Lattice Duality

I. INTRO

Duality plays a key role in understanding how to quantum disorder a superconductor, both in $1+1$ space-time dimensions and in 2+1. The key idea involves exchanging the order parameter phase ϕ for vortex degrees of freedom. In $1+1$ dimensions these are point-like space-time vortices,[1] whereas in $2+1$ there are point like vortices in space which propogate in time.[2] However, the physics of duality is perhaps most accessible when carried out on the lattice. Here we review how lattice duality is implemented in both $1+1$ and $2+1$ dimensions.[1,2] For simplicity we first Wick rotate to Euclidian space, and rescale imaginary time to set the charge velocity to one. The appropriate lattice model is then simply a 2d square lattice or 3d cubic lattice xy model. In the latter case, we also want to include an external gauge field A.

The degrees of freedom which live on the sites of the square or cubic lattice (denoted by a vector of integers $\vec{x}$) are the phases $\phi_x \in [0, 2\pi]$. As usual, the gauge field lives on the links. Discrete lattice derivatives are denoted by

$$\Delta_\mu \phi_x = \phi_{x+\mu} - \phi_x, \tag{1.1}$$

where $\mu = x, y$ for the square lattice and $\mu = x, y, z$ for the cubic lattice and $x + \mu$ denotes the nearest neighbor site to $\vec{x}$ in the $\hat{\mu}$ direction. The gauge field is minimally coupled via,

$$\Delta_\mu \phi_x \to \Delta_\mu \phi_x + A^\mu_x. \tag{1.2}$$

Consider the partition function,

$$Z = \int_0^{2\pi} \prod_x d\phi_x exp[\sum_{x,\mu} V_\kappa(\Delta_\mu \phi_x)]. \tag{1.3}$$

Here the periodic "Villain" potential V_κ is given by,

$$exp[V_\kappa(\Delta\phi)] = \sum_{J=-\infty}^{\infty} e^{-\kappa J^2/2} e^{iJ\Delta\phi}, \tag{1.4}$$

with integer J. When $\kappa >> 1$ only the terms with $J = 0, \pm 1$ contribute appreciably in the sum and this reduces to the more familiar form:

$$V_\kappa(\Delta\phi) = K \cos(\Delta\phi), \tag{1.5}$$

with $K = 2exp(-\kappa/2)$.

The partition function can thus be expressed as a sum over both ϕ and a vector of integers, $\vec{J}_x$, with components J^μ_x living on the *links* of the lattice:

$$Z = \int \prod_x d\phi \sum_{[\vec{J}]} e^{-S} \equiv Tr_{\phi,\vec{J}}\ e^{-S}, \tag{1.6}$$

with action

$$S = S_0 + \sum_x i(\vec{\Delta} \cdot \vec{J}_x)\phi_x, \tag{1.7}$$

$$S_0 = \frac{\kappa}{2} \sum_x |\vec{J}_x|^2. \tag{1.8}$$

In this form the integration over ϕ can be explicitly performed giving

$$Z = Tr'_{\vec{J}}\ e^{-S_0}, \tag{1.9}$$

where the prime on the trace indicates a divergenceless *constraint* at each site of the lattice:

$$\vec{\Delta} \cdot \vec{J}_x = 0. \tag{1.10}$$

In the presence of a gauge field there is an additional term in the action of the form,

$$S_A = i \sum_x \vec{J}_x \cdot \vec{A}_x. \tag{1.11}$$

It is thus clear that the integer of vectors $\vec{J}$ can be interpreted as a conserved electrical current flowing on the links of the lattice. The divergenceless constraint on this electrical 3-current can be imposed automatically by reexpressing $\vec{J}$ as a curl of an appropriate *dual* field. Consider first the 2d case.

A. Two dimensions

To guarantee divergenceless we set the current equal to the (2d) curl of a scalar field, θ_x:

$$2\pi J^\mu_x = \epsilon_{\mu\nu} \Delta_\nu \theta_x, \tag{1.12}$$

so that the action becomes

$$S_0(\theta) = \frac{\kappa}{8\pi^2} \sum_{x,\mu} (\Delta_\mu \theta_x)^2. \tag{1.13}$$

To insure that $\vec{J}$ is an *integer* field, θ must be constrained to be 2π times an integer. This additional constraint can be imposed by introduction of yet another integer field, n_x, which will be interpreted as the (space-time) vortex density. The partition is thereby re-expressed as (dropping an unimportant multiplicative constant),

$$\tilde{Z} = \int_{-\infty}^{\infty} \prod_x d\theta_x \sum_{[n_x]} e^{-S}, \tag{1.14}$$

with

$$S = S_0(\theta) + \sum_x [\frac{\tilde{\kappa}}{2} n_x^2 + i n_x \theta_x]. \quad (1.15)$$

For $\tilde{\kappa} = 0$ the summation over n_x gives a sum of delta functions restricting $\theta_x/2\pi$ to be integer. But we have softened this constraint, introducing a vortex "core" energy $\tilde{\kappa} \neq 0$.

At this stage one could perform the Gaussian integral over θ, to obtain a logarithmically interacting plasma of (space-time) vortices. Alternatively, for $\tilde{\kappa} >> 1$ the summation over n_x can be performed giving,

$$S = S_0(\theta) - u \sum_x cos(\theta_x), \quad (1.16)$$

with $u = 2exp(-\tilde{\kappa}/2)$. Upon taking the continuum limit, $\theta_x \to \theta(x)$, one recovers the (Euclidian) sine-Gordon theory, $S = \int d^2x \mathcal{L}$ with

$$\mathcal{L} = \frac{\kappa}{8\pi^2} (\vec{\nabla}\theta)^2 - u \cos(\theta). \quad (1.17)$$

B. Three dimensions

In three dimensions the divergenceless integer 3-current $\vec{J}$ can be written as the curl of a *vector* field, $\vec{a}$:

$$2\pi \vec{J}_x = \vec{\Delta} \times \vec{a}_x. \quad (1.18)$$

As in 2d one imposes the integer constraint (softly) by introducing an integer vortex field, in this case a 3-vector $\vec{j}$, to express the partition function as,

$$\tilde{Z} = \int_{-\infty}^{\infty} \prod_x d\vec{a}_x \sum_{[\vec{j}_x]} e^{-S}, \quad (1.19)$$

with

$$S = S_0(\vec{a}) + \sum_x [\frac{\tilde{\kappa}}{2} |\vec{j}_x|^2 - i\vec{j}_x \cdot \vec{a}_x], \quad (1.20)$$

$$S_0(\vec{a}) = \frac{\kappa}{8\pi^2} \sum_x |\vec{\Delta} \times \vec{a}_x|^2. \quad (1.21)$$

The integer vector field $\vec{j}$ is the vortex 3-current, "minimally" coupled to $\vec{a}$. To see that the vortex 3-current is conserved, it is convenient to decompose the vector field $\vec{a}$ into transverse and longitudinal pieces: $\vec{a} = \vec{a}_t - \vec{\Delta}\theta$, with θ_x a scalar field. The action becomes,

$$S = S_0(\vec{a}) + \sum_x [\frac{\tilde{\kappa}}{2} |\vec{j}_x|^2 + i\vec{j}_x \cdot (\vec{\Delta}\theta_x - \vec{a}_x)], \quad (1.22)$$

where we have dropped the subscript "t" on $\vec{a}$. The partition function follows from integrating over both $\vec{a}$ *and* θ and summing over integer $\vec{j}$. Integrating over θ leads to the expected condition: $\vec{\Delta} \cdot \vec{j} = 0$. Alternatively, for $\tilde{\kappa} >> 1$ one can perform the summation over $\vec{j}$ to arrive at an action depending on θ and $\vec{a}$:

$$S = S_0(\vec{a}) - K \sum_{x,\mu} \cos(\Delta_\mu \theta_x - a_x^\mu), \quad (1.23)$$

with $K = 2exp(-\tilde{\kappa}/2)$.

In the presence of a gauge field A^μ there is an additional term in the action of the form,

$$S_A = \frac{i}{2\pi} \sum_x (\vec{\Delta} \times \vec{a}_x) \cdot \vec{A}_x, \quad (1.24)$$

which follows directly from Eqn. 1.11 and 1.18.

At this stage one can take the continuum limit, letting $\vec{a}_x \to \vec{a}(x)$ and $\theta_x \to \theta(x)$. Upon expanding the cosine for small argument one obtains $S = \int d^3x \mathcal{L}$ with (Euclidian) Lagrangian

$$\mathcal{L} = \frac{\kappa}{8\pi^2} (\vec{\nabla} \times \vec{a})^2 + \frac{K}{2} (\vec{\nabla}\theta - \vec{a})^2. \quad (1.25)$$

In this dual representation, the vortex 3-current (which follows from $\partial\mathcal{L}/\partial\vec{a}$) is given by $\vec{j}^v = K(\vec{\nabla}\theta - \vec{a})$. Notice that the vortices are minimally coupled to the "vector potential" $\vec{a}$, whose curl equals the electrical 3-current. The field θ can be interpreted as the phase of a vortex operator. In fact it is convenient to introduce such a complex vortex field before taking the continuum limit:

$$e^{i\theta_x} \to \Phi(\vec{x}). \quad (1.26)$$

The continuum limit can then be taking *retaining* the full periodicity of the cosine potential. The appropriate vortex Lagrangian replacing the second term in Eqn. 1.25 is,

$$\mathcal{L}_v = \frac{K}{2} |(\vec{\nabla} - i\vec{a})\Phi|^2 + V_\Phi(|\Phi|). \quad (1.27)$$

The vortex current operator becomes,

$$\vec{j}^v = K Im[\Phi^*(\vec{\nabla} - i\vec{a})\Phi]. \quad (1.28)$$

If the potential is expanded for small Φ as $V_\Phi(X) = r_\Phi X^2 + u_\Phi X^4$, the full dual theory is equivalent to a Ginzburg-Landau theory for a classical three-dimensional superconductor. Inclusion of the original gauge field A^μ leads to an additional term in the dual Lagrangian:

$$\mathcal{L}_A = \frac{i}{2\pi} (\vec{\nabla} \times \vec{a}) \cdot \vec{A}. \quad (1.29)$$

After Wick rotating back to real time and restoring the velocity, $\mathcal{L} + \mathcal{L}_A$ becomes identical to the continuum dual vortex Lagrangian.

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[1] See J.V. Jose, L.P. Kadanoff, S. Kirkpatrick and D.R. Nelson, Phys. Rev. **B16**, 1217 (1978), and references therein.

[2] C. Dasgupta and B.I. Halperin, Phys. Rev. Lett. **47**, 1556 (1981); M.P.A. Fisher and D.H. Lee, Phys. Rev. **B39**, 2756 (1989).