a single probability  $p$  of connexion of lattice sites. The first line corresponds to a one dimensional lattice uniformly occupied initially with three different values of  $p$ : one below, one above and one just at the critical value. The second line is the same but the lattice is only occupied in a single site initially.

figure 1.1. The  $n(x, t) \equiv 0$  state is called an 'absorbing state'. If there are no particles at all the reactions (1.1) can not generate them. In that phase, not only the expected value of the number of particles tends to zero at large times: all fluctuations in the system disappear and the number of particles itself goes to zero. On the other hand, the phase with a non-zero concentration in its final state is called an 'active phase'. The average concentration tends to a non-zero value, but the value of the concentration in a given point continues to fluctuate along time.

The goal of Renormalization Group methods is to go beyond mean-field approximations in a controlled way. For the moment, we will consider only properties with large characteristic times and distances. For those particular kind of properties of the system, one can exploit phenomenological stochastic equations that take into account the diffusion from one point to another and the existence of microscopic *fluctuations* in the local concentration. This establishes a separation between modes that evolve slowly with time that will be described with a local mesoscopic variable  $n(t, \vec{x})$ , and microscopic 'rapid' fluctuations that will be treated as a 'noise'. This can be done with a non-linear Langevin equation:

$$\partial_t n(t, \vec{x}) = D \nabla^2 n(t, \vec{x}) + (\sigma - \mu) n(t, \vec{x}) - 2\lambda n^2(t, \vec{x}) + \zeta(t, \vec{x}) \quad (1.5)$$

where  $\zeta(t, \vec{x})$  represents the effect of the 'noise'.

In order to complete the modeling associated to this Langevin equation it is necessary to specify the form of the noise. Considering the separation of scales and that the noise is supposed to rapidly fluctuate independently from one position to another, one can invoke the Central Limit Theorem in order to argue that the noise should be well described by a centered Gaussian noise with local correlation:

$$\langle \zeta(t, \vec{x}) \rangle = 0 \quad \langle \zeta(t, \vec{x}) \zeta(t', \vec{x}') \rangle = 2\Gamma(t, \vec{x}) \delta(t - t') \delta^{(d)}(\vec{x} - \vec{x}') \quad (1.6)$$

However, there is an important property of this reaction-diffusion problem that can not be neglected for a correct description of the system even at long times and distances intervals: the state  $n(t, \vec{x}) \equiv 0$  is an absorbing state. This means that fluctuations of the system must be suppressed locally if  $n(t, \vec{x})$  is small. We will then suppose that

$$\Gamma(t, \vec{x}) = \gamma n(t, \vec{x}) \quad (1.7)$$

The resulting Langevin equation takes into account, at least phenomenologically, the main properties of the Directed-Percolation universality class. One can solve (at least in principle) this equation by Monte Carlo methods [3, 4]. However we will try to acquire a better comprehension of its behavior by developing approximate semi-analytical methods in order to extract its main properties.

The basic ideas mentioned before can be generalized to any reaction-diffusion system, with one or more species of molecules. We will see, moreover, that one can justify from first principles this kind of Langevin equations directly from the reaction-diffusion set of reactions.

### 1.1.2 The Kardar-Parisi-Zhang equation

The methods that will be presented can also be successfully used when applied to other non-linear Langevin equations that do not originate in reaction-diffusion processes. One example of that is the application to the Kardar-Parisi-Zhang (KPZ) equation [5]. It describes (for example) the interfaces growth between two different media, but it also applies to many other systems including the noisy Burgers equation of fluid dynamics [6]. Suppose an interface between two medias characterized by a height  $\bar{h}(\vec{x}, t)$  as in figure 1.2. In this analysis the 'turning back' effects, as in the figure 1.3, are neglected. Now, suppose a generic relaxational evolution equation of the type:

$$\partial_t \bar{h}(\vec{x}, t) = \nu \nabla^2 \bar{h}(\vec{x}, t) + \chi(\nabla \bar{h}(\vec{x}, t)) + \zeta(\vec{x}, t). \quad (1.8)$$

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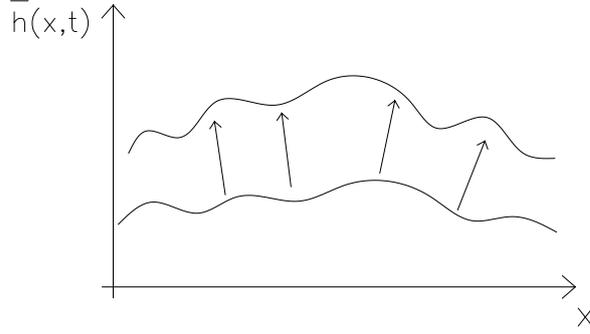


Figure 1.2: Evolution of the height  $\bar{h}(t, \vec{x})$  of an interface

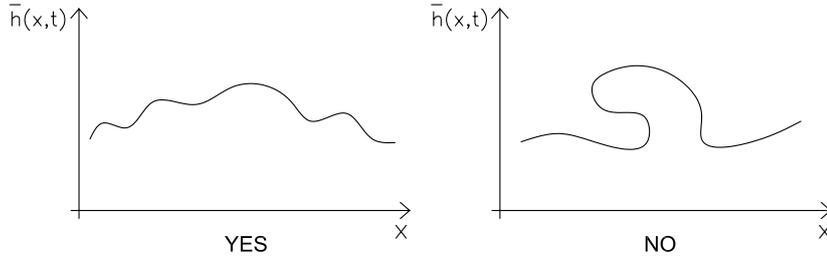


Figure 1.3: Type of interface included and not included in the analysis

Here  $\zeta(\vec{x}, t)$  is the noise term of the equation and the term proportional to  $\nu$  describes the effect of a surface tension. It is important to stress that the deterministic part of the equation depends only on *differences of the field*. Now, suppose that  $\chi(\nabla\bar{h})$  varies slowly:

$$\chi(u = \nabla\bar{h}) = \chi(0) + \nabla\chi|_{u=0} \cdot \nabla\bar{h} + \frac{1}{2} \sum_{ij} \frac{\partial^2 \chi}{\partial u_i \partial u_j} \Big|_{u=0} \partial_i \bar{h} \partial_j \bar{h} + \dots \quad (1.9)$$

Neglecting the dots terms and supposing that the dynamics is *isotropic*:

$$\chi(u = \nabla\bar{h}) = \chi(0) + \nabla\chi|_{u=0} \cdot \nabla\bar{h} + \frac{\lambda}{2} (\nabla\bar{h})^2 + \dots \quad (1.10)$$

Now, if one performs the change of variables  $h(\vec{x}, t) = \bar{h}(x - \nabla\chi|_{u=0} t, t) - \chi(0)t$ , one arrives to the KPZ equation [5]:

$$\partial_t h(\vec{x}, t) = \nu \nabla^2 h(\vec{x}, t) + \frac{\lambda}{2} (\nabla h(\vec{x}, t))^2 + \zeta(\vec{x}, t). \quad (1.11)$$

Again, the form of the noise must be specified. However, one can prove that in this system, the behavior at large distances and large times is extremely universal. Most of the details of the deterministic parts and/or the shape of the noise have no impact on the macroscopic behavior of the surface. Then the simplest form of the noise can be taken: Gaussian and white with:

$$\langle \zeta(\vec{x}, t) \rangle = 0, \quad \langle \zeta(\vec{x}, t) \zeta(\vec{x}', t') \rangle = 2D\delta^{(d)}(\vec{x} - \vec{x}')\delta(t - t') \quad (1.12)$$

### 1.1.3 What kind of objects will be analyzed?

As mentioned, we will be interested in the analysis of the behavior of these out-of-equilibrium systems at long times and at large intervals of distances. It is important to analyze the kind of aspects that will be addressed and compared to experiments and other techniques.

First of all, we will concentrate on quantities that are *averages on the noise*. One could study the behavior of the system for a particular realization of the noise, but here we will concentrate only on quantities that have been averaged on the noise. This kind of quantities are much simpler and much more universal. For example, when the mean on the noise is performed the system tends to depend much less on initial conditions at long times. Moreover, the symmetries of the dynamics are realized on means but are not present on a given particular realization of the noise.

Second, only large distance and long time behavior will be analyzed. Again, in general, this considerably reduces the dependence on initial conditions. The characteristic distances and times are, in most cases, of the same order of magnitude as in the microscopic rules of evolution. However, in some cases the characteristic times and distances are much larger. In such cases, the phenomenon of *universality* takes place for quantities dominated by these macroscopic scales. The corresponding regime is called the *critical regime*. Most part of the analysis to be presented here will refer to the critical regime of some models, but some aspects of the non-critical regimes will be also analyzed. It is important to realize that for non critical properties it is necessary to control not only the macroscopic behavior but also to be able to relate it to the detailed microscopic behavior of the system. This is a much harder task, but some examples of successful applications of the NPRG to that class of problems will be shown. In those cases, we will keep our focus on the large distance and long time behavior even when studying non-universal properties.

Some concrete examples that apply both for reaction-diffusion problems (then  $\varphi = n$ ) or in other Langevin equations (for KPZ,  $\varphi = h$ ) are the following:

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- The mean value of the field  $\varphi$ ,

$$\langle \varphi(t, \vec{x}) \rangle, \quad (1.13)$$

is an extremely important quantity. In the reaction-diffusion problem (1.1) it allows for the determination of the phase diagram of the model. If  $\langle \varphi(t, \vec{x}) \rangle = 0$  the system is in the absorbing phase and if  $\langle \varphi(t, \vec{x}) \rangle > 0$ , the system is in its active phase. It is important to observe that the shape of the phase diagram is not universal and depends on the precise knowledge of the microscopic form of the theory.

Close to the second order phase transition, approached from the active phase one can define a (universal) critical exponent from it:

$$\langle \varphi(t, \vec{x}) \rangle \sim |\mu - \mu_c|^\beta \quad (1.14)$$

Here the rate  $\mu$  has been used as a control parameter, but the other rates can be used as well.

- The correlation function:

$$G(t_1, \vec{x}_1; t_2, \vec{x}_2) = \langle \varphi(t_1, \vec{x}_1) \varphi(t_2, \vec{x}_2) \rangle \quad (1.15)$$

plays a very important role in the analysis of the system, particularly near a second order phase transition. It allows the definition of some critical exponents. One defines also more general correlation functions that are the mean value of products of more than two fields.

- The response function is other important object in this out-of-equilibrium systems. Suppose that one adds a 'source' term  $\tilde{j}(t, \vec{x})$  in the Langevin equation for Directed Percolation:

$$\partial_t n(t, \vec{x}) = D \nabla^2 n(t, \vec{x}) + (\sigma - \mu) n(t, \vec{x}) - 2\lambda n^2(t, \vec{x}) + \zeta(t, \vec{x}) - \tilde{j}(t, \vec{x}). \quad (1.16)$$

The response function is defined as

$$\chi(t_1, \vec{x}_1; t_2, \vec{x}_2) = \left. \frac{\delta \langle \varphi(t, \vec{x}) \rangle}{\delta \tilde{j}(t', \vec{x}')} \right|_{\tilde{j} \equiv 0}. \quad (1.17)$$

As before, it allows the definition of some critical exponents, that a priori are not independent of the previous ones. In some cases, the response function is related by a fluctuation-dissipation theorem to the correlation function, but in many out-of-equilibrium systems they are independent quantities.

In the critical regime, one can consider, for example, the response function averaged on time, whose behavior at large times  $t$  is:

$$\int dt' \chi(t, \vec{x}; t', \vec{y}) \sim \frac{e^{-|\vec{x}-\vec{y}|/\xi_{\perp}}}{|\vec{x}-\vec{y}|^{d-2+\eta}} \quad (1.18)$$

This defines the critical exponent  $\eta$ . The spatial correlation length  $\xi_{\perp}$  diverges in the critical point:

$$\xi_{\perp} \sim |\mu - \mu_c|^{-\nu_{\perp}} \quad (1.19)$$

In the same way, the space average of the response function defines a correlation time  $\xi_{\parallel}$  that diverges near the critical point as

$$\xi_{\parallel} \sim (\xi_{\perp})^z. \quad (1.20)$$

This relation defines a specific out-of-equilibrium exponent.

## 1.2 The Janssen-De Dominicis-Martin-Siggia-Rose procedure

In this subsection we present a formalism that simplifies considerably the analysis for the calculation of mean values on the noise. It was elaborated by Martin-Siggia-Rose [7], Janssen [8] and De Dominicis [9]. Consider the Langevin equation:

$$\partial_t \varphi(t, \vec{x}) = \mathcal{F}[\varphi](t, \vec{x}) + \zeta[\varphi](t, \vec{x}) \quad (1.21)$$

with  $\zeta(t, \vec{x})$  a centered Gaussian noise with

$$\langle \zeta(t, \vec{x}) \rangle = 0 \quad \langle \zeta(t, \vec{x}) \zeta(t', \vec{x}') \rangle = 2\Gamma \delta(t - t') \delta^{(d)}(\vec{x} - \vec{x}') \quad (1.22)$$

where  $\Gamma$  may depend on  $\varphi$  and may also be a differential operator.

Let's call  $\mathcal{P}[\zeta]$  the (non-normalized) probability distribution of the noise. For example, for a Gaussian white noise of variance  $2\Gamma(t, \vec{x})$  it is (see exercise 1):

$$\mathcal{P}[\zeta] \propto \exp\left(-\frac{1}{4} \int d^d x dt \frac{\zeta^2(t, \vec{x})}{\Gamma(t, \vec{x})}\right). \quad (1.23)$$

Then the mean value on the noise of a certain quantity is

$$\langle \mathcal{O}[\varphi] \rangle = \frac{\sum_{\zeta(t, \vec{x})} \mathcal{O}[\varphi_{\zeta}] \mathcal{P}[\zeta]}{\sum_{\zeta(t, \vec{x})} \mathcal{P}[\zeta]} \quad (1.24)$$

where  $\mathcal{O}[\varphi_\zeta]$  is the quantity  $\mathcal{O}[\varphi]$  evaluated on the particular solution of the Langevin equation,  $\varphi_\zeta$  that corresponds to a given realization of the noise  $\zeta(t, \vec{x})$ . The sum in (1.24) over all field configurations may be ambiguous. In order to make it precise one can imagine that the system is discretized in time and space. We will use continuous notations along this manuscript but it should be kept in mind that if any ambiguity takes place it is convenient to consider a discretized version of the model (for example, in a regular lattice).

It is convenient to re-write this expression in a way similar to mean-values of equilibrium statistical mechanics. In order to do so, consider the numerator of the previous expression:

$$\begin{aligned} \mathcal{N} &= \sum_{\zeta(t, \vec{x})} \mathcal{O}[\varphi_\zeta] \mathcal{P}[\zeta] = \sum_{\zeta(t, \vec{x})} \sum_{\varphi(t, \vec{x})} \mathcal{O}[\varphi] \mathcal{P}[\zeta] \prod_{(t, \vec{x})} \delta\left(\varphi(t, \vec{x}) - \varphi_\zeta(t, \vec{x})\right) \\ &\propto \sum_{\zeta(t, \vec{x})} \sum_{\varphi(t, \vec{x})} \mathcal{O}[\varphi] \mathcal{P}[\zeta] \prod_{(t, \vec{x})} \delta\left(\partial_t \varphi(t, \vec{x}) - \mathcal{F}[\varphi](t, \vec{x}) - \zeta[\varphi](t, \vec{x})\right) \end{aligned} \quad (1.25)$$

Here, we should add the following comment: The last relation is only valid if the determinant of the transformation from  $\varphi$  to  $\partial_t \varphi(t, \vec{x}) - \mathcal{F}[\varphi](t, \vec{x}) - \zeta[\varphi](t, \vec{x})$  is a constant. This may or may not be the case depending on the precise definition of the temporal derivative in a discrete form. It can be shown that if a forward derivative is chosen when discretizing (Itô prescription) then the determinant is a constant for any Langevin equation. Here we will suppose that such a discretization has been chosen in order to give a precise meaning to the Langevin equation.

The delta functions, may be re-written via Fourier transformations:

$$\mathcal{N} \propto \sum_{\zeta(t, \vec{x})} \sum_{\varphi(t, \vec{x})} \sum_{\tilde{\varphi}(t, \vec{x})} \mathcal{O}[\varphi] \mathcal{P}[\zeta] \exp\left\{-i \int dt \int d^d x \tilde{\varphi} \left(\partial_t \varphi(t, \vec{x}) - \mathcal{F}[\varphi](t, \vec{x}) - \zeta(t, \vec{x})\right)\right\} \quad (1.26)$$

The sum over the noise is now a Gaussian sum that can be performed exactly:

$$\mathcal{N} \propto \sum_{\varphi(t, \vec{x})} \sum_{\tilde{\varphi}(t, \vec{x})} \mathcal{O}[\varphi] \exp\left(-\mathcal{S}[\varphi, \tilde{\varphi}]\right) \quad (1.27)$$

with

$$\mathcal{S}[\varphi, \tilde{\varphi}] = \int d^d x dt \left\{ 2\Gamma(t, \vec{x}) \tilde{\varphi}^2(t, \vec{x}) + i\tilde{\varphi} \left(\partial_t \varphi(t, \vec{x}) - \mathcal{F}[\varphi](t, \vec{x})\right) \right\} \quad (1.28)$$

Repeating the same procedure for the denominator one concludes that

$$\langle \mathcal{O}[\varphi] \rangle = \frac{\sum_{\varphi(t, \vec{x})} \sum_{\tilde{\varphi}(t, \vec{x})} \mathcal{O}[\varphi] \exp\left(-\mathcal{S}[\varphi, \tilde{\varphi}]\right)}{\sum_{\varphi(t, \vec{x})} \sum_{\tilde{\varphi}(t, \vec{x})} \exp\left(-\mathcal{S}[\varphi, \tilde{\varphi}]\right)} \quad (1.29)$$

$\mathcal{S}$  will be called the 'action' associated with the Langevin equation.

As an example, one can consider the action for the Langevin equation associated to the reactions 1.1 in the class of universality of the Directed Percolation. In this case:

$$\begin{aligned} \mathcal{S}_{DP}[\varphi, \tilde{\varphi}] = \int d^d x dt \left\{ i\tilde{\varphi}(t, \vec{x})(\partial_t - D\nabla^2 - (\sigma - \mu))\varphi(t, \vec{x}) \right. \\ \left. + 2i\lambda\tilde{\varphi}(t, \vec{x})\varphi(t, \vec{x})^2 + 2\gamma\tilde{\varphi}^2(t, \vec{x})\varphi(t, \vec{x}) \right\} \end{aligned} \quad (1.30)$$

It is convenient to re-scale the fields in order to have the same coefficient in the two trilinear terms:  $\varphi \rightarrow \sqrt{\frac{\lambda}{\gamma}}\varphi$  and  $\tilde{\varphi} \rightarrow \sqrt{\frac{\gamma}{\lambda}}\tilde{\varphi}$ . Moreover, the action is usually expressed in terms of the purely imaginary field  $\bar{\varphi} = i\tilde{\varphi}$ , obtaining:

$$\begin{aligned} \mathcal{S}_{DP}^{(MSR)}[\varphi, \bar{\varphi}] = \int d^d x dt \left\{ \bar{\varphi}(t, \vec{x})(\partial_t - D\nabla^2 - (\sigma - \mu))\varphi(t, \vec{x}) \right. \\ \left. + 2\sqrt{\lambda\gamma}(\bar{\varphi}(t, \vec{x})\varphi(t, \vec{x})^2 - \bar{\varphi}^2(t, \vec{x})\varphi(t, \vec{x})) \right\} \end{aligned} \quad (1.31)$$

With fields normalized this way, the Directed-Percolation action presents the 'rapidity symmetry':

$$\varphi(t, \vec{x}) \rightarrow -\bar{\varphi}(-t, \vec{x}), \quad \bar{\varphi}(t, \vec{x}) \rightarrow -\varphi(-t, \vec{x}). \quad (1.32)$$

An expression of the type (1.29) is to be compared to the standard expression for mean values in standard Boltzmann equilibrium. If the configuration variable in that case is a field  $\varphi$  then, the mean value of a quantity  $\mathcal{O}$  is

$$\langle \mathcal{O}[\varphi] \rangle = \frac{\sum_{\varphi(\vec{x})} \mathcal{O}[\varphi] \exp(-\mathcal{H}[\varphi])}{\sum_{\varphi(\vec{x})} \exp(-\mathcal{H}[\varphi])} \quad (1.33)$$

(here the factor  $(k_B T)^{-1}$  has been re-absorbed in the definition of the effective Hamiltonian  $\mathcal{H}$ ).

The expressions (1.31) and (1.33) are very similar. The differences are:

- The sum must be done in (1.29) in twice the variables ( $\varphi$  and  $\tilde{\varphi}$ ) than in (1.33) (only  $\varphi$ ).
- The variables in (1.29) depend on position and *time* and in (1.33) depend only on the position.
- The functional  $\mathcal{S}$  is *complex* and the effective Hamiltonian  $\mathcal{H}$  is real.

This similarity allows us to exploit standard techniques from equilibrium statistical mechanics. In fact, for simplicity, we will analyze equilibrium as examples first. It is important to emphasize that going from the equilibrium Boltzmann distribution sum to the non-equilibrium case is not difficult. We will come back to non-equilibrium problems in the third chapter.

Before considering equilibrium examples, let's comment that Doi and Peliti have proved a more rigorous way to construct mean values with the form (1.29) directly from a reaction-diffusion problem without having to model it with a Langevin equation [10, 11]. For example, one can prove rigorously that mean values in the discretized form of the reaction-diffusion problem (1.1) associated with the Directed Percolation can be calculated with the action

$$\begin{aligned} \mathcal{S}_{DP}^{(DP)}[\varphi, \bar{\varphi}] = & \int d^d x dt \left\{ \bar{\varphi}(t, \vec{x}) (\partial_t - D\nabla^2 - (\sigma - \mu)) \varphi(t, \vec{x}) \right. \\ & \left. + \sqrt{2\sigma\lambda} (\bar{\varphi}(t, \vec{x}) \varphi^2(t, \vec{x}) - \bar{\varphi}^2(t, \vec{x}) \varphi(t, \vec{x})) + \lambda (\bar{\varphi}(t, \vec{x}) \varphi(t, \vec{x}))^2 \right\} \end{aligned} \quad (1.34)$$

Details can be found in e.g. [1]. Here the fields  $\varphi$  and  $\bar{\varphi}$  are *conjugated complex fields*. In contrast, in the J.D.M.S.R procedure these fields are independent,  $\varphi$  is real and  $\bar{\varphi}$  is purely imaginary. If some deformation of contour in the complex plane of the fields allowed us to ignore this point, one observe that the two actions (1.31) and (1.34) have a very similar structure. The only differences are that in the action (1.31) the quartic term is absent and that the value of the coupling is not fixed in terms of the reaction-diffusion rates. This is important for non-universal problems, but for universal quantities as critical exponents one can expect that the obtained values do not depend on the precise form of the action but only in its symmetries. So, in the calculation of universal properties, one can consider both actions as essentially equivalent.

### 1.3 Some examples of physical problems in equilibrium

It is simpler to present the Non-Perturbative Renormalization Group in the context of equilibrium statistical mechanics. As explained in the previous section, the generalization to out-of-equilibrium problems is relatively simple but in order to not superimpose difficulties, the equilibrium case will be presented first. More specifically, we will consider the Ginzburg-Landau model

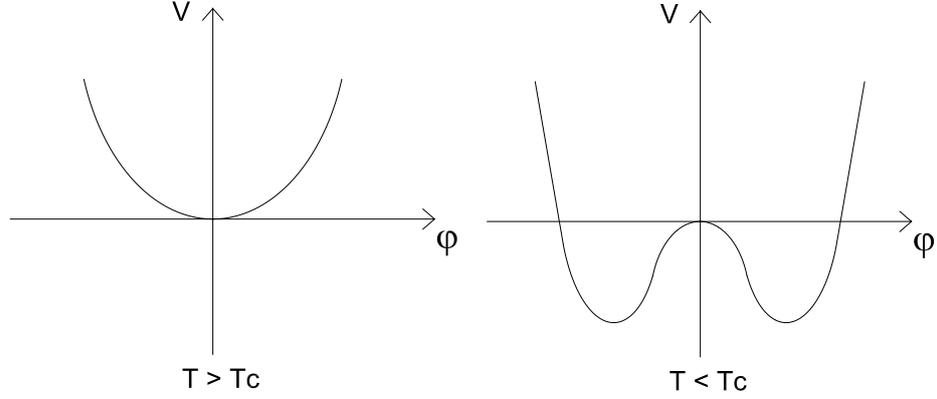


Figure 1.4: On the left: the potential for  $r_0 > 0 \Leftrightarrow T > T_c$ . On the right: the potential for  $r_0 < 0 \Leftrightarrow T < T_c$ .

for the uni-axial ferromagnetic-paramagnetic transition that has an effective Hamiltonian:

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} (\nabla \varphi(\vec{x}))^2 + \frac{r_0}{2} \varphi^2(\vec{x}) + \frac{u_0}{4!} \varphi^4(\vec{x}) \right\} \quad (1.35)$$

The potential of this model is presented in the figure 1.4. In mean-field it presents a second order phase transition at  $r_0 = 0$ .

The kind of objects that can be easily calculated for this model are similar to those that appear in the solution of reaction-diffusion equations or non linear Langevin equations:

- The mean value of the field  $\varphi$ :

$$\langle \varphi(\vec{x}) \rangle \quad (1.36)$$

represents the magnetization of the model. An important point is to know if the symmetry  $\varphi \rightarrow -\varphi$  is spontaneously broken or not in this model. If  $\langle \varphi(\vec{x}) \rangle = 0$  the system is the symmetric/paramagnetic phase and if  $\langle \varphi(\vec{x}) \rangle > 0$ , the system is in its broken/ferromagnetic phase and a spontaneous magnetization is present.

As in the out of equilibrium case, one can define the universal critical exponent  $\beta$  from it.

- The correlation function:

$$\langle \varphi(\vec{x}_1) \varphi(\vec{x}_2) \rangle \quad (1.37)$$

also plays an important role in the analysis of the system. Moreover, in the equilibrium situation, the correlation function is identical to the response function. Critical exponents can be defined in the same way as before, but there is no time dependence and so the  $z$  exponent makes no sense.

All the mean values of products of fields can be expressed in term of *functionals*. As a first example, consider the partition function of the Ginzburg-Landau model in presence of an external magnetic field (or source):

$$Z[J] = \int \mathcal{D}\varphi(\vec{x}) \exp\left(-\mathcal{H} + \int d^d x J(\vec{x})\varphi(\vec{x})\right) \quad (1.38)$$

$W[J] = \log Z[J]$  is (proportional to) the Helmholtz free-energy of this magnetic system. Here the notation  $\int \mathcal{D}\varphi(\vec{x})$  has been introduced. It represents a sum over all field configurations, as discussed before. From  $W[J]$  one can obtain the various correlation functions by taking derivatives of  $W$  with respect to  $J$ . For example, one can calculate the magnetization in presence of the external magnetic field:

$$\phi(\vec{x}) = \langle \varphi(\vec{x}) \rangle_J = \frac{\delta W}{\delta J(\vec{x})} \quad (1.39)$$

One can also calculate the 2-point correlation function:

$$\langle \varphi(\vec{x})\varphi(\vec{y}) \rangle = \frac{\delta^2 W}{\delta J(\vec{x})\delta J(\vec{y})} + \frac{\delta W}{\delta J(\vec{x})} \frac{\delta W}{\delta J(\vec{y})} \quad (1.40)$$

and so on.

It is sometimes convenient to express the free-energy as a function of the magnetization  $\phi(\vec{x})$  instead of  $J(\vec{x})$ . To do so, one performs a Legendre transform, obtaining the Gibbs free-energy  $\Gamma[\phi]$ :

$$\Gamma[\phi] = \int d^d x J(\vec{x})\phi(\vec{x}) - W[J] \quad (1.41)$$

In the previous expression  $J(\vec{x})$  is chosen so that the magnetization takes the value  $\phi(x)$ .

## 1.4 The Landau/mean-field approximation

Before considering sophisticated techniques in order to address non-perturbative issues in out-of-equilibrium systems, it is convenient to analyze the Ginzburg-Landau model in the mean-field approximation, and the perturbative corrections to it. This will help us to acquire an intuition of the meaning of the

mean-field approximation and its limitations. We will first calculate the Gibbs-free energy in the mean field approximation.

Consider the mean field or Landau approximation, which neglects fluctuations of the field around its mean value. In order to do so, one can suppose that in the partition function (1.38) the configuration with maximum probability dominates the sum:

$$Z[J] = \int \mathcal{D}\varphi(\vec{x}) \exp\left(-\mathcal{H}[\varphi] + \int d^d x J(\vec{x})\varphi(\vec{x})\right) \\ \stackrel{\text{Landau}}{\sim} \exp\left(-\mathcal{H}[\varphi_m] + \int d^d x J(\vec{x})\varphi_m(\vec{x})\right) \quad (1.42)$$

where  $\varphi_m(\vec{x})[J]$  minimizes the argument of the exponential:

$$\left. \frac{\delta\mathcal{H}[\varphi]}{\delta\varphi(\vec{x})} \right|_{\varphi \equiv \varphi_m[J]} = J(\vec{x}) \quad (1.43)$$

This gives the Helmholtz free-energy in the mean-field approximation:

$$W[J] = \log Z[J] \stackrel{\text{Landau}}{\sim} -\mathcal{H}[\varphi_m] + \int d^d x J(\vec{x})\varphi_m(\vec{x}) \quad (1.44)$$

The mean-field approximation expresses itself more transparently in terms of the Gibbs free-energy. In order to calculate it, it is necessary to first calculate the magnetization (mean value of  $\varphi$ ) in this approximation:

$$\phi(\vec{x}) = \frac{\delta W[J]}{\delta J(\vec{x})} \stackrel{\text{Landau}}{\sim} - \int d^d y \frac{\delta\mathcal{H}[\varphi_m]}{\delta\varphi_m(\vec{x})} \frac{\delta\varphi_m[J]}{\delta J(\vec{x})} + \varphi_m(\vec{x}) + \int d^d y J(\vec{y}) \frac{\delta\varphi_m[J]}{\delta J(\vec{x})} \\ \stackrel{\text{Landau}}{\sim} \varphi_m(\vec{x}) \quad (1.45)$$

This is reasonable: if only one configuration contributes to the sum, it is to be expected that the mean value of the field is the value of this particular configuration. The Gibbs free-energy can be deduced directly:

$$\Gamma[\phi] = \int d^d x J(\vec{x})\phi(\vec{x}) - W[J] \stackrel{\text{Landau}}{\sim} \mathcal{H}[\phi] \quad (1.46)$$

In words: the mean-field or Landau approximation corresponds to replace the Gibbs free-energy functional for the effective Hamiltonian.

After deriving this general expression, it is convenient to apply it in some simple particular cases. For example one can evaluate the Gibbs free-energy in a constant background magnetization  $\phi(\vec{x}) \equiv \phi$ . In that case

$$\Gamma[\phi] = \mathcal{V}V(\phi) \quad (1.47)$$

where  $\mathcal{V}$  is the space volume and  $\overline{V(\phi)}$  is called the *effective potential*. In the mean field approximation, the effective potential is identical to the potential appearing in the effective Hamiltonian.

From the effective potential one can extract a lot of physical information. For example, taking the first derivative one can extract the relation between the magnetization  $\phi$  and the external magnetic field  $J$ :

$$J = \frac{\partial V}{\partial \phi} = r_0 \phi + \frac{u_0}{3!} \phi^3 \quad (1.48)$$

In particular, when  $J \rightarrow 0$  one can analyze the existence of a spontaneous magnetization. If  $r_0 > 0$ ,  $\phi \rightarrow 0$  when  $J \rightarrow 0$ . On the other hand, when  $r < 0$ ,  $\phi \rightarrow \pm \sqrt{-6r_0/u_0}$  (depending on the sign of  $J$ ). The system presents a second order phase transition at  $r_0 = 0$ . It is reasonable to expect that parameters appearing in the Ginzburg-Landau effective Hamiltonian ( $r_0$  and  $u_0$ ) depend smoothly on the external control parameters as the temperature. One then sees that in mean field approximation,  $r_0(T) \sim \bar{r}_0(T - T_0)$  where  $T_0$  is the mean-field critical temperature. In particular, one realizes that the mean-field  $\beta$  critical exponent is  $1/2$ .

One can also extract from the effective potential the expression for the susceptibility  $\chi$  at zero external magnetic field (supposing that we are in the symmetric phase, then  $\phi = 0$ ):

$$\chi^{-1} = \left. \frac{\partial J}{\partial \phi} \right|_{J=0} = r_0 \quad (1.49)$$

The second derivative of the effective potential is nothing but the inverse of the susceptibility. From this expression one sees that the  $\gamma$  critical exponent (given by  $\chi \sim |T - T_c|^{-\gamma}$ ) is 1 in the mean-field approximation.

From the mean-field expression of the Gibbs free-energy, one can also extract correlation functions (that for equilibrium problems are proportional to response functions). For example, the two-point correlation function can be obtained by

$$G(\vec{x}, \vec{y}) = \langle \varphi(\vec{x}) \varphi(\vec{y}) \rangle = \frac{\delta^2 W[J]}{\delta J(\vec{x}) \delta J(\vec{y})} + \frac{\delta W[J]}{\delta J(\vec{x})} \frac{\delta W[J]}{\delta J(\vec{y})} \quad (1.50)$$

It is convenient to define the *connected* 2-point correlation function by subtracting the product of the expectation values of the fields:

$$G_c(\vec{x}, \vec{y}) = \langle \varphi(\vec{x}) \varphi(\vec{y}) \rangle - \langle \varphi(\vec{x}) \rangle \langle \varphi(\vec{y}) \rangle = \frac{\delta^2 W[J]}{\delta J(\vec{x}) \delta J(\vec{y})} \quad (1.51)$$

Now, observe that in that case

$$G_c(\vec{x}, \vec{y}) = \frac{\delta\phi(\vec{x})}{\delta J(\vec{y})} \quad (1.52)$$

(as mentioned before, in this system the correlation function and the response function are the same). But

$$\delta^{(d)}(\vec{x} - \vec{y}) = \frac{\delta J(\vec{x})}{\delta J(\vec{y})} = \int d^d z \frac{\delta J(\vec{x})}{\delta\phi(\vec{z})} \frac{\delta\phi(\vec{z})}{\delta J(\vec{y})} = \int d^d z \frac{\delta^2\Gamma[\phi]}{\delta\phi(\vec{x})\delta\phi(\vec{z})} G_c(\vec{z}, \vec{y}) \quad (1.53)$$

One concludes that the 2-point function is the *inverse* in the operatorial sense of the second derivative of the Gibbs free-energy.

The 'matrix' of second derivatives of the Gibbs-energy in a constant background field is

$$\Gamma^{(2)}(\vec{x}, \vec{y}) \stackrel{Landau}{\sim} \mathcal{H}^{(2)}(\vec{x}, \vec{y}) = (-\nabla_x^2 + r_0 + \frac{u_0}{2}\phi^2)\delta^{(d)}(\vec{x} - \vec{y}) \quad (1.54)$$

It can be 'diagonalized' by Fourier-transforming, giving the eigenvalues  $q^2 + r_0 + \frac{u_0}{2}\phi^2$ . One concludes that in Landau approximation, and for  $\phi = 0$ ,

$$G_c(\vec{x}, \vec{y}) \stackrel{Landau}{\sim} \int \frac{d^d q}{(2\pi)^d} \frac{e^{i\vec{q}\cdot(\vec{x}-\vec{y})}}{q^2 + r_0} \quad (1.55)$$

By performing the Fourier transform, one deduces (see exercise 2) that for  $T > T_0$ ,  $\xi \sim \sqrt{r_0}$  in mean-field approximation, corresponding to  $\nu = 1/2$ . Moreover, one deduces that the anomalous dimension is  $\eta \sim 0$  in the mean-field approximation.

## 1.5 The Ginzburg-Landau model at one loop

We are particularly interested in cases where the mean-field approximation is not a good approximation. In order to know in which situations the mean-field is a good approximation and in which one it is not, the simplest procedure is to apply the Ginzburg criterion: one calculates the first *corrections* to the mean-field approximation and compares the correction to the supposed dominant term. Only if the corrections are small compared to the supposed dominant term, one can reasonably trust the mean-field expression. So, let's calculate the first correction to the Landau approximation. This can be done by separating in the sum (1.38) the field  $\varphi$  in  $\varphi_m$  and a (supposed 'small') fluctuation  $\tilde{\varphi} = \varphi - \varphi_m$ :

$$Z[J] = \int \mathcal{D}\tilde{\varphi}(\vec{x}) \exp\left(-\mathcal{H}[\varphi_m + \tilde{\varphi}] + \int d^d x J(\vec{x})(\varphi_m(\vec{x}) + \tilde{\varphi}(\vec{x}))\right) \quad (1.56)$$

Given the fact that we are supposing that the fluctuations are small, one can expand the effective Hamiltonian in  $\tilde{\varphi}$ :

$$\begin{aligned} \mathcal{H}[\varphi_m + \tilde{\varphi}] &= \mathcal{H}[\varphi_m] + \int d^d x \left. \frac{\delta \mathcal{H}[\varphi]}{\delta \varphi(\vec{x})} \right|_{\varphi \equiv \varphi_m[J]} \tilde{\varphi}(\vec{x}) \\ &+ \frac{1}{2} \int d^d x \int d^d y \left. \frac{\delta^2 \mathcal{H}[\varphi]}{\delta \varphi(\vec{x}) \delta \varphi(\vec{y})} \right|_{\varphi \equiv \varphi_m[J]} \tilde{\varphi}(\vec{x}) \tilde{\varphi}(\vec{y}) + \dots \end{aligned} \quad (1.57)$$

Replacing in the expression for the partition function:

$$\begin{aligned} Z[J] &\stackrel{1loop}{\sim} \exp \left( -\mathcal{H}[\varphi_m] + \int d^d x J(\vec{x}) \varphi_m(\vec{x}) \right) \\ &\times \int \mathcal{D}\tilde{\varphi}(\vec{x}) \exp \left( -\frac{1}{2} \int d^d x \int d^d y \left. \frac{\delta^2 \mathcal{H}[\varphi]}{\delta \varphi(\vec{x}) \delta \varphi(\vec{y})} \right|_{\varphi \equiv \varphi_m[J]} \tilde{\varphi}(\vec{x}) \tilde{\varphi}(\vec{y}) \right) \end{aligned} \quad (1.58)$$

The remaining integral is a *Gaussian* integral. Gaussian integrals can be calculated analytically, giving the expression (see exercise 1):

$$\int \prod_{k=1}^N (dx_k) \exp \left( -\frac{1}{2} \sum_{ij} x_i a_{ij} x_j \right) = (2\pi)^{N/2} (\det(A))^{-1/2} \quad (1.59)$$

where the  $a_{ij}$  are the components of the matrix  $A$  that is supposed to be positive definite. The expression  $\left. \frac{\delta^2 \mathcal{H}[\varphi]}{\delta \varphi(\vec{x}) \delta \varphi(\vec{y})} \right|_{\varphi \equiv \varphi_m[J]}$  can be seen as a big matrix. It is important to recall that one can always imagine that the system has been put on a periodic lattice and that the indices  $\vec{x}$  and  $\vec{y}$  take their values in a finite set of values. In that case, the integrals are just discrete sums. The determinant can be written in a convenient way by exploiting the expression (see exercise 3)

$$\log \det(A) = \text{Tr} \log(A) \quad (1.60)$$

where the trace of the logarithm of the matrix is nothing but the sum of eigenvalues of the matrix  $A$ . One arrives then to the Helmholtz free-energy with first correction to the Landau approximation ('one-loop approximation'):

$$W[J] \stackrel{1loop}{\sim} -H[\varphi_m] + \int d^d x J(\vec{x}) \varphi_m(\vec{x}) - \frac{1}{2} \text{Tr} \log \left. \frac{\delta^2 \mathcal{H}[\varphi]}{\delta \varphi(\vec{x}) \delta \varphi(\vec{y})} \right|_{\varphi \equiv \varphi_m[J]} \quad (1.61)$$

Please note that an unimportant additive constant (coming from the  $(2\pi)^{N/2}$ ) has been removed in order to obtain a friendlier expression.

As before, the result is much simpler if it is expressed in terms of the Gibbs free-energy. In order to obtain it, one must keep in mind that including these corrections, the mean value of  $\varphi$  is no longer  $\varphi_m$ . The correct mean value will be  $\phi = \varphi_m + \Delta\phi$ , where the correction  $\Delta\phi$  is supposed to be of 'one-loop' order. Performing the Legendre transform of the one-loop expression for  $W[J]$  gives:

$$\Gamma_{1loop}[\phi] = \int d^d x J(\vec{x}) \Delta\phi(\vec{x}) + \mathcal{H}[\varphi_m] + \frac{1}{2} \text{Tr} \log \left. \frac{\delta^2 \mathcal{H}[\varphi]}{\delta\varphi(\vec{x}) \delta\varphi(\vec{y})} \right|_{\varphi \equiv \varphi_m[J]} \quad (1.62)$$

Now,

$$\mathcal{H}[\phi] = \mathcal{H}[\varphi_m] + \int d^d x \left. \frac{\delta \mathcal{H}[\varphi]}{\delta\varphi(\vec{x})} \right|_{\varphi \equiv \varphi_m[J]} \Delta\phi(\vec{x}) + \dots \quad (1.63)$$

where the dots are of 'two-loop order'. One concludes that at one-loop order,

$$\Gamma_{1loop}[\phi] = \mathcal{H}[\phi] + \frac{1}{2} \text{Tr} \log \frac{\delta^2 \mathcal{H}[\phi]}{\delta\phi(\vec{x}) \delta\phi(\vec{y})} \quad (1.64)$$

where in the last expression  $\varphi_m$  has been replaced by  $\phi$  because, by doing so, the error introduced in those terms is of two-loop order.

From this expression, one can extract, for example, the behavior of the effective potential at one-loop order. Evaluating, in a constant background field, and exploiting the form of the second derivative of the effective Hamiltonian obtained in the previous section, one concludes that the effective potential takes the form at one-loop,

$$V(\phi) = \frac{r_0}{2} \phi^2 + \frac{u_0}{4!} \phi^4 + \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \log(q^2 + r_0 + \frac{u_0}{2} \phi^2) \quad (1.65)$$

In absence of some sort of cut-off this integral is divergent because high momenta contributions are not bounded (*ultraviolet divergence*). As it was pointed out before, one can always imagine to be considering the discretized version of the model where there is a natural momentum cut-off of the order of  $\Lambda = 1/a$  where  $a$  is the lattice space. We will suppose that such a regulator exists even if not specifying it explicitly for notation simplicity. Moreover, we will show now that in some cases the system can have low momentum divergences, called *infrared divergences*. These divergences are much more difficult to treat and they will be at the heart of our concerns here.

At this order, the expression for the relation between the magnetization and the external magnetic field becomes:

$$J = \frac{\partial V}{\partial \phi} = r_0 \phi + \frac{u_0}{3!} \phi^3 + \frac{u_0}{2} \phi \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + r_0 + \frac{u_0}{2} \phi^2} \quad (1.66)$$

and the susceptibility,

$$r \equiv \chi^{-1} = \left. \frac{\partial J}{\partial \phi} \right|_{J=0} = r_0 + \frac{u_0}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + r_0} \quad (1.67)$$

One observes that the terms that came from the one-loop correction to the mean field approximation are suppressed by a factor of the coupling constant  $u_0$ . This property is true also at higher orders of the expansion around the mean field approximation: each new term includes higher powers of  $u_0$ . The second order phase transition takes place at the temperature where the magnetic susceptibility diverges. Then  $r(T = T_c) = \chi^{-1}(T = T_c) = 0$ . At first sight this is problematic, because at that temperature  $r_0$  must be negative in order to compensate the second term of the r.h.s. expression, but then the integrand has a pole and is not well defined. This is an artifact of the approximation, because we can safely replace in the second term of the expression  $r_0$  by  $r$  because the correction would be of order  $u_0^2$ :

$$r = \left. \frac{\partial J}{\partial \phi} \right|_{J=0} = r_0 + \frac{u_0}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + r} \quad (1.68)$$

Evaluating then at  $T = T_c$ :

$$0 = r_0(T = T_c) + \frac{u_0}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2} \quad (1.69)$$

Performing the subtraction of these two expressions one arrives to

$$r = r_0(T) - r_0(T = T_c) - \frac{u_0}{2} r \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2(q^2 + r)} \quad (1.70)$$

Two different cases arise:

- If  $d > 4$  the integral is dominated by the ultraviolet range of wave-numbers but is regular when  $T \sim T_c$ . If the integral is cut-off at  $q \sim \Lambda$ , one has  $T = T_c$ ,

$$r = r_0(T) - r_0(T = T_c) - C u_0 r \Lambda^{d-4} \quad (1.71)$$

where  $C$  is a numerical constant. In this case, the correction to the mean-field expression is regular and is small as long as  $u_0 \Lambda^{d-4} \ll 1$ . The critical exponents (as  $\gamma$ ) are the same as their mean-field approximation, at least in a finite neighborhood of  $u_0 = 0$ .

- If  $d < 4$ , the ultraviolet range of wave-numbers is sub-leading (one can safely take the limit  $\Lambda \rightarrow \infty$ ), but the integral is dominated (and divergent) when  $T \rightarrow T_c$  (or, equivalently  $r \rightarrow 0$ ). In that case,

$$r = r_0 - r_0(T = T_c) - rC'u_0r^{(d-4)/2} \quad (1.72)$$

where  $C'$  is another numerical constant. It is clear that the mean field approximation can only be a good approximation as long as  $u_0r^{(d-4)/2} \ll 1$ . In particular, the critical regime  $r \sim 0$  can not be approached by a direct expansion around the mean-field.

Before considering a strategy to avoid this difficulty of the perturbative expansion near a second order phase transition, it is important to recognize the origin of this difficulty. The reason is that all wave-numbers contribute significantly to the loop corrections to the mean field if the system approaches a critical regime. It is difficult to control an approximation when many degrees of freedom with very different typical wave-number scales interact in a significant way. All momentum from the microscopic  $\Lambda^{-1}$  scale to the macroscopic  $r^{-1/2} = \xi$  scale contributes more or less in the same way in the one-loop correction to mean-field approximations. The problem is, of course, magnified when the microscopic coupling  $u_0$  is 'large'. However, even at 'small' microscopic couplings many systems show large correlations between many degrees of freedom that require more sophisticated techniques than mean-field or perturbative approaches.

## 1.6 The Non-Perturbative Renormalization Group equation

The NPRG is the modern version of the Wilsonian Renormalization Group [13, 14, 15, 16, 17, 18]. It is conceptually very similar to it but it is technically much simpler and efficient [19, 20, 21, 22, 23, 24]. The main idea of these approaches is to treat each shell of wave-numbers in one step. That is, the Renormalization Group (RG) abandons the idea of trying to formulate approximation schemes, that are valid from microscopic wave-number scales as the inverse lattice spacing in a ferromagnetic system, to the macroscopic sizes of the crystal. Instead, the RG is based on the construction of a family of effective theories with a variable scale and in each step the RG employs an effective theory for a given scale in order to construct the effective theory of the next scale. Of course, in the rare cases of interacting systems where exact solutions are available, the RG is normally useless. The RG purpose is

to give a good background for the formulation of precise and robust approximations. This is why in these notes the expression 'Exact Renormalization Group' is employed only once, even if it is an expression frequently used in the literature. Another expression employed for the same techniques is 'Functional Renormalization Group'.

We will present an heuristic introduction to the ideas of the RG and more particularly of the NPRG, without proving the main equation of the NPRG. The reader who wants a detailed proof is encouraged to read the appendix. The lecture notes of B. Delamotte [1] and the review of J. Berges, N. Tetradis and C. Wetterich [25] and references therein are certainly a good complement.

The goal of Wilsonian RG and of NPRG is to avoid expressions like (1.68) where modes with all scales of wave-numbers contribute at the same time. In order to do so, the NPRG strategy includes two main points:

- The first one is to introduce a 'regulator' that filters modes with low wave-numbers from the sums contributing to the free-energy. Only modes with wave numbers higher than a certain parameter  $k$  will be included in the corresponding Gibbs free-energy what will be noted  $\Gamma_k[\phi]$ . When  $k \rightarrow 0$ , all modes are included and  $\Gamma_k[\phi]$  becomes the standard non-regulated Gibbs free-energy. On the other hand, when  $k$  becomes of the order of the microscopic scale  $\Lambda$ , no modes at all are included and Landau approximation becomes essentially exact  $\Gamma_k[\phi] \sim \mathcal{H}[\phi]$ .
- The second one, is to obtain an equation giving the dependence of  $\Gamma_k$  with  $k$  that is more or less *local* in wave-numbers. One will look for an equation for  $\partial_k \Gamma_k$  where integrals of the type (1.68) appear but only modes with wave-numbers in a small shell around  $k$  contribute. The expressions will be finite in the ultraviolet and in the infrared and the construction of approximations will be much easier. The equation must be local in another sense: we will look for an equation where  $\partial_k \Gamma_k$  can be calculated in terms of  $\Gamma_k$  but do not require the knowledge of  $\Gamma_{k'}$  for other values of  $k'$ .

There are other points to be added to these ones from the practical point of view. One needs reasonably simple equations in order to be able to perform concrete calculations. In particular, we want to exploit all the information we know in an exact way (as the presence of symmetries) in order to simplify calculations and motivate better approximations. So we will try, whenever it is possible, to have a simple equation that respects all the symmetries of the model.

The simplest way to filter the low wave-numbers modes from the contributions to a partition function is to add a quadratic term to the effective Hamiltonian. One will require of this quadratic term to not affect the modes with wave-numbers higher than  $k$  and to bring the correlation length of modes with wave-length smaller than  $k$  to a value of the order of  $k^{-1}$ . The corresponding extra term in the Hamiltonian will look like

$$\Delta\mathcal{H}_k[\varphi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \varphi(-\vec{q}) R_k(q^2) \varphi(\vec{q}) \quad (1.73)$$

In order to satisfy the properties mentioned before, we will require that

$$R_k(q^2) \sim \begin{cases} Z_k k^2 & \text{if } q \lesssim k \\ 0 \text{ rapidly} & \text{if } q \gtrsim k \end{cases} \quad (1.74)$$

The factor  $Z_k$  varies slowly with  $k$  and will be explained in detail below. Moreover it is necessary that  $R_k(q^2)$  tends to infinity if  $k \rightarrow \infty$  at fixed  $q$ .

From the Hamiltonian  $\mathcal{H} + \Delta\mathcal{H}_k$  a regulated Helmholtz free-energy  $W_k[J]$  can be calculated. From it one can perform a (modified) Legendre transform in order to obtain a regulated Gibbs free-energy:

$$\Gamma_k[\phi] = \int d^d x J(\vec{x}) \phi(\vec{x}) - W_k[J] - \Delta\mathcal{H}_k[\phi] \quad (1.75)$$

The subtraction of the last term is essential so that  $\Gamma_k$  tend to  $\mathcal{H}[\phi]$  when  $k \rightarrow \infty$ . In that limit the Landau approximation is well justified because all modes have a very short correlation length. According to the Ginzburg criterion of validity of the Landau approximation, the Landau approximation becomes asymptotically exact. In this limit, the standard Legendre transform of  $W_k[J]$  goes to  $\mathcal{H} + \Delta\mathcal{H}_k$ . Given the fact that we want a definition of  $\Gamma_k$  that goes to  $\mathcal{H}$  in that limit, one must subtract the last term.

It is shown in the Appendix that this definition of  $\Gamma_k$  gives an exact RG equation satisfying all the properties required before. Here, we will arrive to it by an heuristic argument based in a one-loop analysis. In order to do so, let's suppose that in a certain regime, the one-loop approximation for the regulated Gibbs free-energy is well justified. One deduces that

$$\Gamma_k[\phi] = \mathcal{H}[\phi] + \frac{1}{2} \text{Tr} \log \left( \mathcal{H}^{(2)}[\phi] + R_k \right) \quad (1.76)$$

where  $\mathcal{H}^{(2)}$  is the matrix of second derivatives of  $\mathcal{H}$ .

Now, one can calculate from it the derivative with respect to  $k$ :

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left( \partial_k R_k (\mathcal{H}^{(2)} + R_k)^{-1} [\phi] \right) \quad (1.77)$$

This expression has good news and bad news. The good news is that thanks to the regulator  $R_k$ , the equation is dominated by a small shell of momenta  $q \sim k$ . The reason for that is, first, that  $\partial_k R_k(q^2)$  goes to zero very rapidly when  $q \gg k$  and, second, the term  $R_k$  added to  $\mathcal{H}^{(2)}$  ensures that the contributions coming from  $q \ll k$  are suppressed. The bad news is that one observes that in (1.77), the derivative of  $\Gamma_k$  with respect to  $k$  is not expressed here in terms of  $\Gamma_k$  but in terms of  $\mathcal{H} \sim \lim_{k \rightarrow \infty} \Gamma_k$ . This is not in the spirit of the NPRG as expressed before where, in order to be able to construct robust approximations, one wants to analyze a single scale at each time.

Fortunately, at one-loop order, one can avoid the bad news, by replacing  $\mathcal{H}$  by  $\Gamma_k$  in the r.h.s. of (1.77). In fact the difference would induce a correction of 2-loop orders. The corresponding equation

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left( \partial_k R_k (\Gamma_k^{(2)} + R_k)^{-1} [\phi] \right) \quad (1.78)$$

has all the good properties we were looking for and is called the Wetterich equation [22]. Now, magic happens: this equation turns out to be *exact!* Even if we have motivated the equation by looking for an equation with very convenient properties and a one-loop argument, one can prove that it is an exact expression. The corresponding proof can be found in the Appendix.

In the next chapter, this equation is put at work. We will show that a very simple approximation scheme is able, at leading order, to reproduce almost all the critical properties of Ginzburg-Landau models in *any dimension* with reasonable precision. We will show, moreover, that going beyond the leading order one can systematically improve the precision of the predictions and results seem to converge to the well-established results in any case where they exist in any dimension. Even more, the same approximations allow us to treat notoriously non-perturbative problems as the convexity properties of the effective potential in the broken phase.

In the last chapter, these techniques are generalized for nonequilibrium systems. It will be shown that the NPRG is also able to analyze successfully that type of systems. In particular, universal and non-universal properties of some models in the Directed Percolation universality class will be calculated quantitatively. In some cases these results go beyond those obtained *at any order* of perturbation theory.

## 1.7 Appendix: Proof of the NPRG equation

Here we prove the NPRG equation without any approximation. Consider the expression for the regulated Helmholtz free-energy:

$$\exp(W_k[J]) = \int \mathcal{D}\varphi(\vec{x}) \exp\left(-\mathcal{H} - \Delta\mathcal{H}_k + \int d^d x J(\vec{x})\varphi(\vec{x})\right) \quad (1.79)$$

One can take the derivative with respect to  $k$ , obtaining:

$$\partial_k W_k[J] = -\langle \partial_k \Delta\mathcal{H}_k \rangle_J = -\frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \partial_k R_k(q^2) \left\{ \frac{\delta^2 W_k}{\delta J(\vec{q}) \delta J(-\vec{q})} + \frac{\delta W_k}{\delta J(\vec{q})} \frac{\delta W_k}{\delta J(-\vec{q})} \right\} \quad (1.80)$$

Now, consider  $\tilde{\Gamma}_k[\phi]$  the standard Legendre transform of  $W_k[J]$ , and take the derivative with respect to  $k$  at fixed  $\phi$  on it:

$$\begin{aligned} \partial_k \tilde{\Gamma}_k[\phi]|_\phi &= \int d^d x \partial_k J(\vec{x})|_\phi \phi(\vec{x}) - \int d^d x \frac{\delta W_k[J]}{\delta J(\vec{x})} \partial_k J(\vec{x})|_\phi - \partial_k W_k[J]|_J \\ &= -\partial_k W_k[J]|_J \end{aligned} \quad (1.81)$$

Now, it has been shown in (1.53) that the 'matrices'  $W_k^{(2)}$  and  $\tilde{\Gamma}_k^{(2)} = \Gamma_k^{(2)} + R_k$  are inverses, one deduces that

$$\partial_k \tilde{\Gamma}_k[\phi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \partial_k R_k(q^2) \left\{ \left( \Gamma_k^{(2)} + R_k \right)_{\vec{q}, -\vec{q}}^{-1} + \phi(q)\phi(-q) \right\} \quad (1.82)$$

Subtracting  $\Delta\mathcal{H}_k$  from  $\tilde{\Gamma}_k$  one deduces finally,

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \partial_k R_k(q^2) \left( \Gamma_k^{(2)} + R_k \right)_{\vec{q}, -\vec{q}}^{-1} \quad (1.83)$$

## 1.8 Exercises

1) Prove that

$$\begin{aligned} &\int \prod_{k=1}^N (dx_k) \exp\left(-\frac{1}{2} \sum_{ij} x_i A_{ij} x_j + \sum_i x_i j_i\right) \\ &= (2\pi)^{N/2} (\det(A))^{-1/2} \exp\left(\frac{1}{2} \sum_{ij} j_i (A^{-1})_{ij} j_j\right) \end{aligned} \quad (1.84)$$

Deduce that

$$\langle x_l \rangle = \frac{\int \prod_{k=1}^N (dx_k) \exp\left(-\frac{1}{2} \sum_{ij} x_i A_{ij} x_j\right) x_l}{\int \prod_{k=1}^N (dx_k) \exp\left(-\frac{1}{2} \sum_{ij} x_i A_{ij} x_j\right)} = 0 \quad (1.85)$$

and

$$\langle x_l x_m \rangle = \frac{\int \prod_{k=1}^N (dx_k) \exp(-\frac{1}{2} \sum_{ij} x_i A_{ij} x_j) x_l x_m}{\int \prod_{k=1}^N (dx_k) \exp(-\frac{1}{2} \sum_{ij} x_i A_{ij} x_j)} = (A^{-1})_{lm} \quad (1.86)$$

- 2) Consider the Ginzburg-Landau model with  $N$  scalar fields  $\varphi_i$  with  $O(N)$  symmetry:

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} \sum_i \nabla \varphi_i(\vec{x}) \cdot \nabla \varphi_i(\vec{x}) + \frac{r_0}{2} \sum_i \varphi_i^2(\vec{x}) + \frac{u_0}{4!} \left( \sum_i \varphi_i^2(\vec{x}) \right)^2 \right\} \quad (1.87)$$

Generalize the expressions for the Gibbs free-energy in the Landau and in the one-loop order for this model. Deduce the expression for the 2-point function in this model in the symmetric phase without external magnetic field in mean-field approximation. Extract from it the critical exponents  $\eta$  and  $\nu$ .

- 3) Show that for a positive definite symmetric matrix  $A$ ,

$$\log \det(A) = \text{Tr} \log(A) \quad (1.88)$$

- 4) Consider the action (1.34) for the reaction-diffusion process (1.1). Deduce from it the mean-field exponents  $\eta$ ,  $\nu_\perp$  and  $z$  in that model.

