Frustrated Quantum Magnets

From semi-classical to purely quantum phases

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2-dimens. Spin-1/2 Antiferromagnets

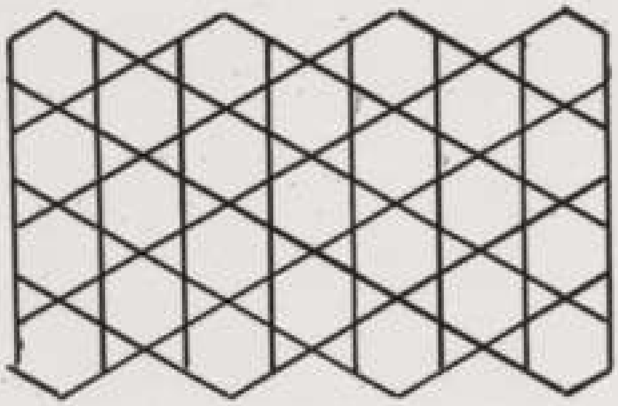
- **Néel states**
  - T=0 spin-spin LRO, SU(2) symmetry breaking, excitations: gapless magnons
- **Valence Bond Crystals**
  - T=0 dimer-dimer long range order, No SU(2) symmetry breaking, gapped excitations
- **Resonating Valence Bond Spin Liquids**
  - No long range order in any local order parameter, No SU(2) symmetry breaking (different types !)

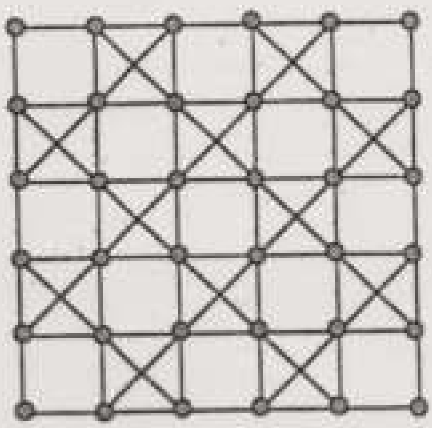
Lecture 1: Semi-classical Néel order versus Quantum dimerization

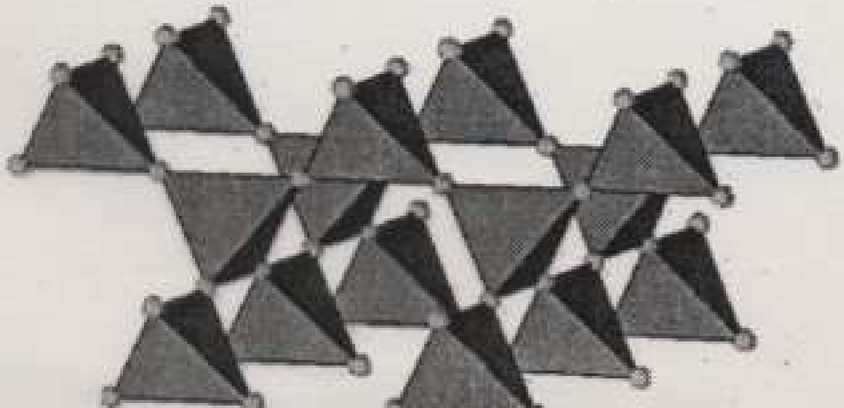
- History of frustration for Ising spins
  - degeneracy of the g.-s.
  - order by disorder (Villain 80)
- Frustrated Heisenberg spins
  - Non colinear order
    - Magnetization plateaux (lecture 3)
    - Quantum Spin Liquids (lecture 2)
  - Selection of colinear Néel order by quantum fluctuations (Shender 82)
  - Quantum alternative: formation of short range singlets
- Semi-classical Néel ordered g.-states
  - Degeneracy of the g.-s. in therm. limit, Anderson tower of states
  - Gapless excitations: magnons,
    - another point of view on order by disorder,
    - reduction of the order parameter by quantum fluctuations

Lecture 3: Spin models with infinite local degeneracy in the classical limit. Integer spins versus half integer ones

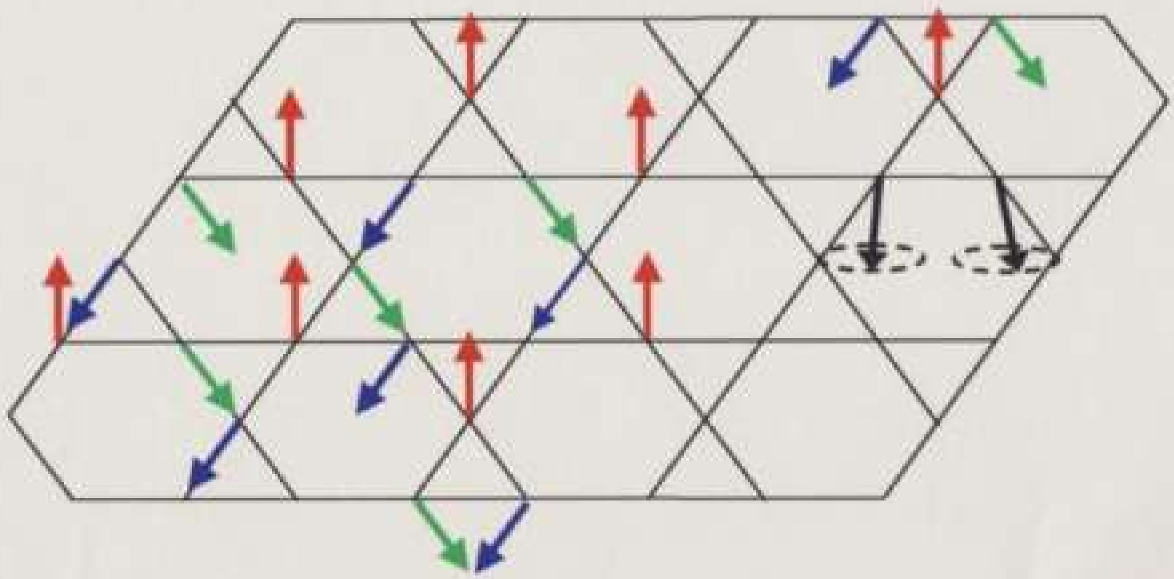
- Classical Heisenberg model on the kagomé, checkerboard and pyrochlore lattices
  - Infinite local degeneracy
  - Extensive entropy
  - Thermal fluctuations
- Quantum spins
  - Checkerboard Spin ½ : VBC
  - Kagomé Spin 1 VBS
  - Kagomé Spin ½ ?
- Lieb, Schultz, Mattis theorem
- 2D conjecture:
  - Haldane, Affleck, Oshikawa, Misguich & CL, Hastings….







Heisenberg Hamiltonian on the kagomé lattice: a semi-classical picture



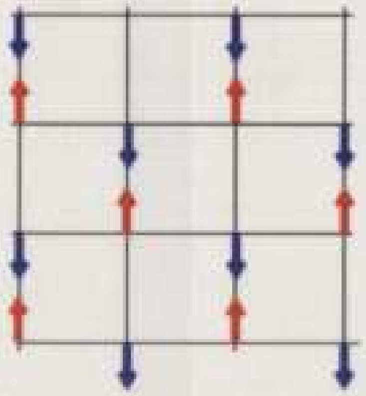
An infinite number of soft modes, an infinite T=0 degeneracy

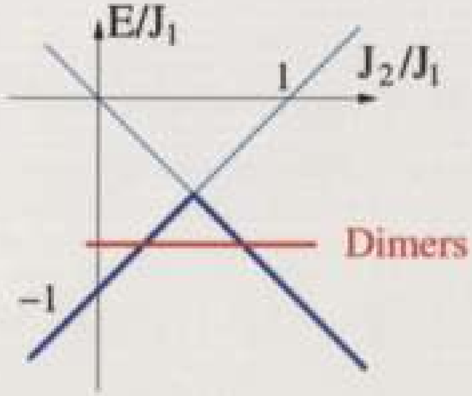
Néel order versus dimer Pairing

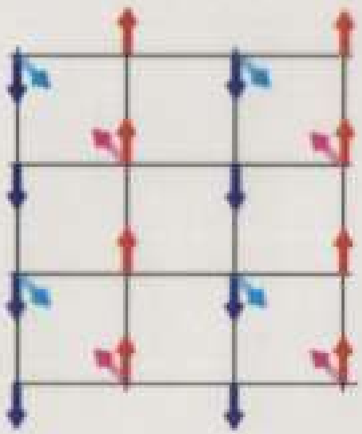
Role of competing interactions

$$\mathcal{H} = 2\,J_1 \sum_{<ij>} \mathbf{S}_i.\mathbf{S}_j + 2\,J_2 \sum_{<<ij>>} \mathbf{S}_i.\mathbf{S}_j$$

$$\begin{aligned}\mathcal{H} \;&=\; 2\,(J_1+J_2)\,\mathbf{S}^2_{tot}/N\,+\\ &+\;2\sum_{k\neq(0,0)} \mathbf{S}_k.\mathbf{S}_{-k}\,\{J_1(cos(k_x)+cos(k_y))+J_2(cos(k_x+k_y)+cos(k_x-k_y))\}\end{aligned}$$



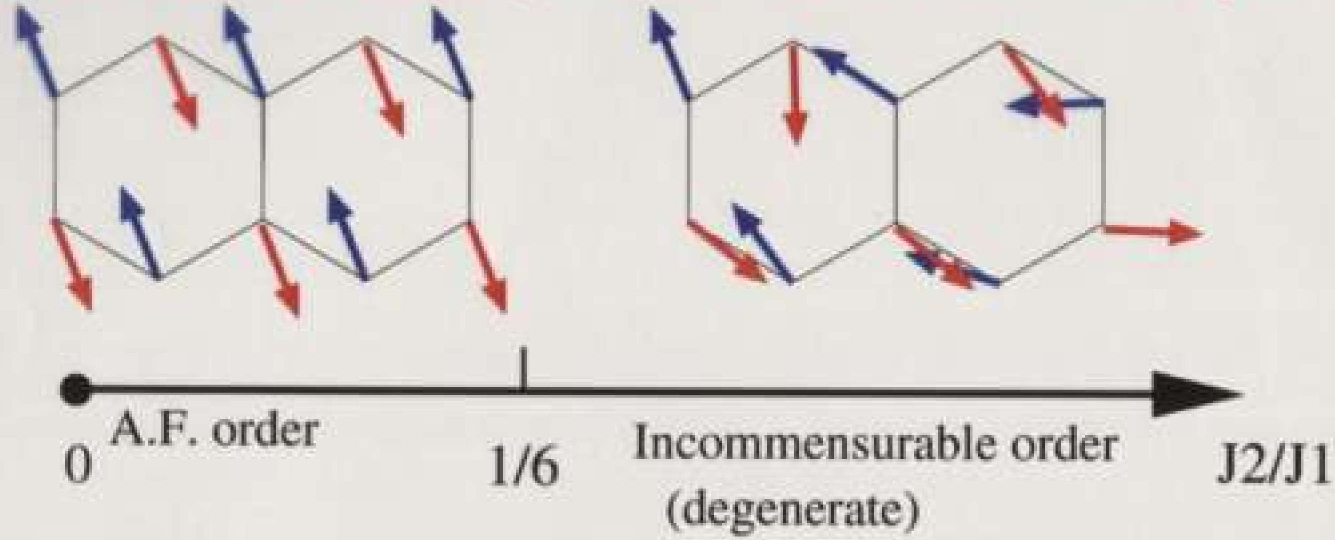




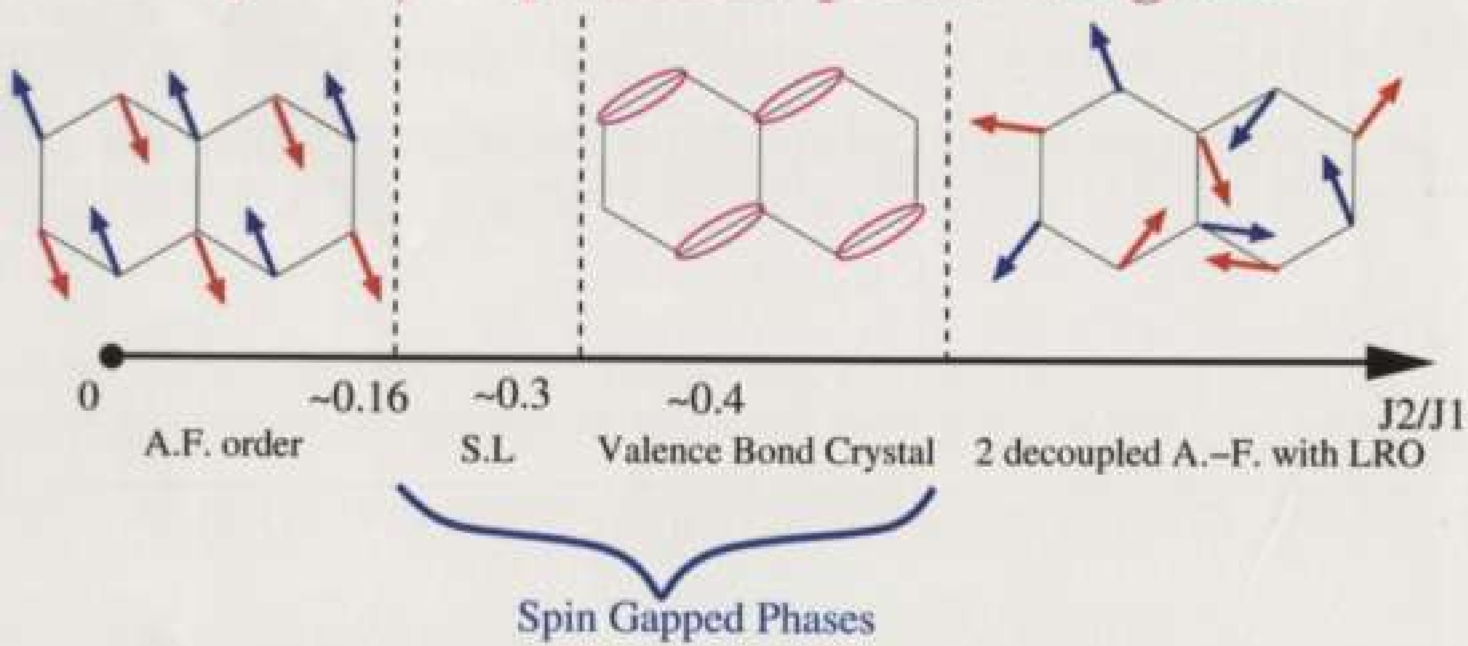
Selection of order by disorder

Partial Restoration of symmetry by fluct.

$J_1 - J_2$ model on honeycomb lat.: classical phase diag.



Spin-1/2 quantum phase diagram



Fouet et. al.: Eur.Phys. J. B. 2001

Symm. breaking, order param. and susceptibility

$$\mathcal{A} = \sum_{j=1}^{N} e^{i\mathbf{K}_1 \cdot \mathbf{r}_j} O(\mathbf{r}_j) \qquad (1)$$

Order parameter $\mathcal{P}$:

$$\mathcal{P}^2 = <\psi_0|\mathcal{A}^\dagger \mathcal{A}|\psi_0> /N^2 \qquad (2)$$

Linear response theory:

$$H_\delta = H_0 - (\delta \mathcal{A} + h.c.) \,. \qquad (3)$$

T=0, intensive susceptibility:

$$\chi = \frac{2}{N} <\psi_0|\mathcal{A}^\dagger \frac{1}{H_0 - E_0} \mathcal{A}|\psi_0> \quad (4)$$

$$\frac{4\mathcal{P}^4 N^2}{f} < \chi \qquad (5)$$

$$f = \frac{1}{N} \langle\psi_0| \left[\mathcal{A}, [H_0, \mathcal{A}]\right] |\psi_0\rangle \qquad (6)$$

Demons:

$$\rho(\omega) = \frac{1}{N} \sum_{n \neq 0} |<\psi_0|\mathcal{A}|n>|^2 \, \delta(\omega - \omega_n) \qquad (7)$$

where $\omega_n = E_n - E_0$

$$\mathcal{P}^2 = \frac{1}{N} \int \rho(\omega) d\omega \qquad (8)$$

Cauchy Schwartz inequality:

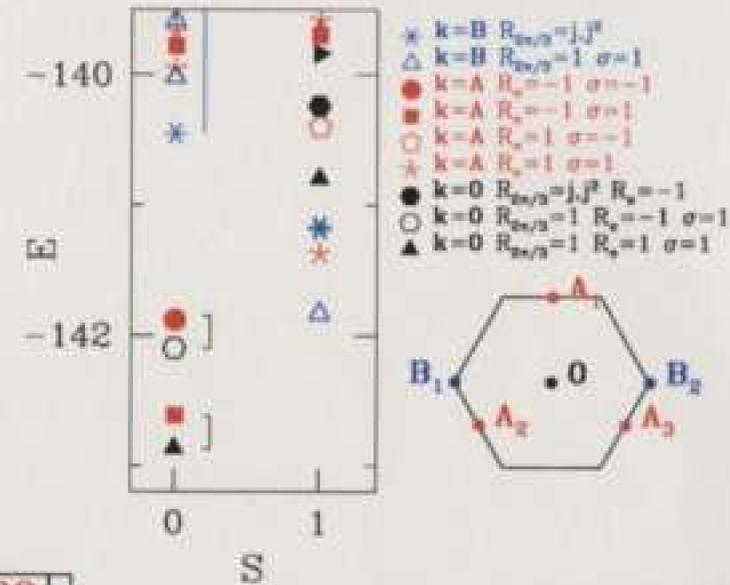
$$\mathcal{P}^4 \leq \frac{1}{N^2} \int \omega \rho(\omega) d\omega \int \omega^{-1} \rho(\omega) d\omega$$

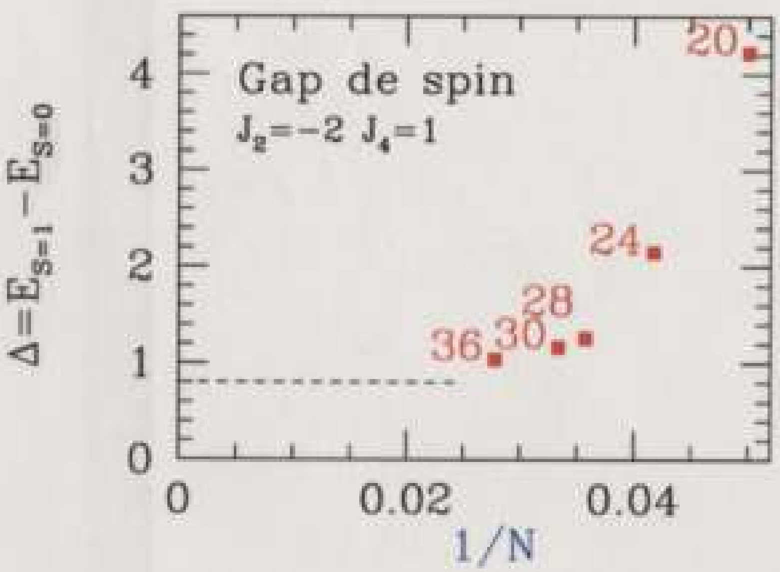
where

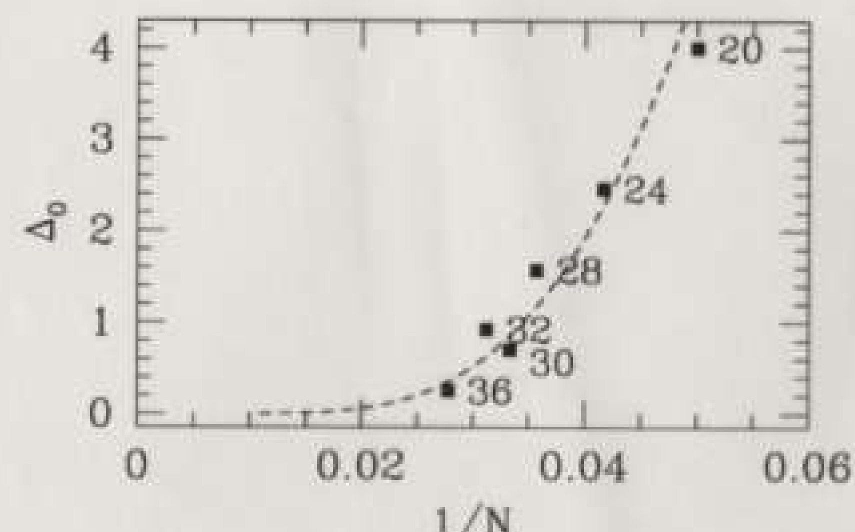
$$\int \omega \rho(\omega) d\omega = f/2 \qquad (9)$$

$$\int \omega^{-1} \rho(\omega) d\omega = \chi/2 \qquad (10)$$

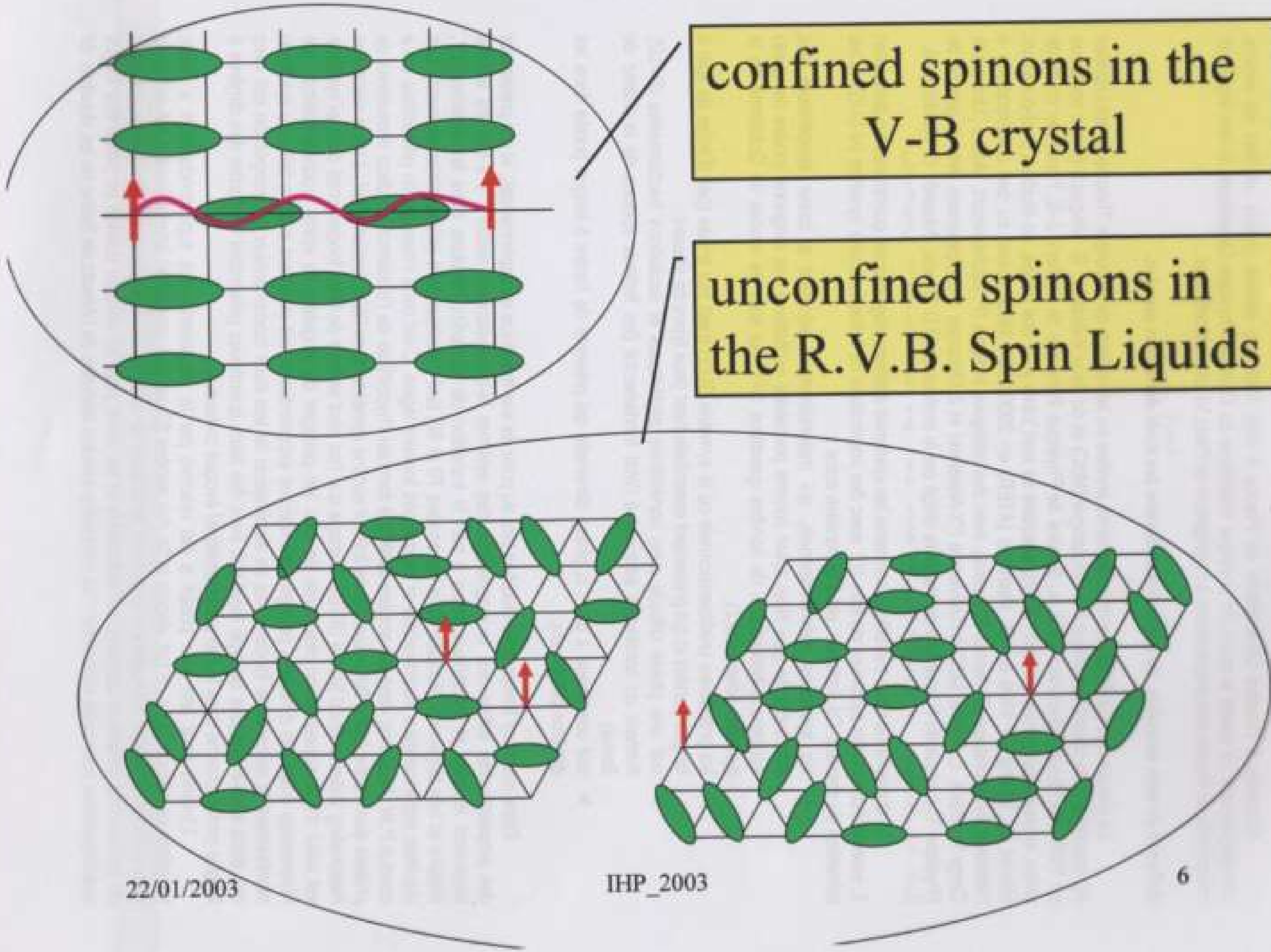
A True Spin Liquid: Spin Gap & Topological Degeneracy



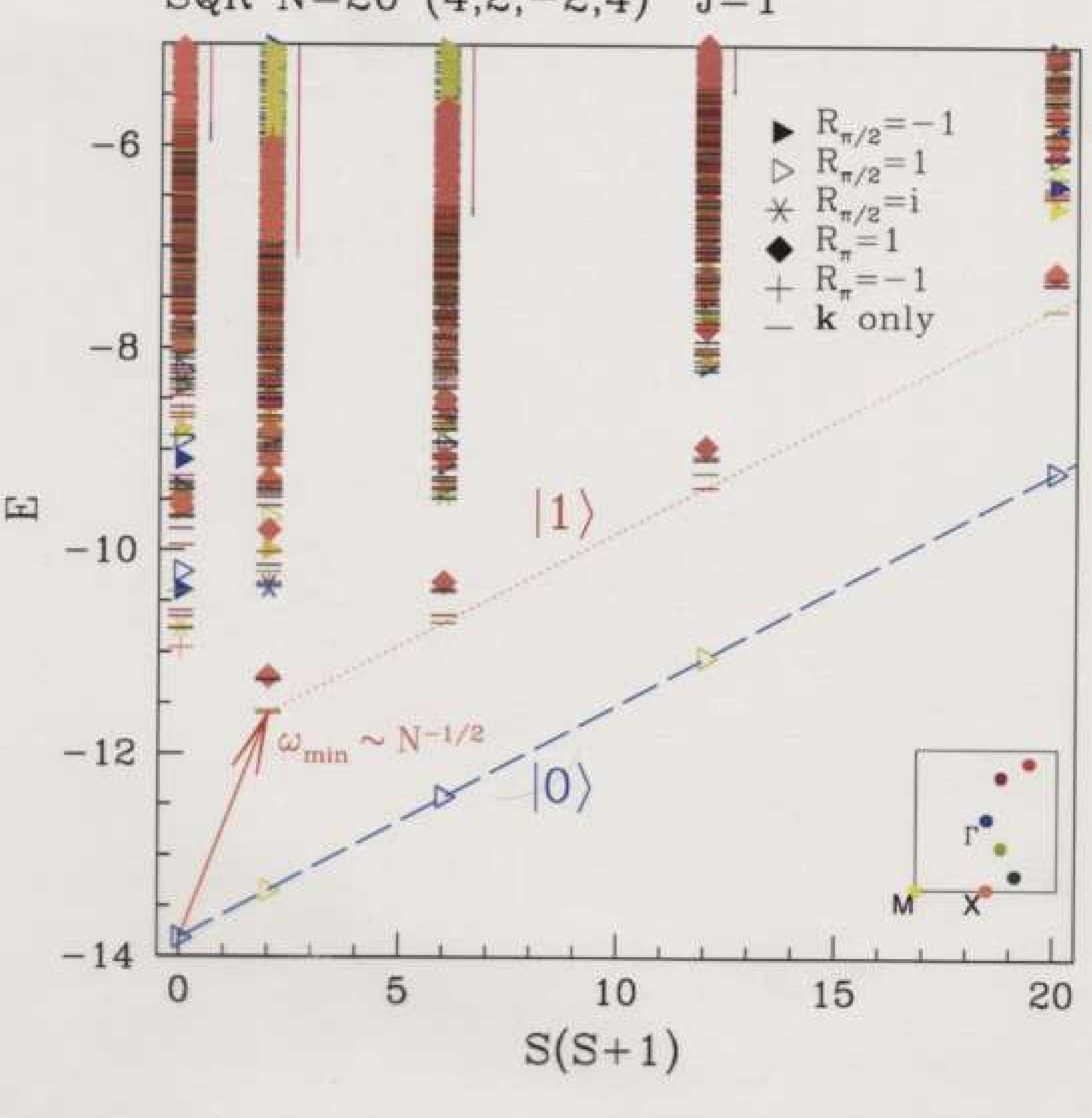




$exp(-L/\xi)$, $\xi = 0.6.$



confined spinons in the V-B crystal
unconfined spinons in the R.V.B. Spin Liquids



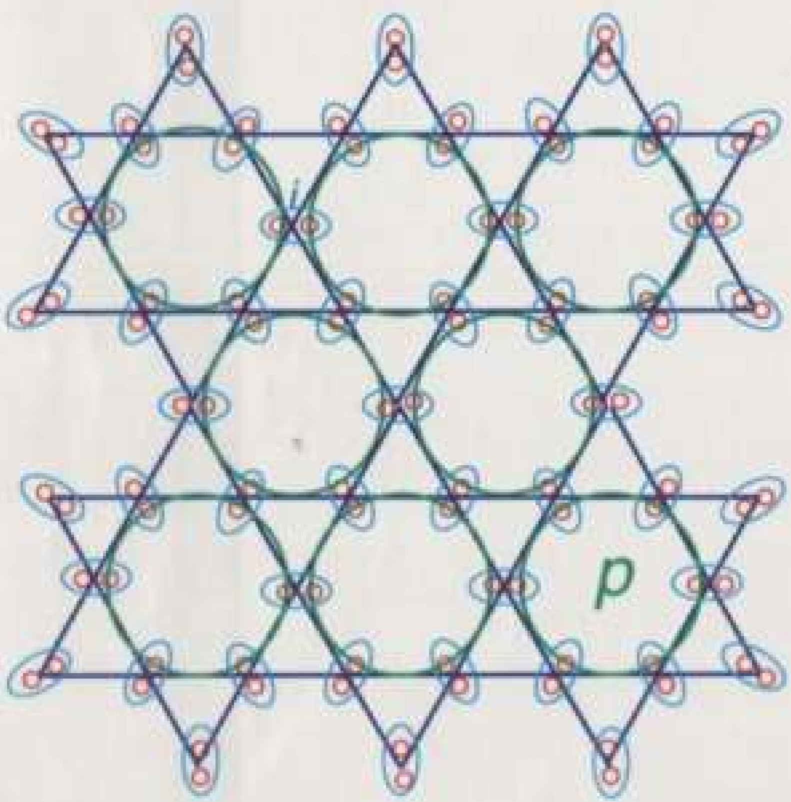
SQR N=20 (4,2,−2,4) J=1
E
S(S+1)
$R_{\pi/2}=-1$
$R_{\pi/2}=1$
$R_{\pi/2}=i$
$R_{\pi}=1$
$R_{\pi}=-1$
k only
$|1\rangle$
$|0\rangle$
$\omega_{min} \sim N^{-1/2}$
Γ
M
X
−6
−8
−10
−12
−14
0
5
10
15
20

Quantum magnets with infinite local degeneracy in the classical limit

Spin-1/2 kagome Waldtmann 99	Spin-1 kagome Hida 01	Spin-1/2 checkerboard Fouet 03	Spin ½ pyrochlore Fouet 03, Berg 03
Spin gap ~ 1/20	Spin gap ~ 0.3	Spin gap ~ 0.7	Spin gap ~ 0.7
No Gap in singlet sector	Gap in the singlet sector ~0.3	Gap in the singlet sector ~0.3	Gap in the singlet sector ~ 1/100 Berg ~ 0.3 Fouet
VBC ? Syromyatnikov Yes Nikolic Yes Our work No	VBS	VBC	VBC

Spin-1 Heisenberg AF on the kagomé lattice

Hida J. phys. Soc. Jpn **69**, 4003, 2000



A spin 1 is a symmetrized combination of two spins 1/2:

$$\otimes(\alpha_i, \beta_i) = [|\alpha_i, \beta_i\rangle + |\beta_i, \alpha_i\rangle] / (\sqrt{2}\, or\, 2)$$

$$|VBS\rangle = \bigotimes_i (\alpha_i \beta_i) \prod_p |i_p, j_p, k_p, l_p, m_p, n_p\rangle_{S=}$$

Journal of the Physical Society of Japan
Vol. 66, No. 4, April, 1997, pp. 961-964

Observation of Spin-Gap State in Two-Dimensional Spin-1 *Kagomé* Antiferromagnet *m*-MPYNN·BF_4

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(Received January 8, 1997)

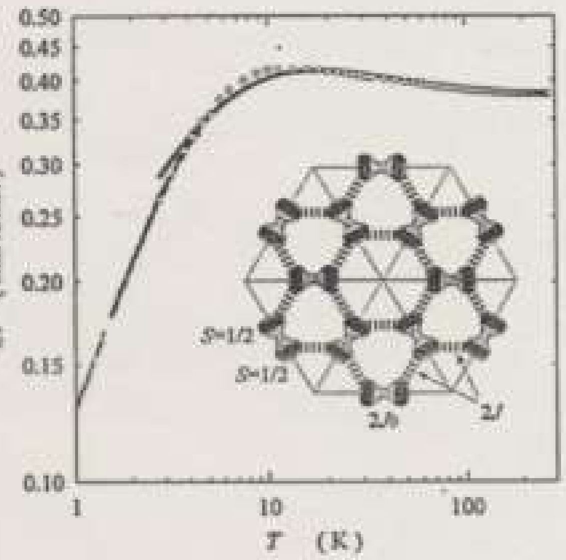


Fig. 1. Paramagnetic susceptibility of *m*-MPYNN·BF_4. The solid curve shows the calculated susceptibility using the intradimer ferromagnetic interaction $2J_0$ and the interdimer antiferromagnetic interaction $2J$ on the spin-1 *Kagomé* lattice shown in the inset (see the text).

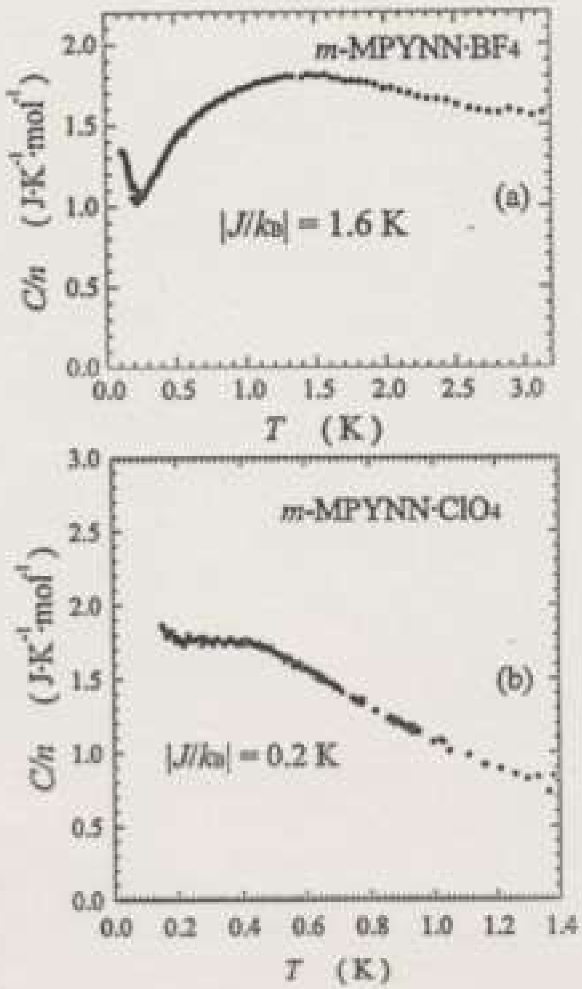


Fig. 2. Molar heat capacities of (a) *m*-MPYNN·BF_4 and (b) *m*-MPYNN·ClO_4. The heat capacities are maximum at temperatures of about $|J|/k_B$.

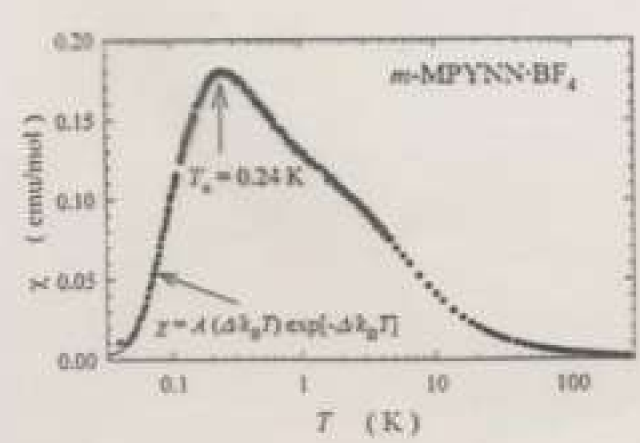
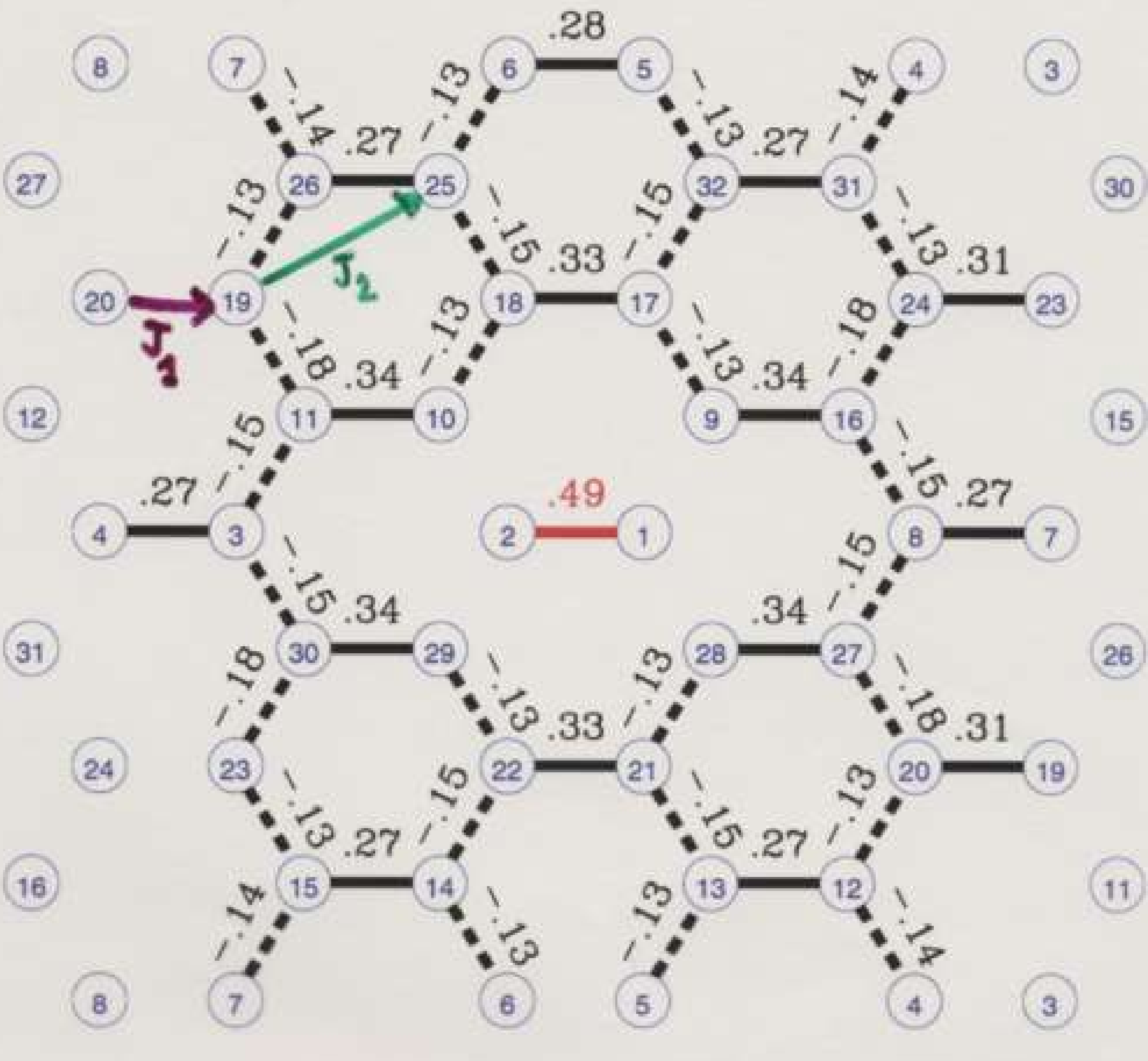


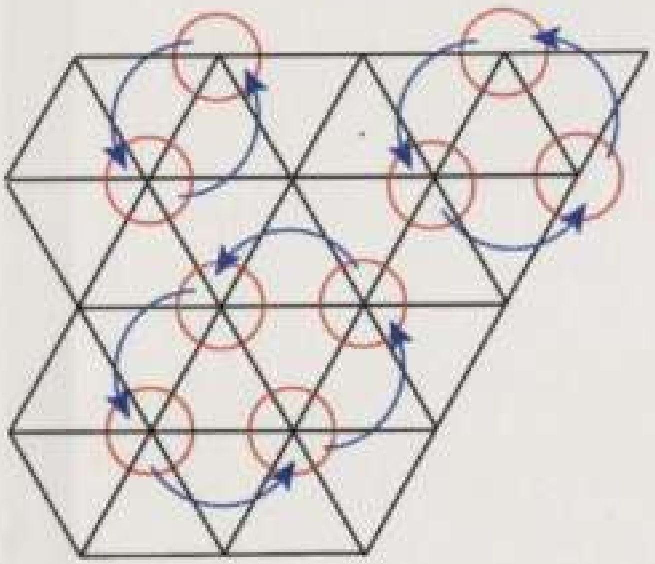
Fig. 4. Low-temperature susceptibility χ of *m*-MPYNN·BF_4. The solid curve between 0.04 and 0.1 K shows activation-type temperature dependence with a gap energy of Δ/k_B=0.23 K.

Dimer-dimer correlations in the Valence-Bond Crystal on the honeycomb lattice
($J_1 - J_2$ model, $J_1 = 1.$, $J_2 = 0.4$)



J.-B. Fouet, P. Sindzingre and C. Lhuillier, E. P. J. B (2001)

A True Spin Liquid: Spin Gap & S. R. Correlations

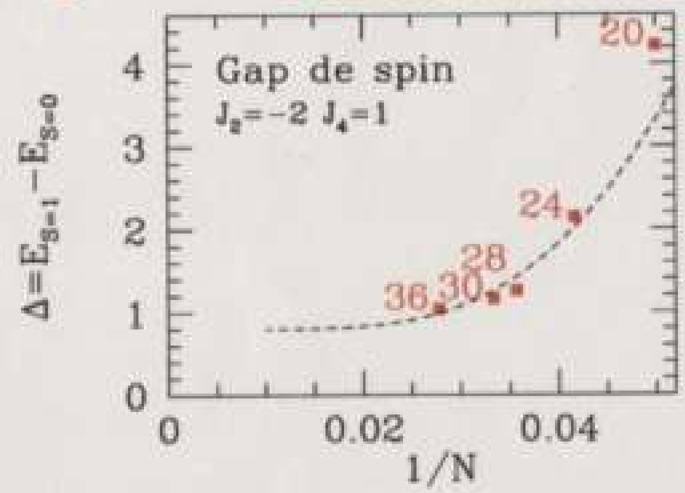


$$H = \underbrace{(+J_2 - 2J_3)}_{J_2^{\text{eff}}} \sum P_{1,2}$$

$$+J_4 \sum (P_{1\ldots4} + P_{4\ldots1})$$

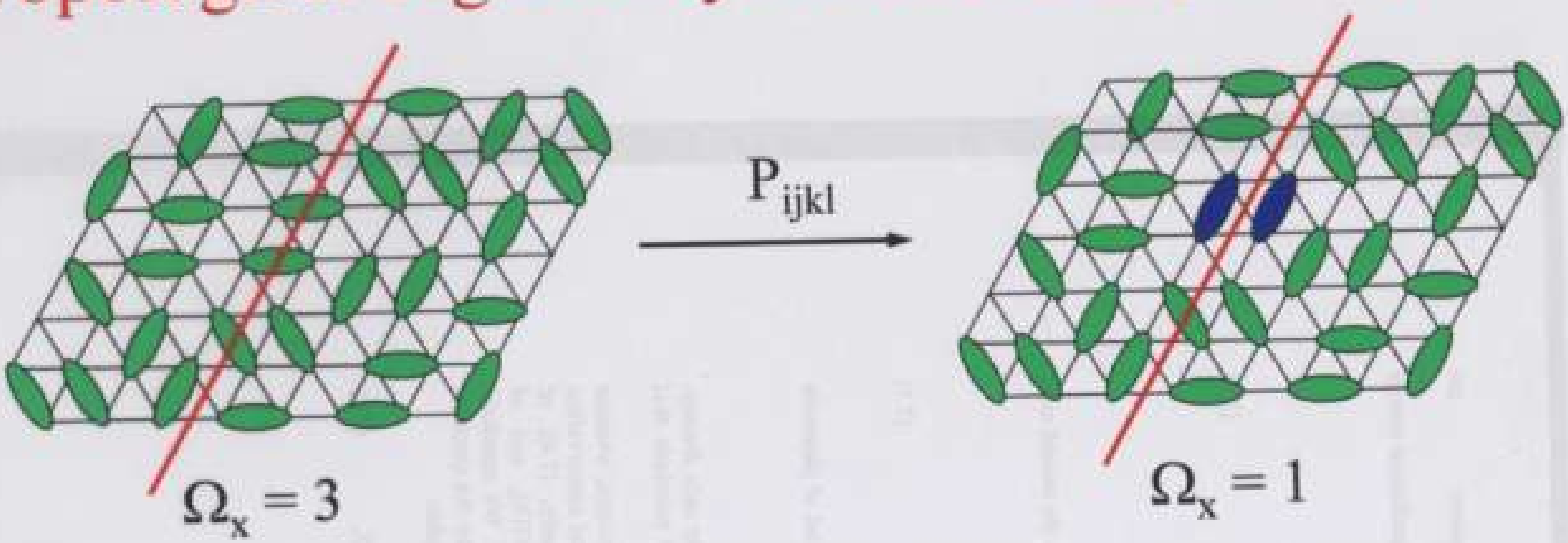
$$P_{1,2} = 2\vec{S}_1 \cdot \vec{S}_2 + \frac{1}{2}$$

$J_2 = -2 \;\; J_4 = 1$ All correlations are short ranged



Misguich et al 98, 99

Topological degeneracy in SRRVB Spin Liquids

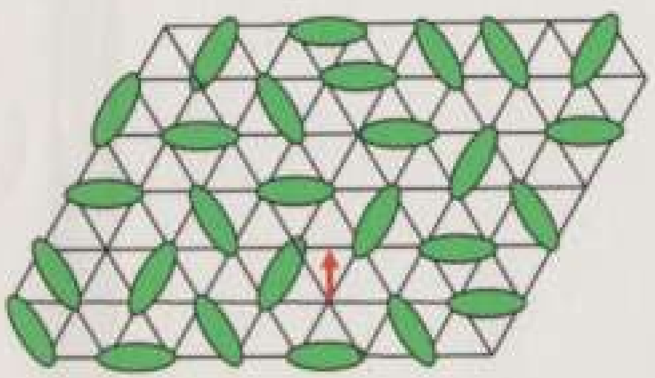


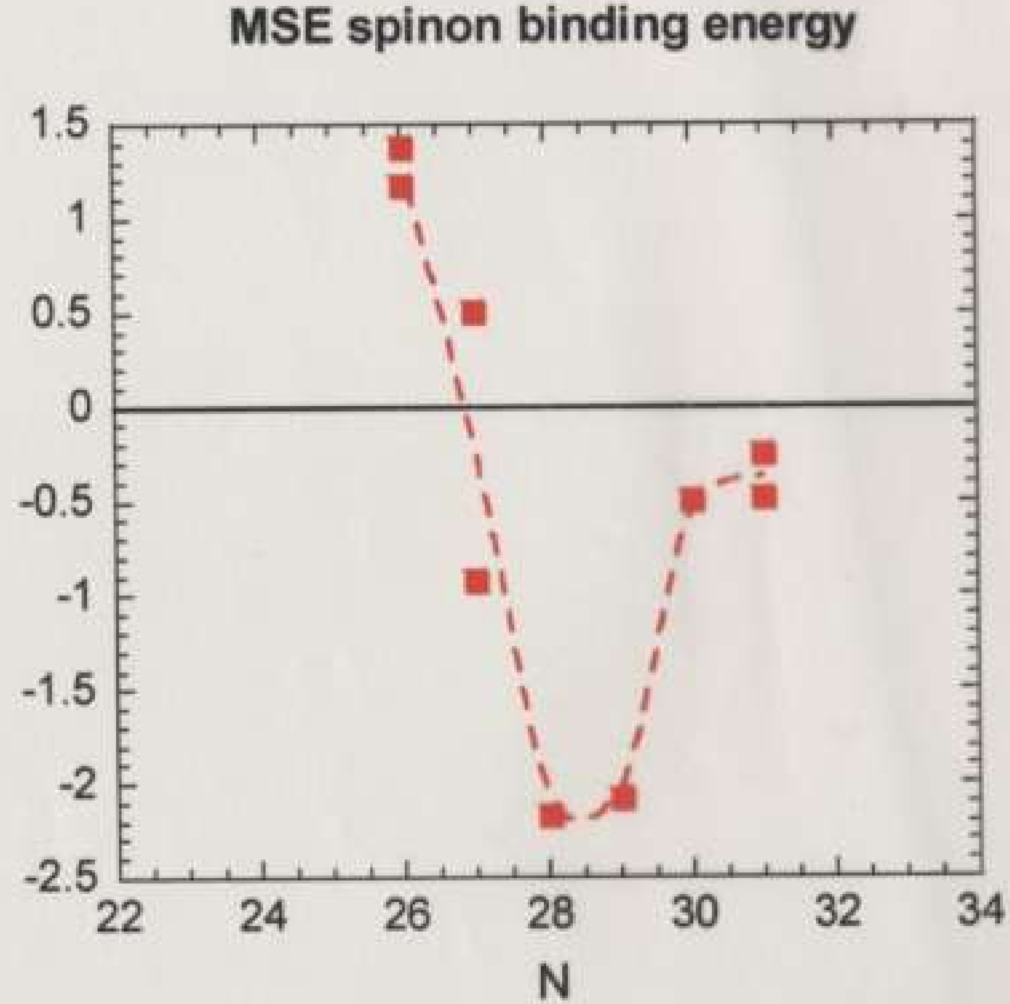
Parities of winding numbers (Ω_x, Ω_y) are good quantum numbers:

- 4 unconnected topological subspaces on a 2-torus
  - degenerate in the thermodynamic limit

↳ 4-fold degeneracy of low lying singlets

$$E(2N+1, S=1/2) = (2N+1)/(2N)\ E(2N, S=0) + \Delta_{spinon}$$
$$E(2N, S=1) = E(2N, S=0) + 2\ \Delta_{spinon} + E_{binding}$$



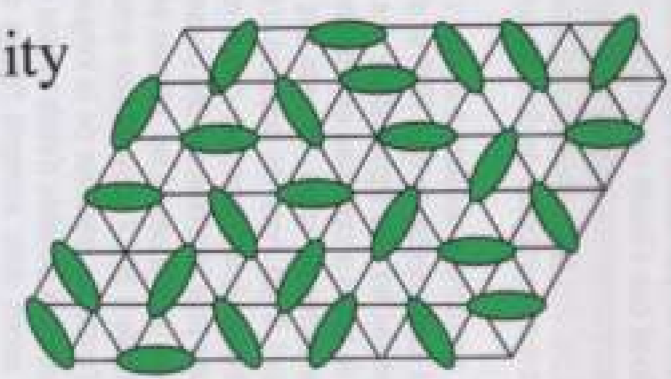


Type I SRRVB Spin Liquids

- ➢ S=0 ground-state, a Spin Gap
- No <u>local</u> order parameter, no Symmetry Breaking
- All correlations functions are short ranged
- All excitations are gapped: χ , C_v thermally activated

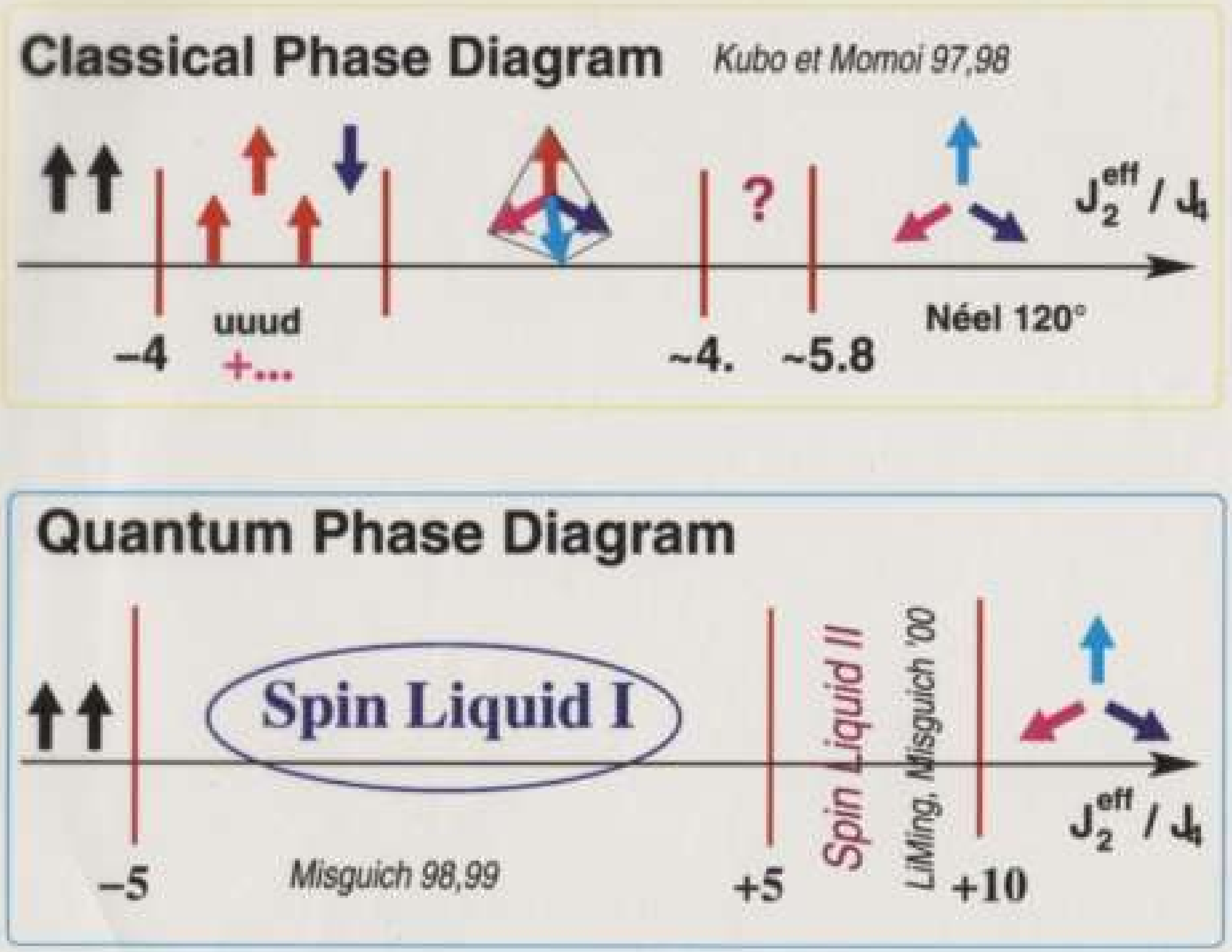
- ➢ 4-fold topological degeneracy on a torus
- ➢ Continuum of deconfined spin-1/2 excitations

•competing interactions near a ferro. instability
•triangular pattern for the singlet bonds
(dimer model: Moessner & Sondhi 2001)



•Experimental realizations:
3He on graphite: Collin et al PRL 2000 and refs. therein

Classical and Quantum phase diagrams of the MSE Hamiltonian



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Entropy Balance and Evidence for Local Spin Singlets in a Kagomé-Like Magnet

A. P. Ramirez,[1] B. Hessen,[2] and M. Winklemann[3]

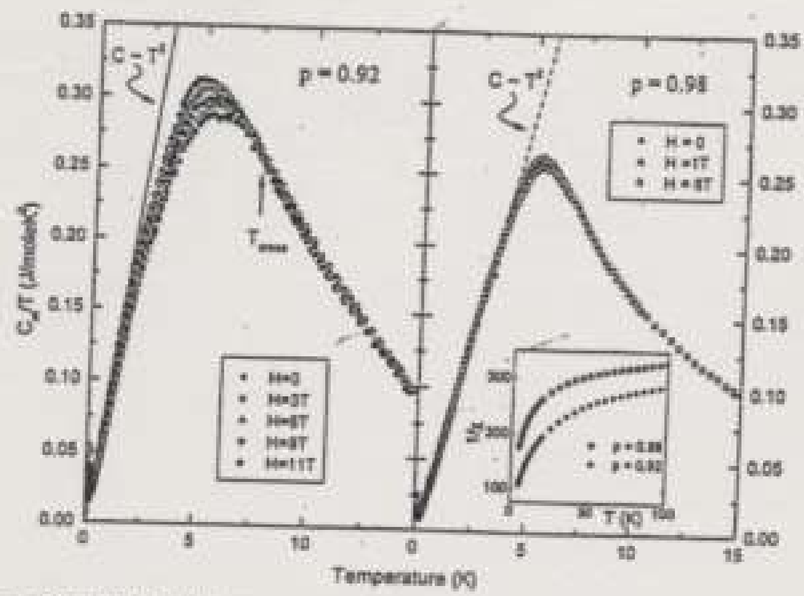


FIG. 2. The magnetic specific heat, divided by temperature, $C(H,T)/T$, for $SrCr_{9p}Ga_{12-9p}O_{19}$, $p = 0.92$ (left) and right), as a function of temperature, at several different values of fixed magnetic field. The inset shows the crossing ter T_{max} for the $p = 0.92$ sample. The inset also shows the inverse susceptibility $1/\chi$ ($H = 50$ G) for the two samples.

CAL REVIEW B VOLUME 43, NUMBER 13 1 MAY 199

Magnetothermal behavior of the two-dimensional triangular-lattice compounds RCuO$_2$ [R = La, Pr, Nu, Eu, and (La$_{0.2}$Gd$_{0.8}$)]

A. P. Ramirez, R. Jager-Waldau,* and T. Siegrist

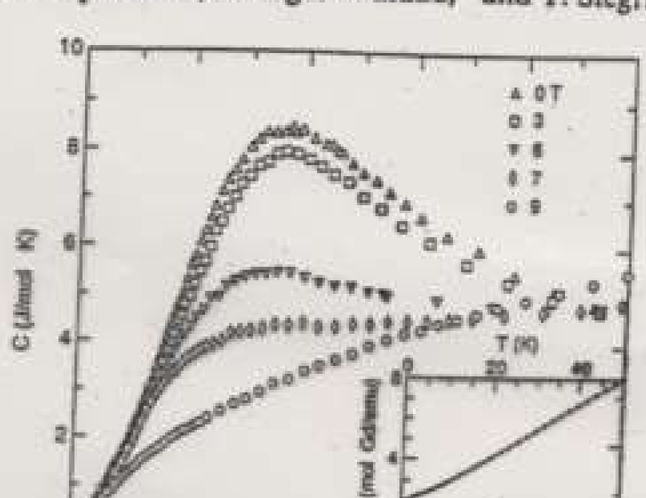
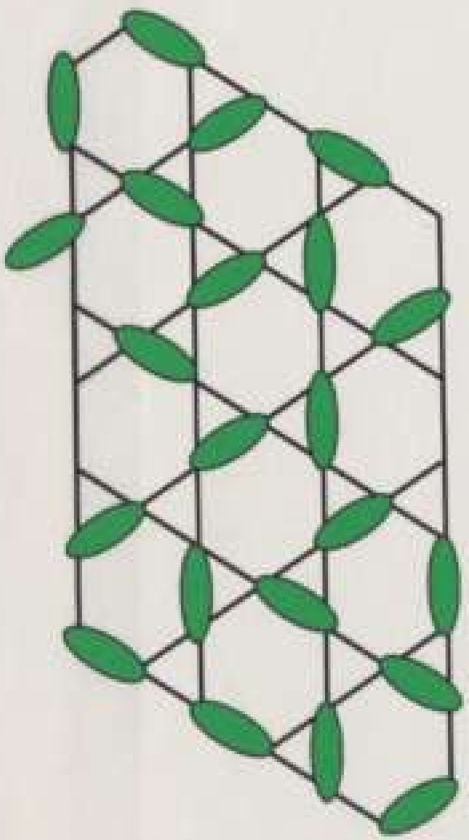
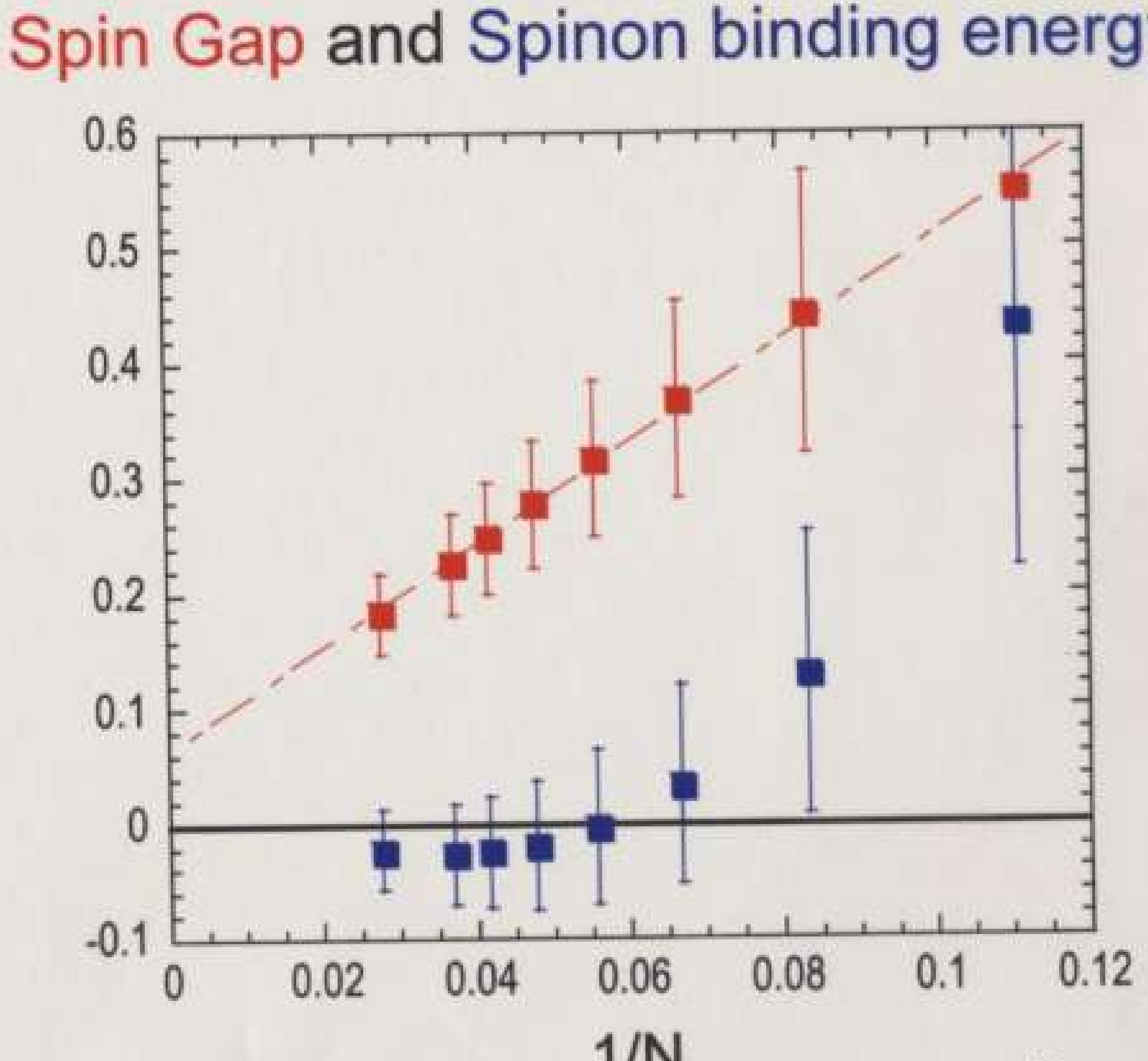


FIG. 4. Specific heat of $La_{0.2}Gd_{0.8}CuO_2$ vs temperature for several values of applied field. The decrease of the peak temperature with field indicates antiferromagnetic short-range order. The inset shows the inverse susceptibility for the same compound. The straight line is a least-squares fit, and yields $\Theta_{CW} = -12.5$ K and a moment of $7.912\mu_B$.

Heisenberg model on the kagomé lattice:





Long Range Order or Spin Liquids ?

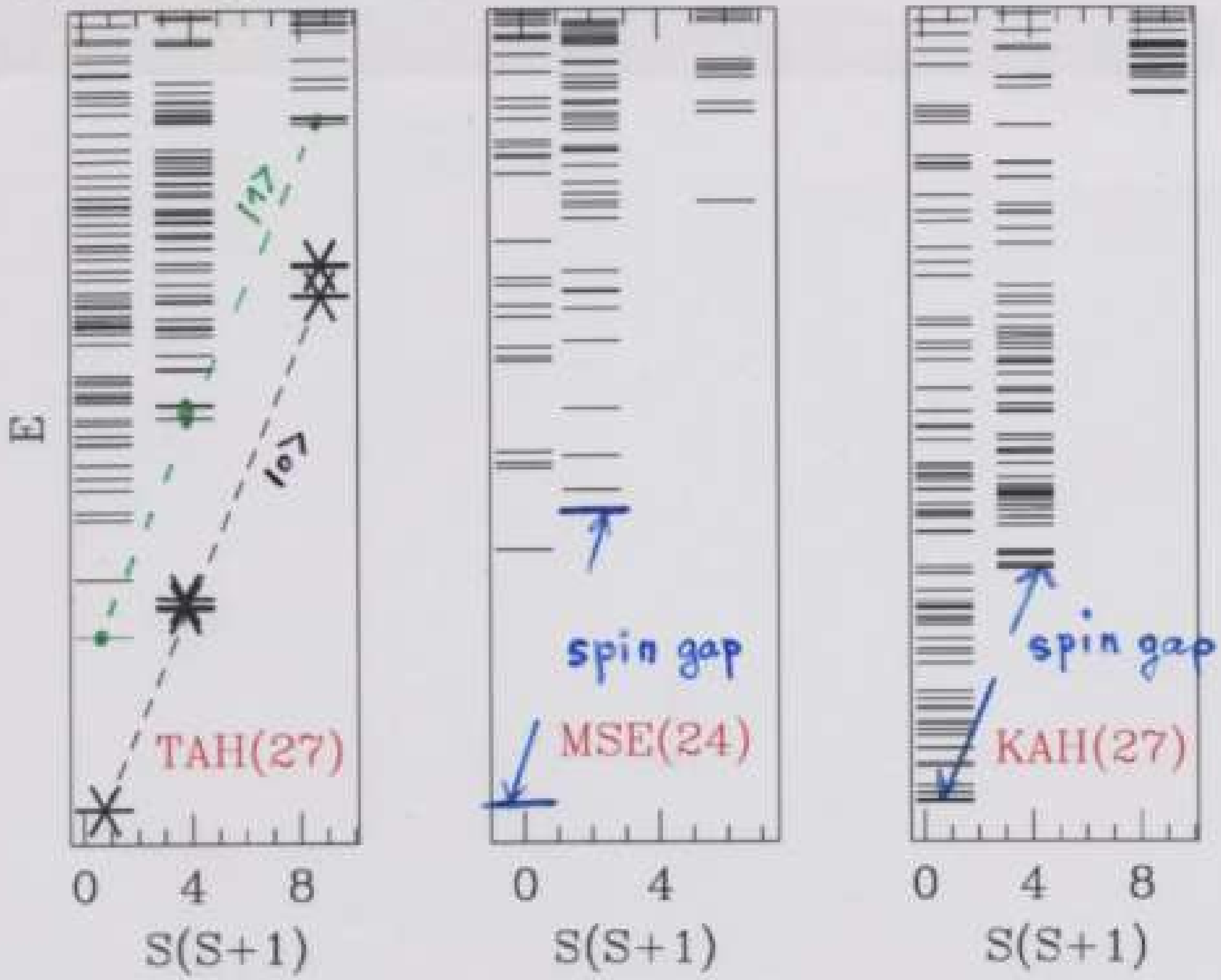
Degeneracy of the ground-state	Nature of the phase	Symmetry Breaking				First excitations	Thermodynamics
		SU(2)	Translat	Point Gr.	Time Rev		
$O(N^p)$	Néel Order p Sublattices.	X	X	X	X	Gapless magnons	$C_v \alpha T^2$
At least 4	Valence Bond Crystal	0	X	X	0	All excit. gapped	C_v & χ thermally activated
2	Chiral Spin Liquid	0	0	0	X	All excit. gapped	C_v & χ thermally activated
1	Short Range R.V.B. Spin Liquid	0	0	0	0	All excit. gapped	C_v & χ thermally activated
1.15^N	KAFM: Short Range Spin Liquid	0	0	0	?	Triplet excitations gapped! Singlet excit. NO!	χ thermally activated $C_v \alpha T^{2x}$ insensitive to H at low T

Lecture 1: Semi-classical Néel order versus Quantum dimerization

- Valence Bond Crystals
  - The Majumdar Gosh chain

    Ground-states and excitations
  - 2-dimensional VBC
    - J_1-J_2 model on the honeycomb lattice
    - Heisenberg model on the checkerboard lattice
    - Shastry Sutherland model: $Sr\,Cu_2\,(BO_3)_2$
  - Conclusions on VBC
    - A local quantum scenario to overcome frustration
    - Tensorial product wave-functions are good variational descriptions
    - Excitations
      - Itinerant triplet (and singlets)
      - Domain walls in the singlet subspace between the different plaquettized ground-states
  - A toy model for VBC: the quantum dimer model on the square lattice (Rokhsar & Kivelson 88)

2-dimensional geometrically frustrated anti-ferromagnets: 3 scenarii

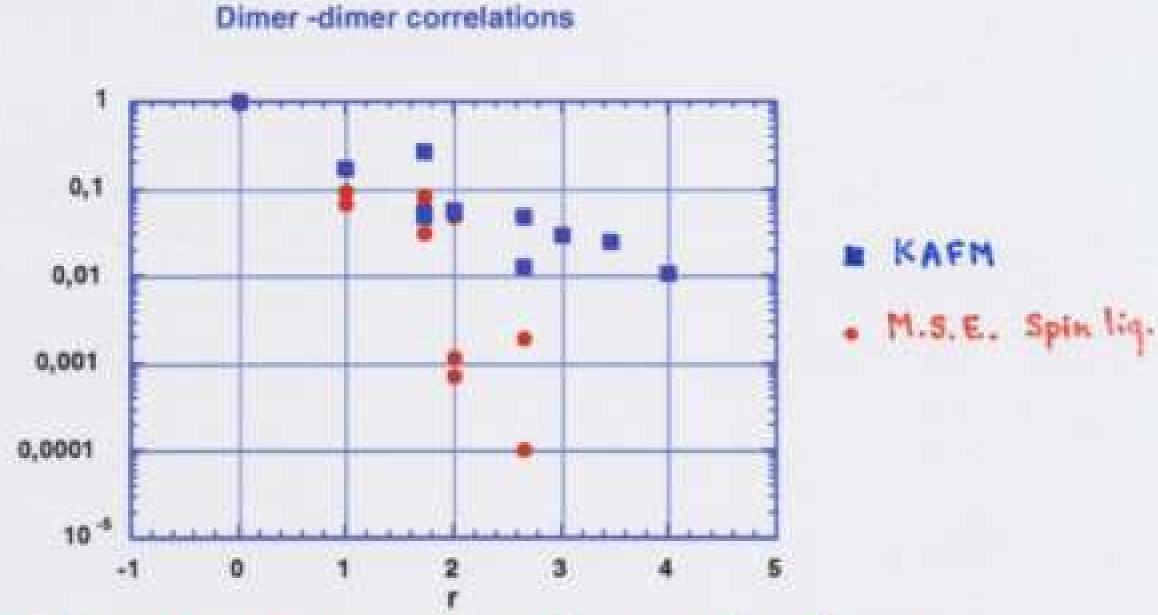


T=0 Néel order	T=0 Q.S.L	T=0 Q.S.L
first excitations: magnons	first excitations are gapped	first excitations: gapless singlets
$C \propto T^2$	$C \propto e^{-\frac{\Delta}{T}}$	$C \propto T^\alpha$ ($\alpha \neq 2$)
	$\chi \propto e^{-\frac{\Delta}{T}}$	$\chi \propto e^{-\frac{\Delta}{T}}$
		spin 1/2 chiral excitations?

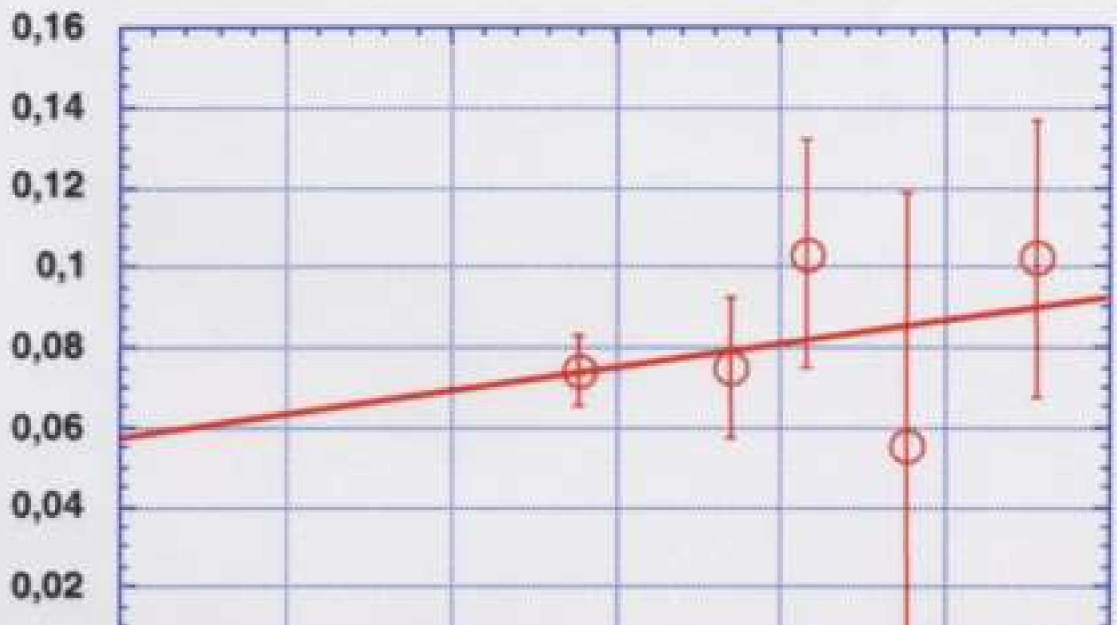
The Heisenberg A.F. on the kagomé lattice

- short range magnetic correlations

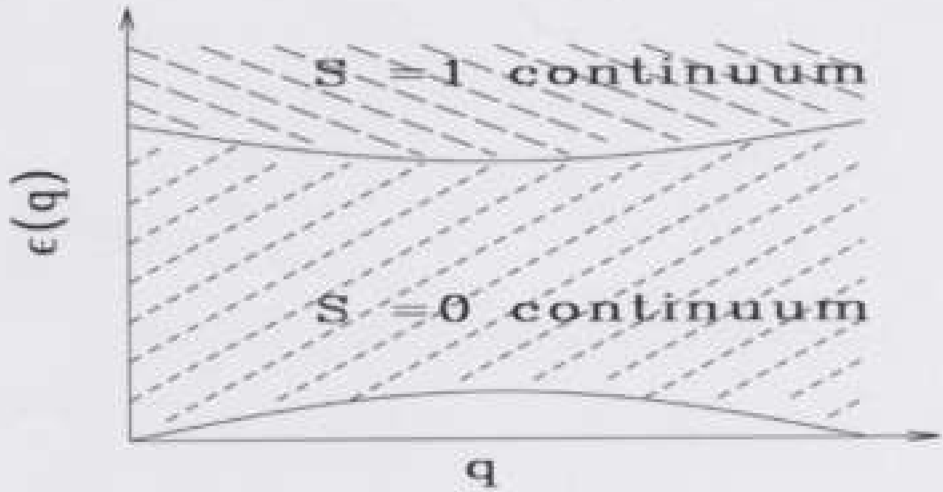
spin-spin, dimer-dimer, chiral-chiral correlations go to 0 exponentially with distance



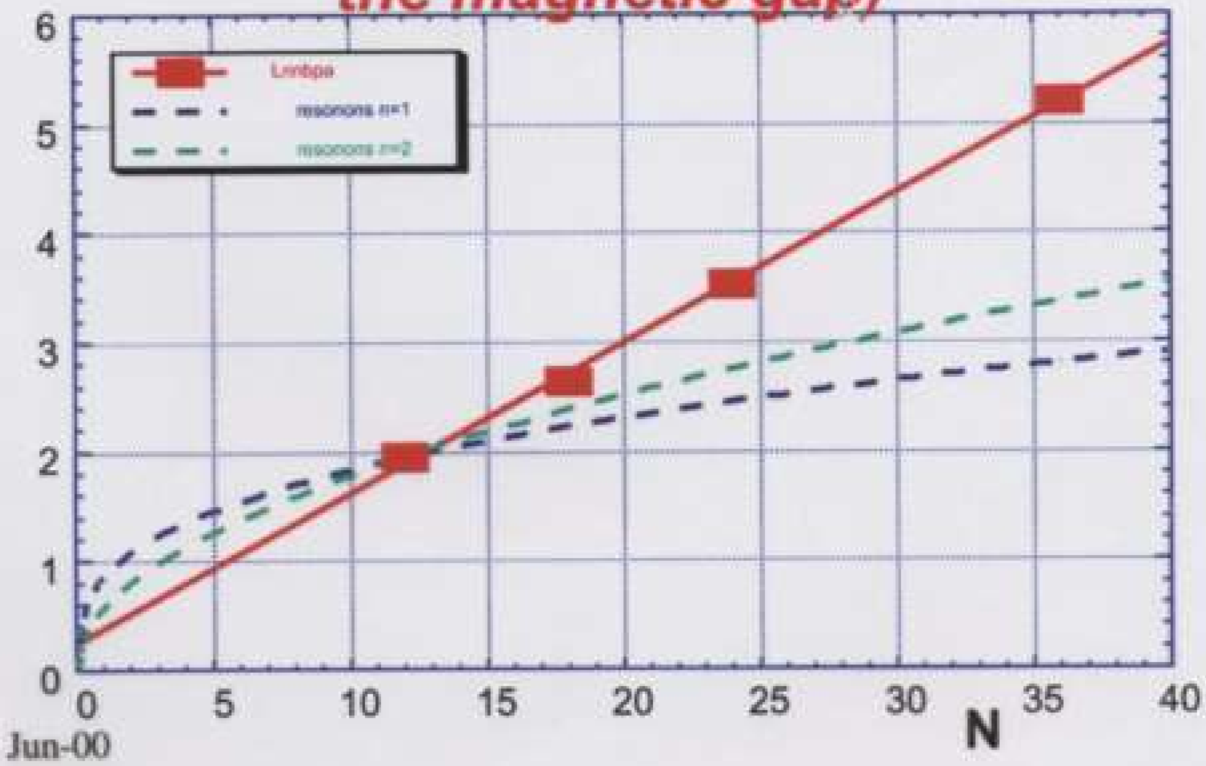
- a small gap to magnetic excitations



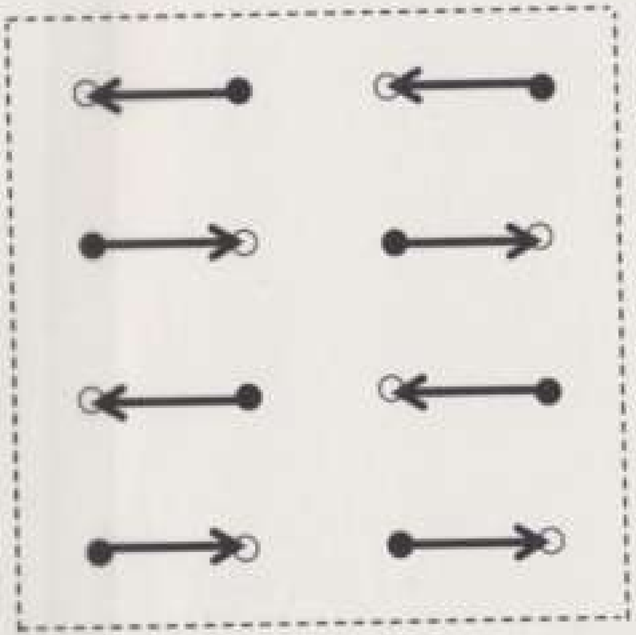
Magnetic and non magnetic excitations form continua



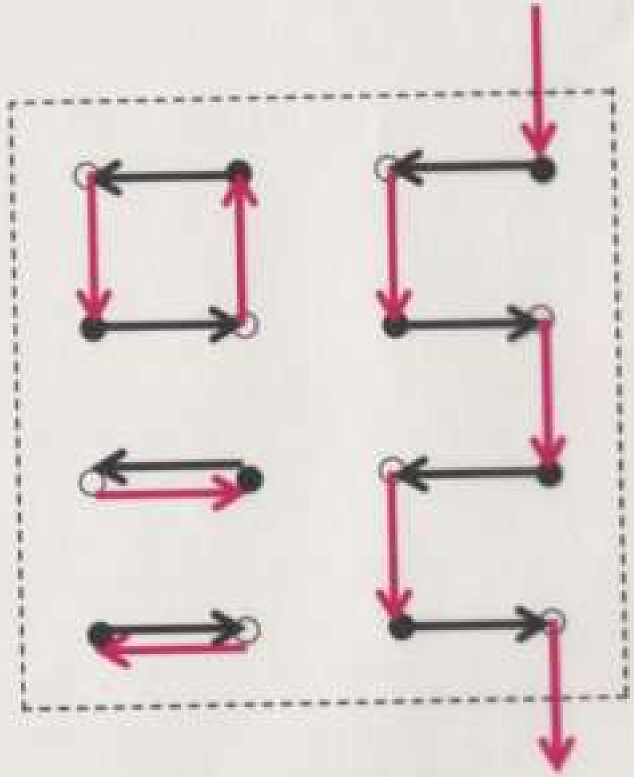
Ln (number of singlet states below the magnetic gap)



Transition graph, Winding numbers and Topological sectors



Ref. Configuration



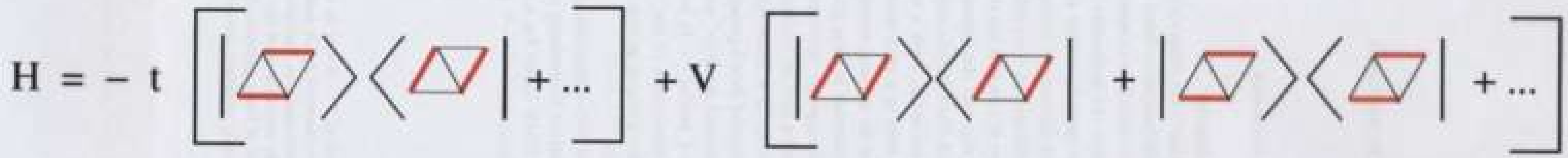
Transition graph

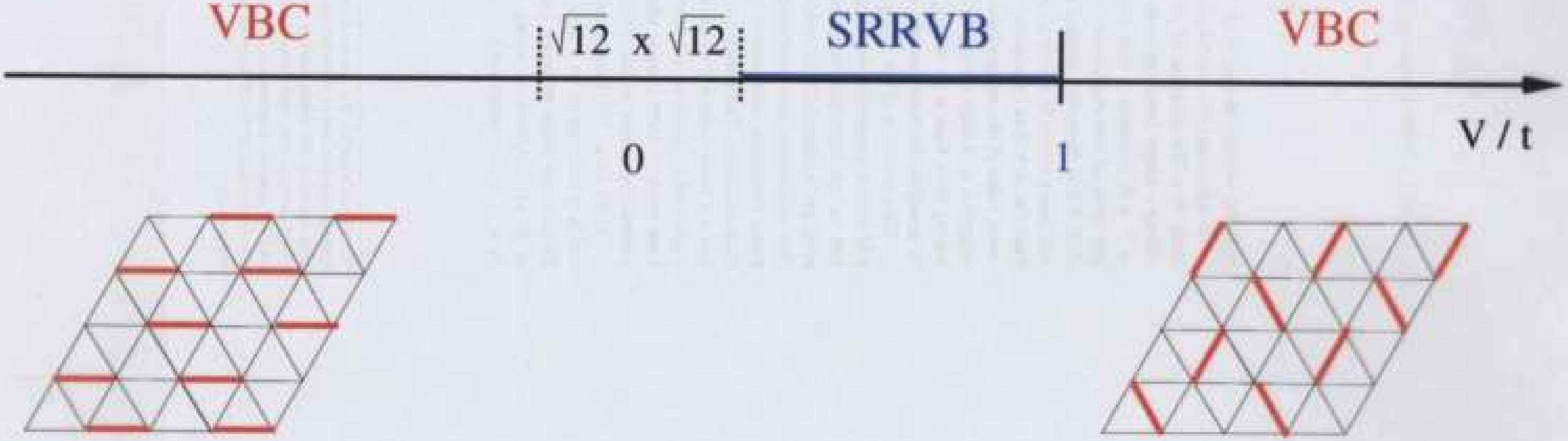
Winding numbers of C

0, -1

A toy model with VBC and a SRRVB ground-states

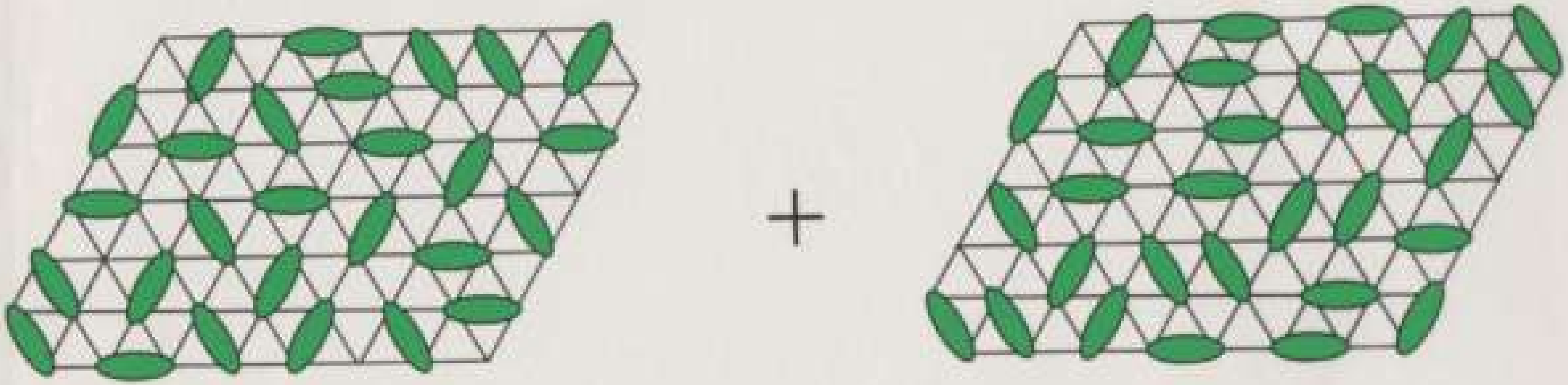
Moessner-Sondhi hard core quantum dimer model on the triangular lattice (2000)





Resonating Valence Bond Spin-Liquid

P.W. Anderson 1973

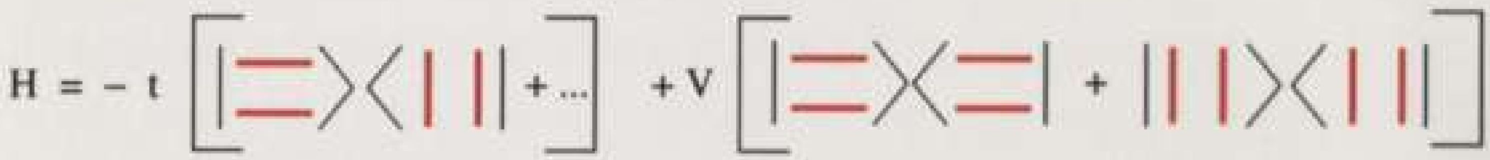


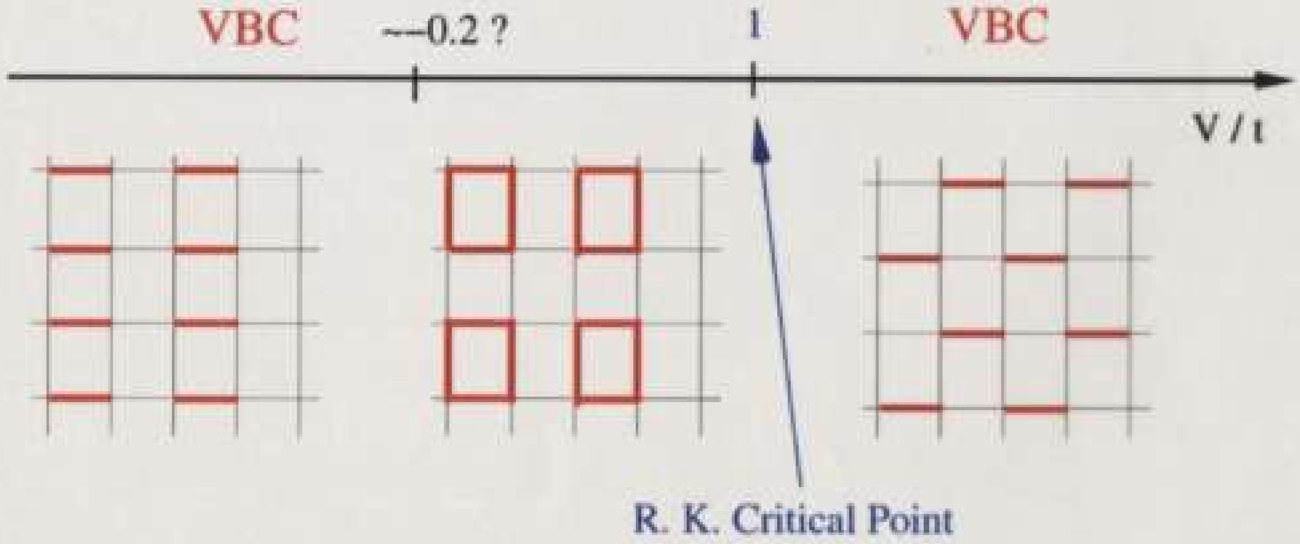
$+ \cdots\cdots$

α^{N} *terms* $(1 < \alpha < 2)$

A toy model with VBC ground-states

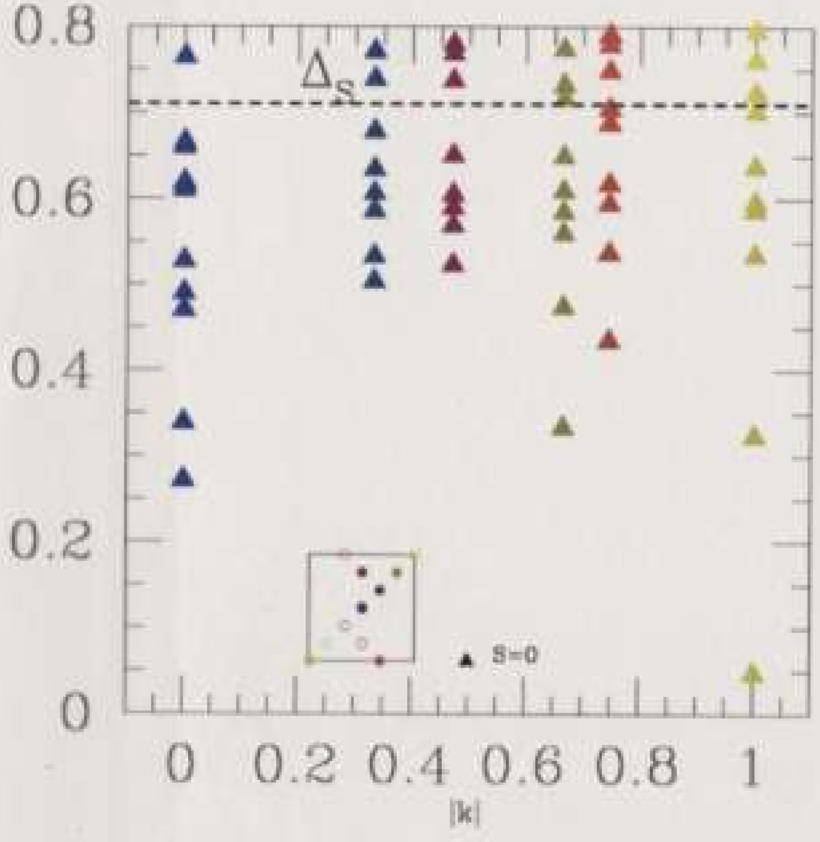
Rokhsar–Kivelson hard core quantum dimer model on the square lattice '88



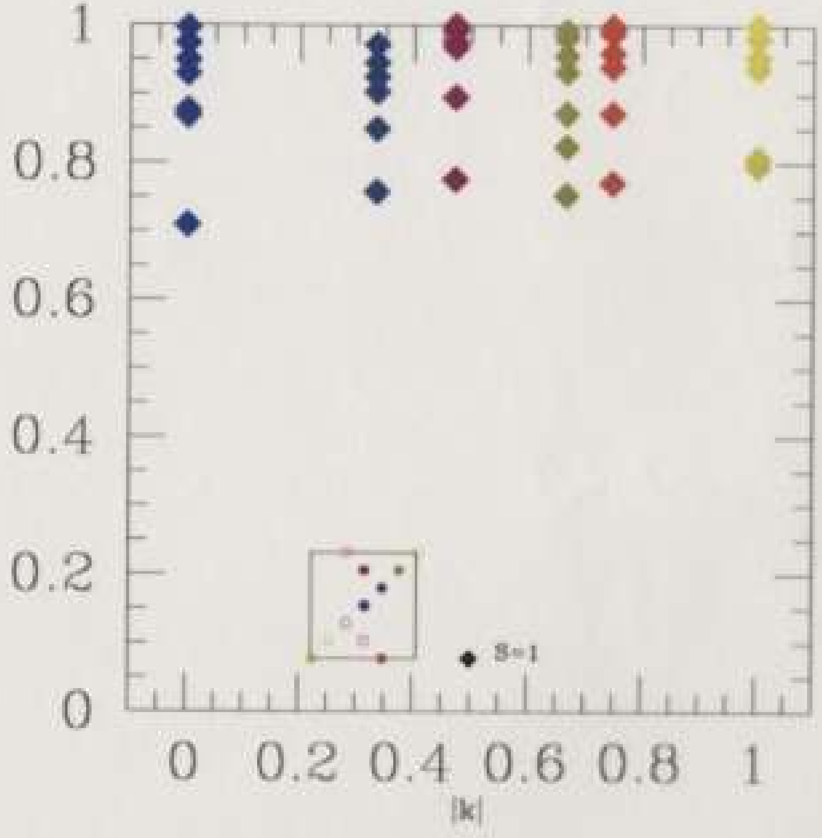


Similar model on the hexagonal lattice: Moessner and Sondhi '01

Excitations of the VBC on the checkerboard lattice

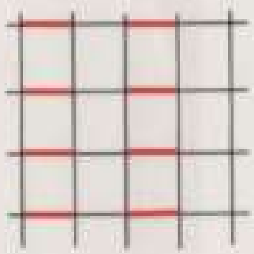


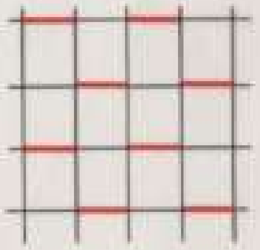
Dispersion relation
in the singlet sector



Dispersion relation
in the triplet sector

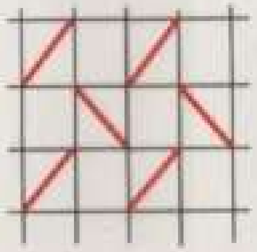
Valence-Bond Crystals

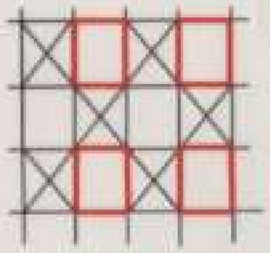




- Short range order in spin-spin correlations
  L.R.O. in dimers or larger S=0 plaquettes
  - ➢ Spontaneous Symmetry Breaking
    & degenerate S=0 ground-states

- A gap to any excitations
  - ➢ First excitations: $\Delta S=1, 0$ excitations
    branches of "optical magnons"





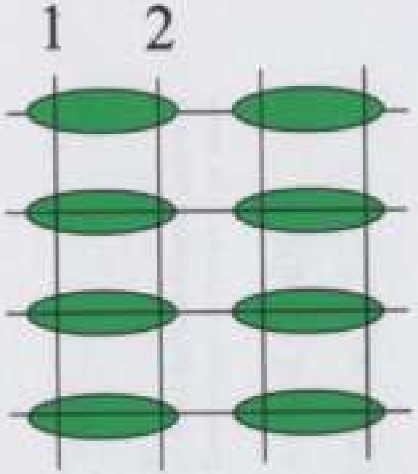
- J_1-J_2 model on the square (?), honeycomb,
  Shastry-Sutherland, 2-d pyrochlore lattices

- Exp: CaV_4O_9, $SrCu_2(BO_3)_2$

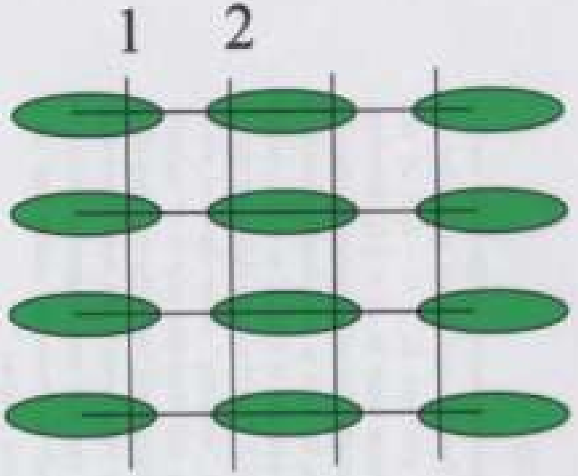
Lecture 2: Resonating Valence Bond Spin liquids (RVB SL)

- Quantum dimer models
  - Square lattice (Rokhsar & Kivelson 88):
    Phase diagram, topological sectors, RK point
  - Triangular lattice (Moessner & Sondhi 01)
    VBC and a true liquid phase
  - Kagomé lattice (Misguich, Serban , Pasquier 02)
    A solvable model in the spin liquid phase, coherence of the ground-state (RVB), deconfined visons excitations, spinons
- A realistic spin model with a RVB SL: the MSE model
  - Hamiltonian
  - Short range correlations and spin gap
  - Topological degeneracy
  - Spinons
- Other models with RVB SL g.-s.
  - J_1-J_2 model on the honeycomb lattice
  - Sachdev & Read large N approach

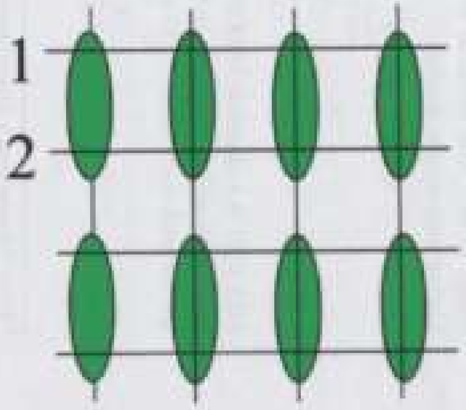
V.B.C. ground-state wave-function



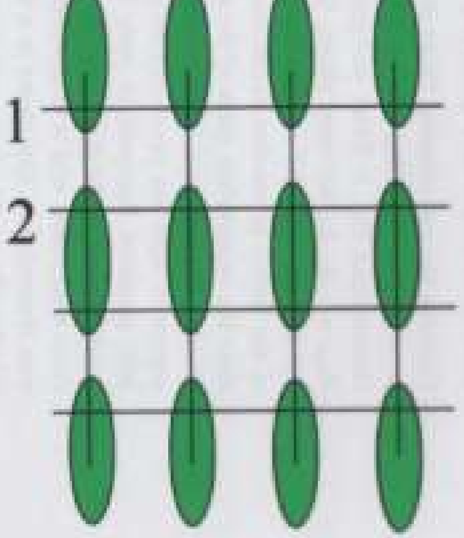
$+ z$



$+ z^2$



$+ z^3$



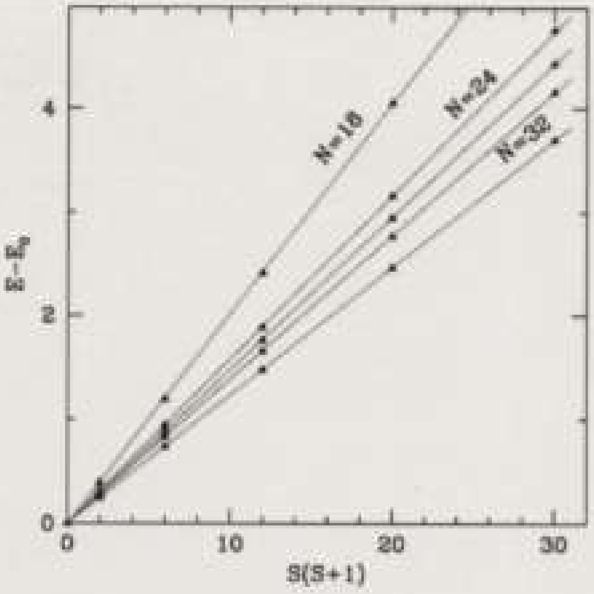


Figure 2.5: AF Heisenberg model on the honeycomb lattice, scaling of the QDJS with S and N for $N = 18, 24, 26, 28, 32$ (taken from ref. [20]).

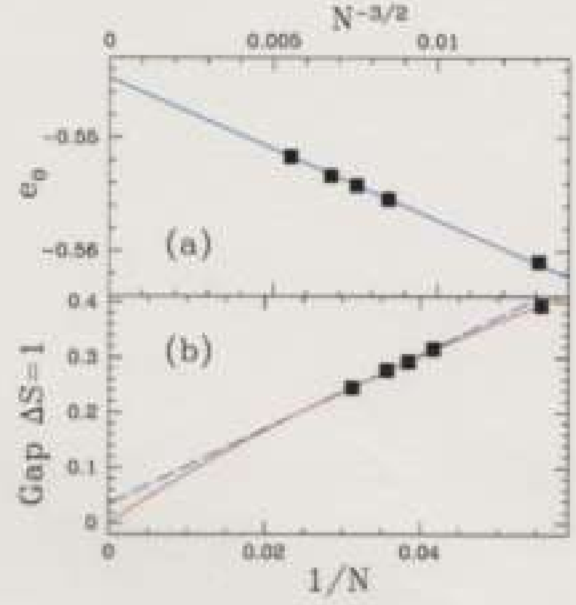
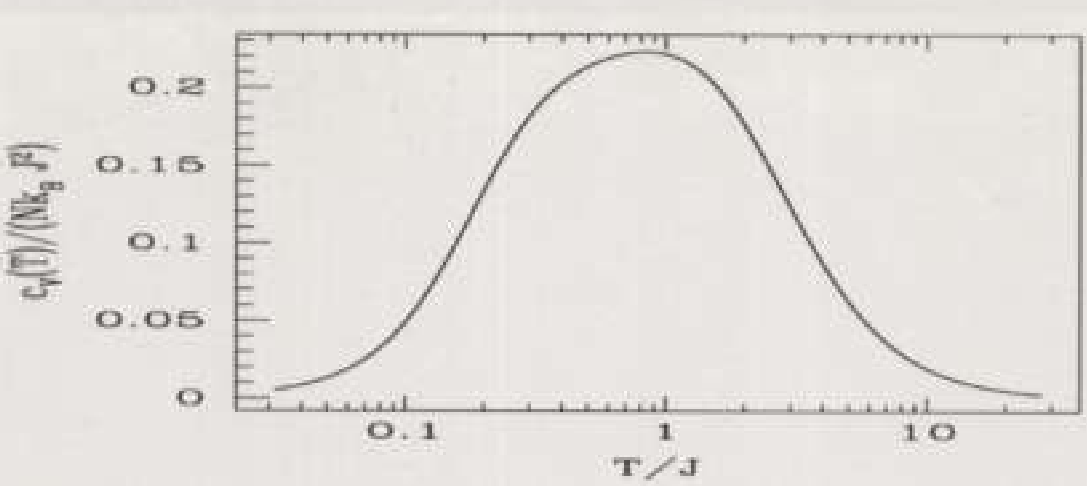


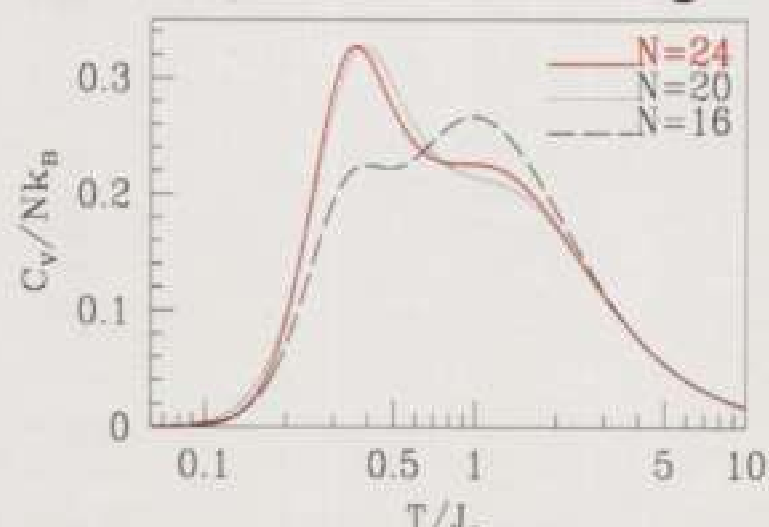
Figure 2.6: AF Heisenberg model on the honeycomb lattice, (a) energy per site e_0 versus $N^{-\frac{3}{2}}$ (b) spin-gap: The dashed line is a linear fit in $1/N$: for the sizes of interest the restriction to the leading term of the finite size expansion is insufficient. The full line is a fit to eq. [29, 31]: $\Delta(N) = \frac{1}{4\chi N}(1 - \beta\frac{c}{\rho\sqrt{N}}) + \mathcal{O}(\frac{1}{N^2})$ where χ is the spin susceptibility, c is the spin-wave velocity, ρ the spin stiffness and β is a number of order one (taken from ref. [20]).

Downward shift of the spectral weight

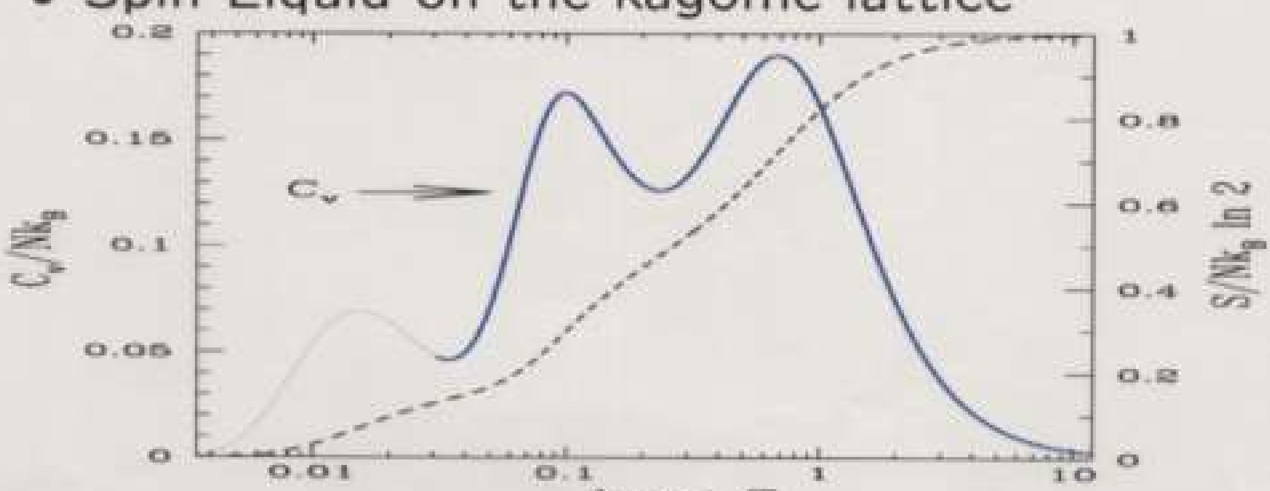
- Néel ordered A.F. on the triang. latt.



- M.S.E. Spin Liquid on the triang. latt.



- Spin Liquid on the kagomé lattice

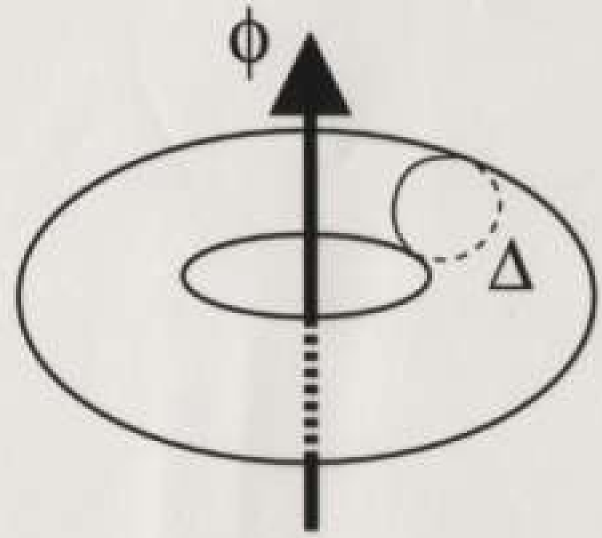


"Next excitations : In two dimensions, it is not at all clear that the two spins in a singlet necessarily ever separate by any appreciable distance, in which case there may be an energy gap to the lowest triplet excitation, so that the state need be only weakly paramagnetic if at all.

But especially if it is a Bose state, it will probably have low-energy excitations, which may in this case be purely singlet. Two puzzling results to which this is relevant on TaS_2 (1T) are the low paramagnetic susceptibility and the large, not quite linear, specific heat".

(P.W. Anderson - 1973).

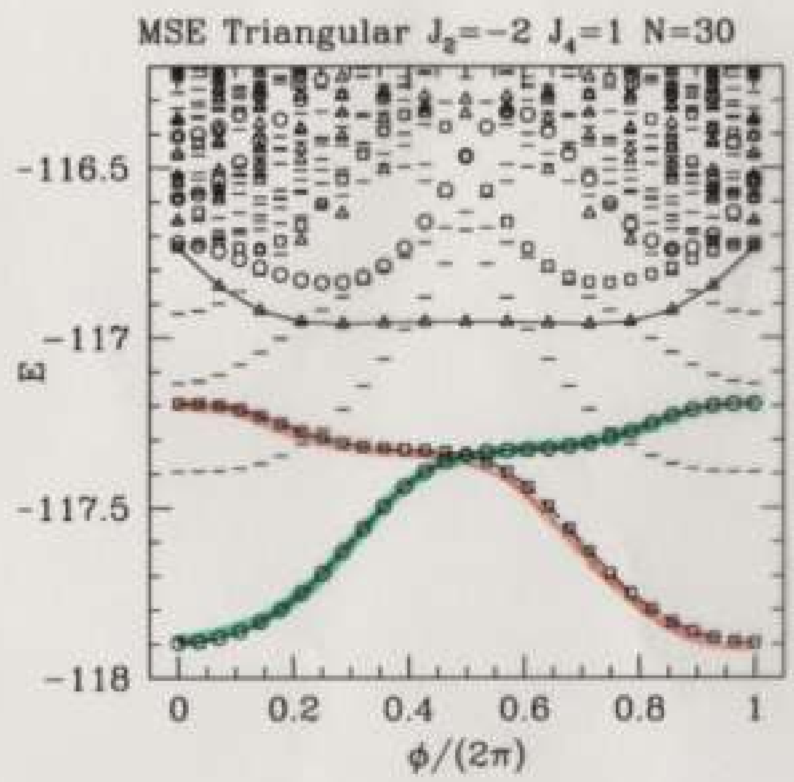
Topological Degeneracy



$$\mathcal{H}_0 = \sum \left(\mathbf{S}_{n,p} \cdot \mathbf{S}_{n+1[L_x],p} + \mathbf{S}_{n,p} \cdot \mathbf{S}_{n,p+1[L_y]} \right)$$

$$\mathbf{S}(\mathbf{R}_i + \mathbf{T}_j) = e^{i\phi_j S^z(\mathbf{R}_i)} \mathbf{S}(\mathbf{R}_i) e^{-i\phi_j S_z(\mathbf{R}_i)}$$

$$\begin{aligned} \tilde{\mathcal{H}}_\phi &= \mathcal{H}_0 \\ &+ \frac{1}{2} \sum \left((e^{i\frac{\phi}{L_x}} - 1) S^-_{n,p} S^+_{n+1,p} + \text{h.c.} \right) \end{aligned}$$



⇒ Fractionalization: unbound spinons

Misguich et al. Eur. Jour. Phys. B 26, 167-183 (2002)

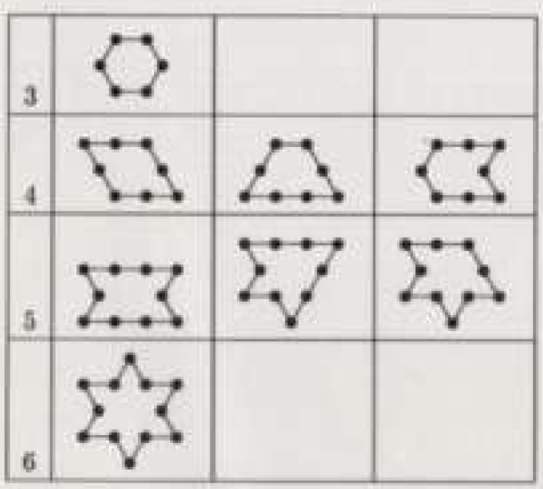
A solvable model: Quantum dimer model on the kagomé lattice

$$\mathcal{H} = -\sum_h \sigma^x(h)$$

$$\text{where } \sigma^x(h) = \sum_{\alpha=1}^{32} |d_\alpha(h)\rangle \langle \bar{d}_\alpha(h)| + |\bar{d}_\alpha(h)\rangle \langle d_\alpha(h)|$$

$$\mathcal{H} = -N_h + \sum_h \sum_{\alpha=1}^{32} P_\alpha(h)$$

$$P_\alpha(h) = \left[\, |d_\alpha(h)\rangle - |\bar{d}_\alpha(h)\rangle \,\right] \left[\, \langle d_\alpha(h)| - \langle \bar{d}_\alpha(h)| \,\right]$$



⟸ The 8 inequivalent loops around which the dimers are moved

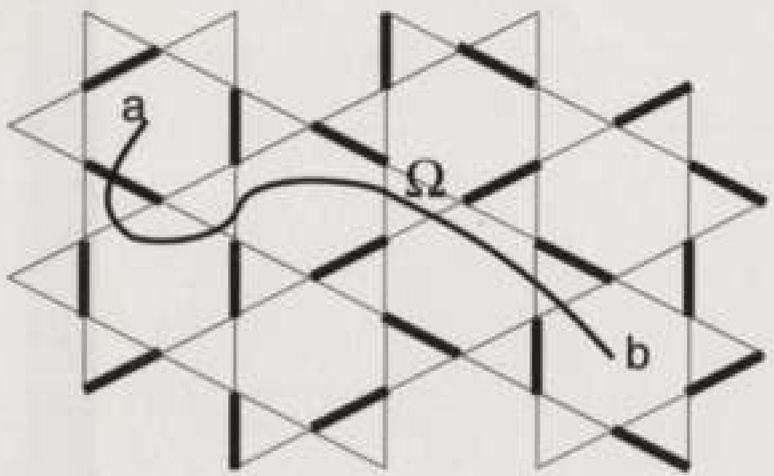
$P_\alpha(h))$ are projectors.

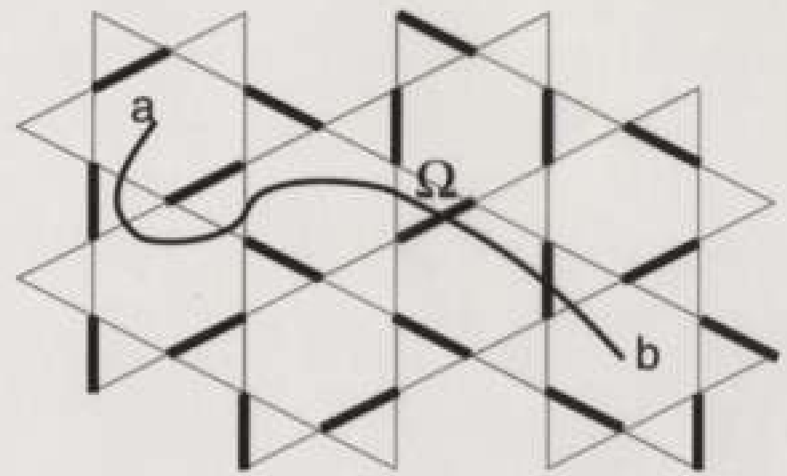
The groundstate is the symmetric superposition of coverings defined by $\sigma^x(h) = 1$

Visons

$$\sigma^x(a)\Omega(a,b) = -\Omega(a,b)\sigma^x(a) \rightarrow \Omega(a,b) \text{ flips } \sigma^x(b) \text{ and } \sigma^x(b)$$

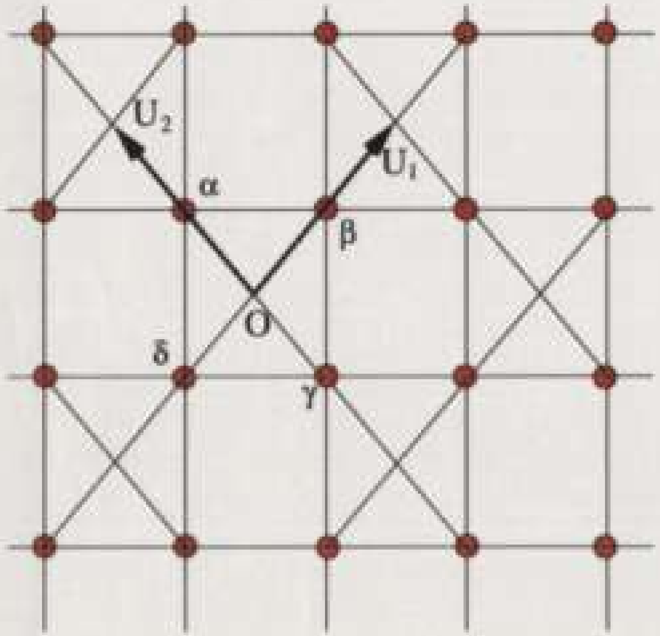
$\Omega(a,b)|0\rangle$ is a two-vison state.

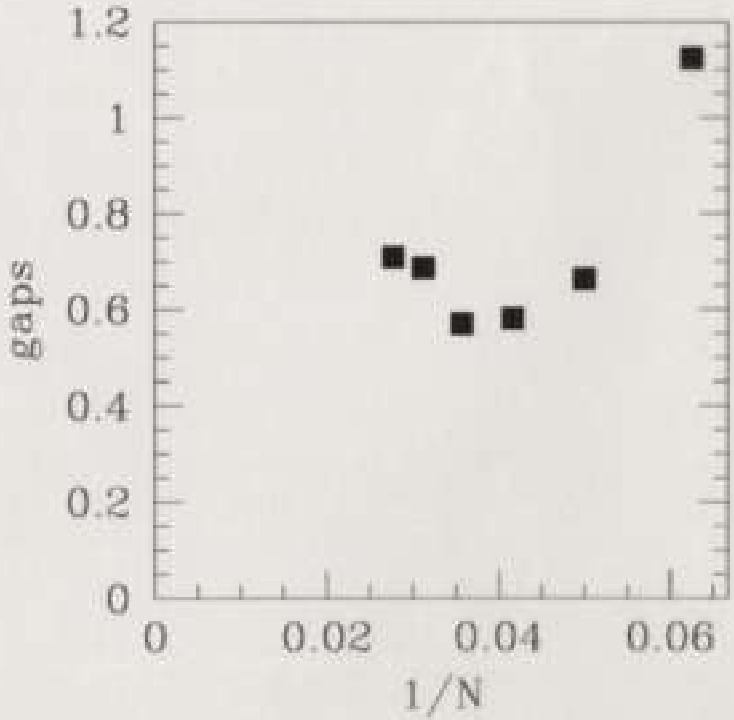




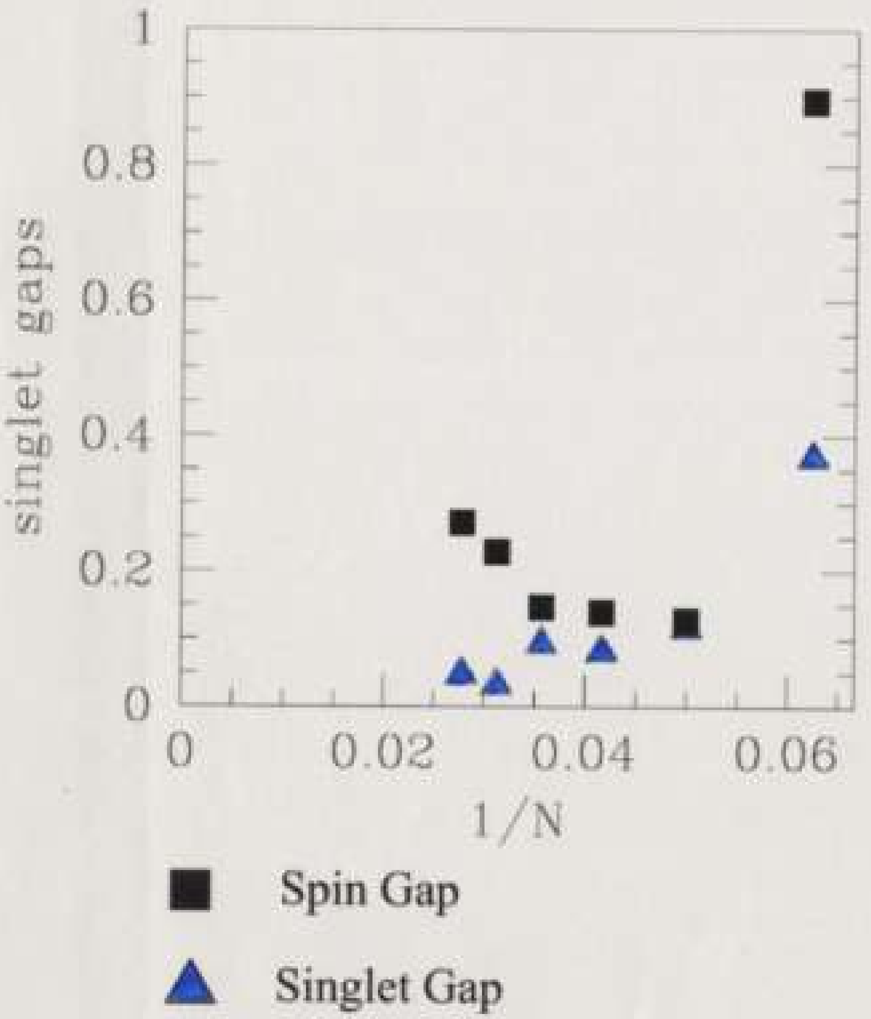
A pair of visons (located in a and b) is created by applying to the RK wave-function a factor (-1) for each dimer crossing the string Ω.
The dimerization shown there on the left appears in the linear superposition of the two-vison state with the sign -1 whereas the one on the right has the sign $+1$

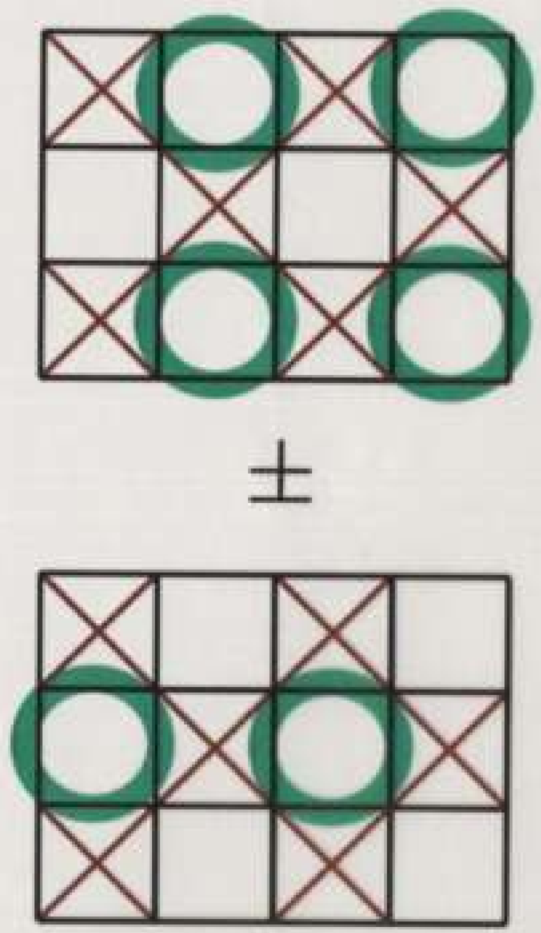
Valence Bond Crystal on the checkerboard lattice, Spin Gap





Degeneracy in the singlet sector and symmetry breaking





$$H = J \sum_{<i,j>} (S_i^z S_j^z + \frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+))$$

$$\forall i \in A, \begin{cases} S_i^+ &= \sqrt{2s - n_i}\, b_i \\ S_i^- &= b_i^\dagger \sqrt{2s - n_i} \\ s_i^z &= s - n_i \end{cases} \qquad \forall i \in B, \begin{cases} S_i^+ &= b_i^\dagger \sqrt{2s - n_i} \\ S_i^- &= \sqrt{2s - n_i}\, b_i \\ S_i^z &= -s + n_i \end{cases}$$

with $n_i = b_i^\dagger b_i$. Linearization, and Diagonalization lead to

$$H = -2JNs^2 + Js \sum_{i \in A, \delta} (n_i + n_{i+\delta} + b_i b_{i+\delta} + b_i^\dagger b_{i+\delta}^\dagger)$$

$$H = H_0^{cl} - 2JsN + \sum_{\mathbf{q} \in 1zB^*} \frac{1}{2} \omega_{\mathbf{q}} + \sum_{\mathbf{q} \in 1zB^*} N_{\mathbf{q}} \omega_{\mathbf{q}}$$

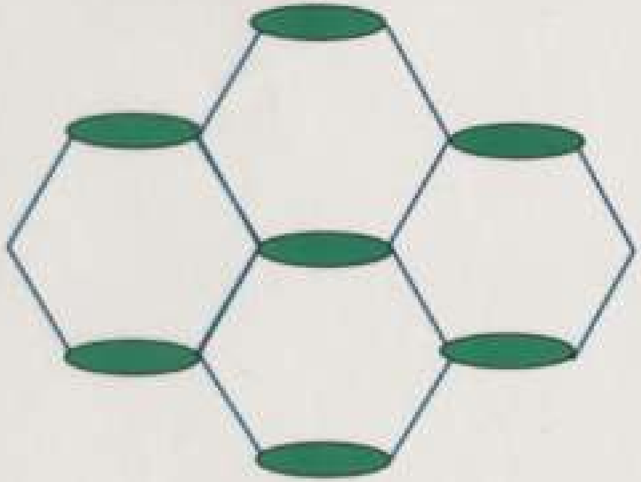
with $\omega_{\mathbf{q}} = 4Js\sqrt{1 - \gamma_{\mathbf{q}}^2}$, $\gamma_{\mathbf{q}} = (cosk_x + cosk_y)/2$. Reduct. of the order param.:

$$\delta m = -\frac{1}{N} \sum_{\mathbf{q} \in 1zB^*} \frac{1}{\sqrt{1 - \gamma_{\mathbf{q}}^2}})$$

Valence Bond Crystals

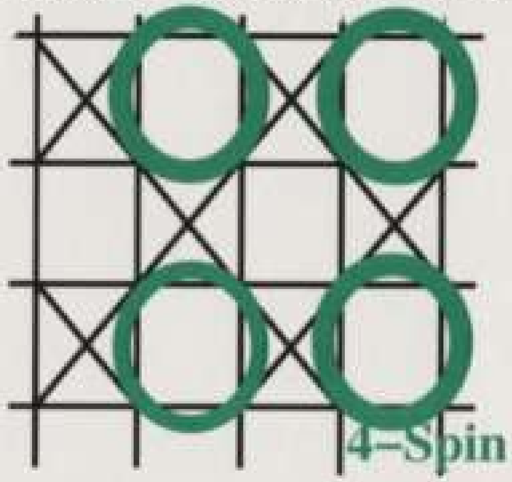
Bipartite lattices

$J_1 - J_2$ model
on the hexagonal lattice



Fouet et al Eur. Phys. J. B 2001

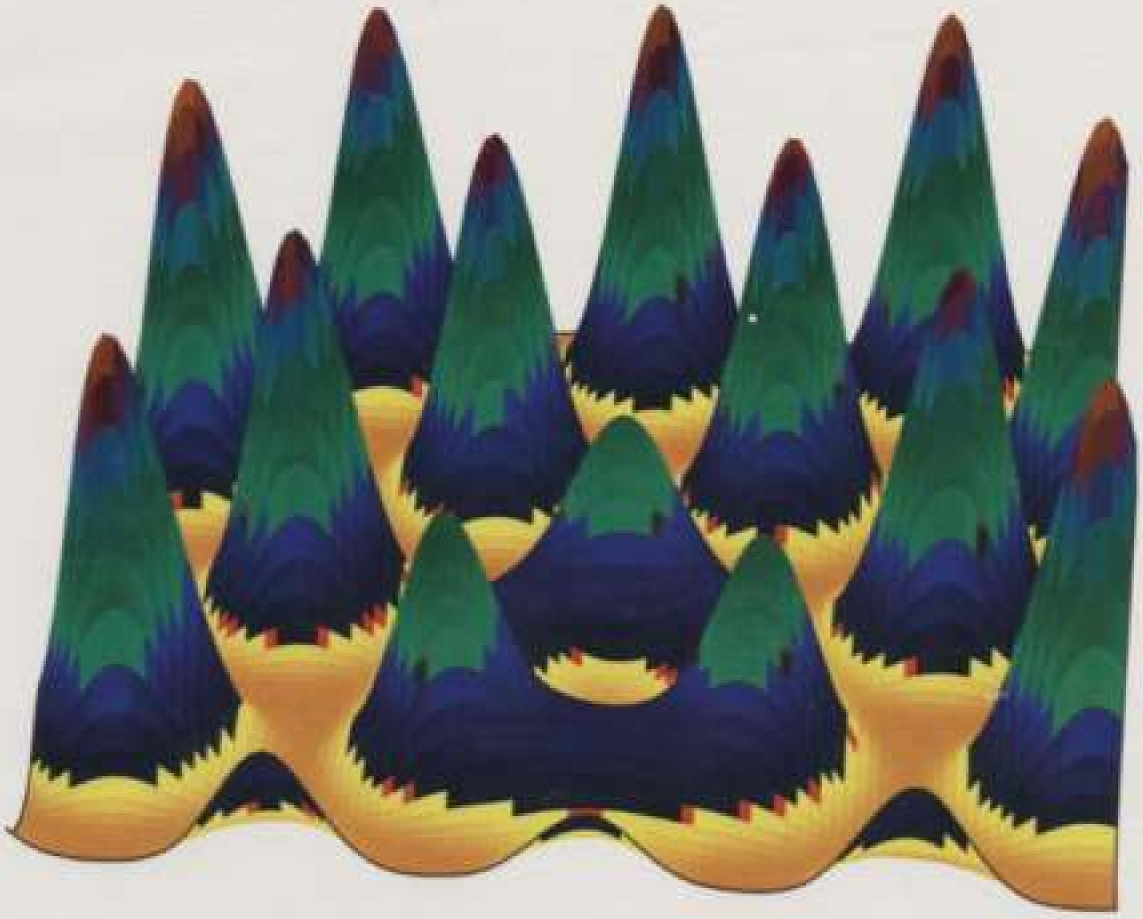
Heisenberg model
on the checkerboard lattice



Fouet et al condmat 2001
PRB 2003

Branches of gapped $\Delta S = 1,\ 0$ excitations

3-body exchange process in an ^3He monolayer

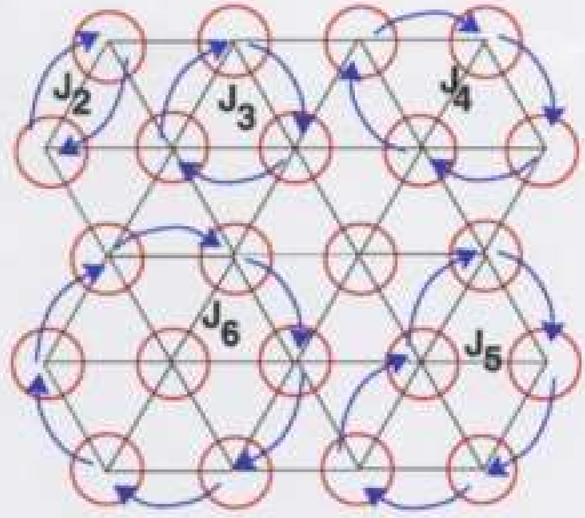


Path Integral Monte Carlo simulations:

^3He monolayer on Graphite: $\rho = 7.0 nm^{-2}$
Exchange frequencies: J_2=0.42mK, J_3=1.6mK, J_4 = 0.35mK, J_5 =0.26mK et J_6=0.21mK

Bernu et Ceperley (1999)

M. S. E. Hamiltonian for localized fermions: ^{3}He, Wigner Crystal...



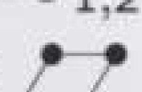
$$H = \underbrace{(+J_2 - 2J_3)}_{J_2^{\text{eff}}} \sum P_{1,2} + J_4 \sum (P_{1\ldots4} + P_{4\ldots1})$$

$$-J_5 \sum (P_{1\ldots5} + P_{5\ldots1}) + J_6 \sum (P_{1\ldots6} + P_{6\ldots1})$$

Cyclic permutations and spin operators:

② $P_{1,2} = \bullet\!-\!\bullet = 2\vec{S}_1 \cdot \vec{S}_2 + \frac{1}{2}$

③ $P_{1,2,3} + P_{3,2,1} = P_{1,2} + P_{2,3} + P_{3,1} - 1$

④ $P_{1,2,3,4} + \text{h.c} =$ 

Competition of interactions and Frustration.