The Kosterlitz-Thouless Phase Transition

Steven M. Girvin
Department of Physics
Indiana University
Bloomington, IN 47405

August 1, 2000

1 Outline

- 2D XY Model

- Continuous Symmetry, Mermin-Wagner Theorem

- Vortices

- Electrostatics and Coulomb Gas Representations

- Experiments on ^{4}He films and superconducting films

- Duality Transformations

- Sine-Gordon Representation and RNG Analysis (Still Under Construction)

The reader is directed to Refs. [1–9] which are a useful general background.

2 Gauge Invariance and Continuous Symmetry

In the previous lecture I discussed the nature of symmetry breaking in superconductors. The superconducting state is described by a complex order

parameter whose properties are essentially those of a macroscopic wave function

$$|\Psi| \, e^{i\varphi(\vec{r})} \equiv \left\langle \psi^\dagger_\uparrow(\vec{r}) \psi^\dagger_\downarrow(\vec{r}) \right\rangle . \tag{1}$$

In the Ginzburg-Landau mean field theory one neglects fluctuations and the mean field transition temperature is defined by the point at which the amplitude of the order parameter first becomes non-zero. In reality there are always thermal fluctuations so that the rms value of the magnitude of the order parameter is always non-zero. It turns out that for the special case of two dimensions, the true transition temperature can be substantially below the mean-field T_c and is controlled primarily by fluctuations in the phase of the order parameter. In our discussion, we will therefore take the order parameter amplitude to be unity everywhere.

The microscopic Hamiltonian H commutes with electron number

$$[H, \hat{N}] = 0, \tag{2}$$

where

$$\hat{N} \equiv \sum_\sigma \psi^\dagger_\sigma \psi_\sigma. \tag{3}$$

Let us define the unitary transformation

$$\mathcal{U} \equiv e^{+i\theta\hat{N}}. \tag{4}$$

Particle number conservation is represented by the fact that H is invariant under this transformation

$$\mathcal{U} H \mathcal{U}^\dagger = H. \tag{5}$$

The transformation $\mathcal{U}$ is a special case of the more general case of a gauge transformation in which the phase angle θ varies with position. See Appendix A for further details. Hence gauge invariance is equivalent to particle number conservation. Note however that the order parameter has less symmetry than the Hamiltonian and it is *not* invariant under this transformation and therefore is not gauge invariant

$$\left\langle \mathcal{U} \psi^\dagger_\uparrow(\vec{r}) \psi^\dagger_\downarrow(\vec{r}) \mathcal{U}^\dagger \right\rangle = e^{2i\theta} \left\langle \psi^\dagger_\uparrow(\vec{r}) \psi^\dagger_\downarrow(\vec{r}) \right\rangle = e^{i[\varphi(\vec{r}) + 2\theta]}. \tag{6}$$

The factor of two in the exponential tells us that we have a 'charge two' order parameter.

Charge conservation guarantees that the energy is invariant under the continuous global $U(1)$ transformation

$$\varphi(\vec{r}) \longrightarrow \varphi(\vec{r}) + 2\theta. \tag{7}$$

Neglecting fluctuations in the amplitude of the order parameter, the simplest form consistent with this symmetry is

$$U = \frac{1}{2}\rho_s \int d^2r \, |\nabla \varphi|^2 \,, \tag{8}$$

where ρ_s is called the (bare) 'spin stiffness' or 'superfluid density' and in 2D has units of energy. It therefore sets the characteristic temperature scale for the system.

This 2D XY model is used to describe superfluid ^{4}He and ^{3}He films as well as superconducting films. [In the case of superconducting films, it is important that there be sufficient disorder and/or that the films be sufficiently thin that the magnetic penetration length is much larger than the system size. Otherwise magnetic screening needs to be included in the model and the Kosterlitz-Thouless transition is destroyed.] The model is also relevant to some high T_c materials like BSSCO which has extremely weak interlayer coupling. The actual phase ordering transition is 3D but there is a temperature regime in which the layers act approximately independently.

Because this model has a continuous symmetry, the Mermin-Wagner theorem guarantees that it can not show true spontaneous symmetry breaking in 2D. That is, true long-range order

$$\lim_{r \to \infty} \langle e^{-i\varphi(\vec{r})} e^{i\varphi(\vec{0})} \rangle \neq 0 \tag{9}$$

is impossible at any finite temperature. To see why this is so, consider the partition function at inverse temperature β

$$Z = \int \mathcal{D}\varphi \, e^{-\beta\frac{1}{2}\rho_s \int d^2r \, |\nabla\varphi|^2}, \tag{10}$$

where $\int \mathcal{D}\varphi$ indicates a functional integral or partition sum over all possible configurations of the field φ. The first suspicious thing we notice is that this seems to be a simple gaussian model which can't possibly have any phase transition at all. We will come back to this subtlety shortly. For now let us take it at face value and use that fact that for a gaussian model

$$G(r) \equiv \langle e^{-i\varphi(\vec{r})} e^{i\varphi(\vec{0})} \rangle = e^{-\frac{1}{2}\langle [\varphi(\vec{r}) - \varphi(\vec{0})]^2 \rangle}. \tag{11}$$

Using the fact that the quadratic form is diagonal in Fourier space we have

$$\langle \varphi_{-\vec{q}} \varphi_{\vec{q}} \rangle = \frac{1}{\beta \rho_s q^2} \tag{12}$$

From this

$$\left\langle \varphi(\vec{r}) \varphi(\vec{0}) - \varphi(\vec{0}) \varphi(\vec{0}) \right\rangle = \int \frac{d^2 q}{(2\pi)^2} \frac{1}{\beta \rho_s q^2} [e^{i\vec{q} \cdot \vec{r}} - 1] \tag{13}$$

The momentum integral has an ultraviolet cutoff $1/a$ set by some microscopic length in the problem (such as the Cooper pair size). In the infra-red the logarithmic divergence is cut off by $1/r$. Thus for $r \gg a$ we can approximate the integral as

$$\frac{1}{2\pi \beta \rho_s} \int_{1/r}^{1/a} dq \, \frac{1}{q} [0 - 1]. \tag{14}$$

That is, we take the exponential to cancel the -1 term for $qr < 1$ and assume that the exponential oscillates so rapidly for $qr > 1$ that we can replace it by zero. The result for the correlation function is then

$$\left\langle \varphi(\vec{r}) \varphi(\vec{0}) - \varphi(\vec{0}) \varphi(\vec{0}) \right\rangle = -\frac{1}{2\pi \beta \rho_s} \ln \left(\frac{r}{a} \right). \tag{15}$$

Finally we obtain (for $r \gg a$) the result that G decays to zero algebraically

$$G(r) \sim \left(\frac{a}{r} \right)^{\eta(T)} \tag{16}$$

where

$$\eta(T) = \frac{k_{\mathrm{B}} T}{2\pi \rho_s}. \tag{17}$$

This result has several important implications. First it demonstrates that there is no long-range order for $T > 0$, in agreement for with the Mermin-Wagner theorem. Second it seems to indicate that the system is 'critical' at any temperature–there is no characteristic length scale associated with exponential decay since G decays algebraically for all $T > 0$.

Something has to be wrong however. Surely at temperatures $T \gg \rho_s$, the quasi-long-range correlations should be destroyed and G should decay exponentially. Indeed, there is something wrong. Our gaussian model has neglected the existence of topological defects, vortices, in the order parameter. There are two ways to remedy this problem. We can include in our

partition sum singular configurations of the φ field such as those illustrated in Figs.(1-2). The second way is to modify the model so that these excitations appear naturally by introducing a lattice regularization

$$H = -J \sum_{\langle ij \rangle} \cos(\varphi_i - \varphi_j), \tag{18}$$

where i and j label lattice sites on (say) a square lattice of lattice constant a and the sum is over near neighbors. Defining 2D spin vectors $\vec{S}$ by $S_x + iS_y = e^{i\varphi}$, we can map this onto a model of a 2D magnet with easy plane anisotropy

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j. \tag{19}$$

If we assume $T \ll J$ so that the spins are nearly parallel on neighboring sites then we can expand the cosine to second order to obtain the lattice gaussian model

$$H \approx \frac{1}{2} J \sum_{\langle ij \rangle} (\varphi_i - \varphi_j)^2. \tag{20}$$

The continuum approximation to this model is identical to Eq.(8) if we take

$$J = \rho_s. \tag{21}$$

This is the reason that ρ_s is often referred to as the 'spin stiffness' instead of the superfluid density.

The key feature of the lattice regularization of the model used in Eq. (18) is that it has an additional symmetry not present in the gaussian approximation used in Eqs. (8) and (20). In addition to being invariant under the continuous *global* $U(1)$ transformations in Eq.(7), it is also invariant under discrete *local* transformations which change any single spin

$$\varphi_i \longrightarrow \varphi_i \pm 2\pi. \tag{22}$$

This is a key feature because it permits the existence of vortices. A vortex is a topological defect in which the phase winds by $\pm 2\pi$ in going around the defect as illustrated in Figs. (1-2)

$$\oint d\vec{r} \cdot \vec{\nabla} \varphi = 2\pi n_W \tag{23}$$

5

where $n_W = \pm 1$ is the topological 'charge' or winding number. [1] In the presence of a vortex there is a discontinuity in φ where the 2π value is adjacent to the $\varphi = 0$ value. Of course this is totally missed in the gaussian approximation.

To put it in more formal language, it is crucial to recognize that φ is a *compact* variable living on the unit circle. The gaussian model implicitly takes it to be non-compact and living on the real line.

We can now ask ourselves how much energy it costs to introduce a vortex into the system. In the continuum limit the phase field configuration for a right-handed vortex centered on the origin as shown in Fig. (1) is simply

$$\varphi(\vec{r}) = \theta(\vec{r}) + \theta_0 \tag{24}$$

where $\theta = \arctan(y/x)$ is the azimuthal angle at position $\vec{r}$ and θ_0 is an arbitrary constant. Hence we have $\vec{\nabla}\varphi = \frac{\hat{\theta}}{r}$ and the energy cost in a system of size L is

$$E = \int_a^L dr\, 2\pi r \frac{1}{2}\rho_s \frac{1}{r^2} \sim \pi\rho_s \ln\left(\frac{L}{a}\right) + E_c. \tag{25}$$

The integration at short distances is cutoff by the core size $\xi_0 \sim a$ of the vortex, and E_c is the core energy. The core energy in the XY spin model is on the scale of ρ_s and is not an independent parameter. In a model with more than near-neighbor couplings or in the continuum Ginzburg-Landau model where the diameter of the core region depends on the GL coherence length ξ_0, the core energy is an adjustable parameter that has to be fit to experiment.

The integration at large distances diverges logarithmically with system size. We thus see that the vortex costs an infinite amount of energy in the thermodynamics limit, so perhaps we were justified after all in neglecting vortices in our original calculation. This is not the case however, as can be seen by computing the entropy. The number of independent places where the vortex can be located is $\sim L^2/a^2$ and so the entropy is

$$S = k_{\mathrm{B}} \ln \frac{L^2}{a^2}. \tag{26}$$

The free energy cost to introduce a single vortex is therefore

$$F = \pi\rho_s \ln\left(\frac{L}{a}\right) + E_c - 2k_{\mathrm{B}}T \ln \frac{L}{a}. \tag{27}$$

---

[1]In principle it is possible to have vortices with higher winding numbers but these are generally expensive energetically and can be safely ignored.

In the thermodynamic limit, this changes sign from positive to negative as T increases through the Kosterlitz-Thouless temperature

$$T_{\mathrm{KT}} = \frac{\pi}{2}\rho_s. \tag{28}$$

At this point vortices begin to proliferate and G(r) begins to decay exponentially on a length scale given by the typical spacing between vortices

$$G(r) \sim e^{-r/\xi(T)} \tag{29}$$

where $\xi(T)$ scales with the inverse square root of the vortex density. It can be shown from a renormalization group analysis that the decay length diverges extremely rapidly near the critical

$$\xi(T) \sim e^{-b|T-T_{\mathrm{KT}}|^{-1/2}}. \tag{30}$$

[This divergence is faster than any power law and so the correlation length exponent (defined by $\xi \sim |T - T_{\mathrm{KT}}|^{-\nu}$) obeys $\nu = \infty$.]

Below T_{KT} vortices can exist only in bound pairs with opposite vorticity [see Fig. (3)] held together by a logarithmic confining potential (discussed in the next section). Eq. (28) is not exact because it considers only a single vortex rather than many interacting vortices. As we will see, the bound pairs of vortices at short distances renormalize the bare spin stiffness ρ_s measured on long length scales. If however we use this spin stiffness measured on long length scales then this expression for the critical temperature is exact.

Above T_{KT} the finite density of unbound vortices causes the spin stiffness to drop discontinuously to zero. It is a peculiar feature of this 2D transition, that the stiffness is discontinuous even though the transition itself is continuous. Notice also that in addition to a 'universal jump'

$$\frac{\rho_s}{T_{\mathrm{KT}}} = \frac{2}{\pi} \tag{31}$$

in the superfluid density, the algebraic decay exponent in Eq. (17) takes on the universal value

$$\eta = 1/4 \tag{32}$$

as T_{KT} is approached from below. The universal jump in superfluid density is beautifully illustrated in the data of Bishop and Reppy [10] reproduced in Fig. (4).

All of this remarkable phenomenology can be understand within an electrostatics analogy discussed in the next section.

3 Electrostatics Representation

It is very useful to develop a representation in which vortex singularities in the φ field play the role of point charges in a 2D electrostatics analogy. Let us therefore define a 'displacement field' by

$$\vec{D} = \vec{\nabla}\varphi \times \hat{z}. \tag{33}$$

Examples of this are shown in Figs. (1-2). Using Eq. (24) we see that

$$\vec{\nabla} \cdot \vec{D} = \vec{\nabla} \times \vec{\nabla}\varphi = 2\pi n_w \, \delta^2(r). \tag{34}$$

This is simply the 2D version of the Poisson equation from electrostatics for a point charge whose strength is the vortex 'topological charge' (winding number). The fact that the coefficient of the delta function is 2π rather than the familiar 4π is simply from the fact that the circumference of a circle is $2\pi r$ while the area of a sphere is $4\pi r^2$. The electric (displacement) field from a point charge of unit strength in 2D (equivalent to a line charge in 3D) is

$$\vec{D} = \frac{\hat{r}}{r} \tag{35}$$

and the flux passing through a circle of radius r is 2π.

Defining the 'dielectric constant'

$$\epsilon = \frac{1}{2\pi\rho_s}, \tag{36}$$

and defining the electric field via

$$\vec{D} = \epsilon\vec{E}, \tag{37}$$

Eq. (8) for the energy of the XY model becomes the usual (2D) electrostatics expression

$$U = \frac{1}{4\pi} \int d^2r \, \vec{E} \cdot \vec{D}. \tag{38}$$

The electrostatic potential V produced by a charge can be found from integrating

$$\vec{E} = n_w \frac{\hat{r}}{\epsilon r} \tag{39}$$

to obtain (after a somewhat arbitrary but convenient choice of integration constant)

$$V(r) = -\frac{n_w}{\epsilon} \ln\left(\frac{r}{a}\right). \tag{40}$$

The statistical mechanics of a collection of interacting vortices is thus that of a 2D plasma of charges with potential energy

$$U = -\frac{1}{\epsilon} \sum_{i<j} q_i q_j \ln\left|\frac{\vec{r}_i - \vec{r}_j}{a}\right|, \tag{41}$$

where $q_i = \pm 1$ is the winding number of the ith vortex.

It is a little strange that the magnitude of the interaction *increases* with distance, but still we see that opposite charges attract. It costs an infinite energy to pull a bound pair apart and so the systems exhibits charge confinement at low temperatures and is an 'insulator' in the electrostatics analogy. In the language of the superconductor however this 'insulator' is the ordered dissipationless superconducting phase. For $T > T_{\mathrm{KT}}$ the bound pairs ionize due to the effects of entropy and mutual screening and the system becomes a conducting plasma. This 'metal' in the electrostatics analogy is highly dissipative due to the vortex motion and so is the 'normal' non-superconducting phase in the language of the superconductor.

Having established the electrostatics analogy it is now extremely easy to see what the effect of a uniform supercurrent is on a bound vortex pair. The uniform supercurrent density is established by a background phase gradient in the order parameter

$$\vec{J}_s = \frac{2e}{\hbar} \rho_s \vec{\nabla}\varphi. \tag{42}$$

This expression can be derived from general gauge invariance considerations (see Appendix A), but here we simply note that the prefactor has the correct engineering units. The pseudo 'electric field' established by this phase gradient is simply

$$\vec{E} = \frac{\hbar}{2e} \frac{1}{\epsilon \rho_s} \vec{J}_s \times \hat{z} = \frac{h}{2e} \vec{J}_s \times \hat{z}. \tag{43}$$

The force that this produces on a (stationary) vortex with 'charge' $q = n_w$ is

$$\vec{F} = n_w \frac{h}{2e} \vec{J}_s \times \hat{z} \tag{44}$$

is at right angles to the current and is called the Magnus force (or in a different interpretation, the Lorentz force). This force is opposite in sign for each

9

member of a bound pair of vortices and so the bound pair can polarize but will not otherwise respond (see below however). This polarization is exactly like electric polarization in a medium and contributes to the renormalization of the dielectric constant. The increase in ϵ in this case corresponds to a *decrease* in the spin stiffness. The weakly bound pairs become infinitely polarizable at T_{KT} and drive $1/\epsilon$ and therefore ρ_s discontinuously to zero.

If there are unbound vortices (as is the case above T_{KT}), they drift under the influence of the Magnus force, dissipating energy in their normal cores and in the lattice, thereby destroying the superfluid or superconducting state. For weak applied force we expect linear response to hold and the drift velocity will obey

$$\vec{v} = \mu \vec{F} \tag{45}$$

where the 'mobility' μ is determined by various microscopic details and the normal state resistivity of the material.

The phase difference between the top and bottom of the sample of height L slips by 2π each time a vortex crosses the sample horizontally. The time it takes a vortex to drift across the sample of width W is W/v and the number of vortices in the sample is $n_V LW$. Hence it follows that the rate of phase slip due to an applied current J is

$$\dot{\varphi} = 2\pi L n_V \mu \frac{h}{2e} J, \tag{46}$$

where n_V is the total vortex density $(n_+ + n_-)$. From the Josephson relation

$$\hbar\dot{\varphi} = 2eV, \tag{47}$$

the (true) voltage drop is

$$V = L n_V \mu \left(\frac{h}{2e}\right)^2 J, \tag{48}$$

and the (true) resistivity of the material is

$$\rho = \left(\frac{h}{2e}\right)^2 n_V \mu = \sigma_V, \tag{49}$$

where the flux quantum $\frac{h}{2e}$ can be interpreted as the vortex 'charge' and σ_V as the vortex 'conductivity' in analogy to the usual expression for electron

conductivity $\sigma = e^2 n\mu$. This is an example of a 'duality relation' in which the resistivity of the particles is related to the conductivity of the 'dual particles,' the vortices.

Using the fact that the vortex density $n_V \sim \xi^{-2}$ and Eq. (30) we see that above T_{KT} the resistance varies extremely rapidly with temperature

$$R \sim e^{2b|T-T_{\mathrm{KT}}|^{-1/2}}. \tag{50}$$

Experimental confirmation of this is shown in Fig. (5).

I remarked above that bound pairs of vortices don't do much. This isn't quite correct. It turns out that instead of simply polarizing, each bound pair has a small probability of being 'ionized' by the 'electric field' (supercurrent). This means that the critical current is actually zero and there is always some dissipation in a 2D film (although the dissipation can be exceedingly weak). To see how this works, consider the energy of a bound pair of separation d under the combined influence of their mutual attraction and the applied current

$$U = 2\pi\rho_s \ln\left(\frac{d}{a}\right) - \frac{h}{2e}Jd. \tag{51}$$

The second term is linear in the separation because the force from the external 'field' is constant. Consider the case of very tiny J. Then for small d the second term is not important and the interaction energy rises logarithmically with distance. This confining potential is eventually overwhelmed however (even for tiny J) because the second term varies linearly with d while the first term rises only logarithmically. Extremizing this expression with respect to d shows that the maximum in the energy cost

$$U(d^*) = 2\pi\rho_s \ln\left(\frac{J_0}{J}\right) \tag{52}$$

occurs at a distance

$$d^* = a\frac{J_0}{J} \tag{53}$$

where

$$J_0 \equiv \frac{2e\rho_s}{\hbar a} \tag{54}$$

is a measure of the maximum possible scale of the current density. Random fluctuations will thermally activate this pair over the energy barrier at a rate given by the Arrhenius law

$$R_{\mathrm{ionization}} \propto e^{-\beta U^*}. \tag{55}$$

11

From the law of mass action, we expect that the rate of recombination of unbound pairs of vortices obeys

$$R_{\text{recombination}} \propto n_+ n_- \propto n_V^2. \tag{56}$$

Equating these two rates in steady state shows that the non-equilibrium density of free (unbound) vortices is

$$n_V \propto \sqrt{R_{\text{ionization}}} \propto e^{-\beta U^*/2} \propto \left(\frac{J}{J_0}\right)^{\frac{1}{2\eta(T)}}, \tag{57}$$

where $\eta(T)$ is the exponent defined in Eq. (17). Since the resistivity is proportional to this quantity we immediately have a scaling law for the non-linear current voltage curve

$$V \sim J^a \tag{58}$$

where from Eqs. (17) and (28), the exponent a has the simple form

$$a = 1 + 2\frac{T_{\text{KT}}}{T}. \tag{59}$$

For $T > T_{\text{KT}}$ the system is in the normal state (since unbound pairs can be thermally excited even without a driving current) and the behavior will be linear (ohmic) for small J. Thus we expect a universal jump in the exponent

$$a = 3 \rightarrow a = 1, \tag{60}$$

as we pass through T_{KT}. This is in fact observed experimentally as illustrated in Fig. (6) and Fig. (7) reproduced from Mooij's article in Ref. [2].

A   Gauge Invariance

We are used to thinking about gauge invariance in first quantization terms. If we make a local phase change on a single-particle wave function

$$\psi(\vec{r}) \longrightarrow \mathcal{U}\psi(\vec{r}) \tag{61}$$

where

$$\mathcal{U} = e^{i\theta(\vec{r})} \tag{62}$$

then the velocity operator transforms according to

$$\mathcal{U} \left(\vec{p} + \frac{e}{c}\vec{A} \right) \mathcal{U}^\dagger = \vec{p} + \frac{e}{c}\vec{A} - \hbar\vec{\nabla}\theta \tag{63}$$

which we can interpret as a gauge change

$$\vec{A} \longrightarrow \vec{A} - \frac{\hbar c}{e}\vec{\nabla}\theta. \tag{64}$$

The analogous transformation in second quantized language is

$$\mathcal{U} = \exp\left(+i\int d^2r\,\theta(\vec{r})\hat{n}(\vec{r}) \right) \tag{65}$$

where

$$\hat{n}(\vec{r}) = \psi^\dagger(\vec{r})\psi(\vec{r}) \tag{66}$$

is the local density operator. The analog of Eq.(61) is

$$\psi^\dagger(\vec{r}) \longrightarrow \mathcal{U}\psi^\dagger(\vec{r})\mathcal{U}^\dagger = e^{i\theta(\vec{r})}\psi^\dagger(\vec{r}). \tag{67}$$

For the case of constant θ the vector potential is unchanged and the Hamiltonian will remain invariant, *provided that it conserves particle number*

$$\left[H, \int d^2r\,\hat{n}(\vec{r}) \right] = 0. \tag{68}$$

Thus gauge invariance and charge conservation are one and the same. See Schrieffer's book for further discussion of this point. [11]

From the transformation properties of the charge two order parameter in Eq. (6) we see that the XY model in Eq. (18) transforms under a gauge change according to

$$U \longrightarrow \frac{1}{2}\rho_s \int d^2r\, \left(\vec{\nabla}\varphi - \frac{2e}{\hbar c}\vec{A}. \right)^2 \tag{69}$$

It follows that this must be the correct form in general, even when $\vec{A}$ has a curl and is therefore physical. The definition of the current density is the functional derivative of the energy (or free energy) with respect to the vector potential

$$J^\alpha(\vec{r}) \equiv -c\frac{\delta U}{\delta A^\alpha(\vec{r})}. \tag{70}$$

Eq. (42) follows directly from this result (in the limit of zero vector potential).

Figure 1: (a) φ field configuration for a vortex with winding number $+1$. (b) $\vec{\nabla}\varphi$ for this vortex. (c) The field $\vec{D} \equiv \vec{\nabla}\varphi \times \hat{z}$ used in the electrostatics representation. (e-f) corresponding configurations after a global $U(1)$ transformation $\varphi \longrightarrow \varphi + \pi$. Notice that the winding number has *not* changed. It is still $+1$ and the $\vec{\nabla}\varphi$ and $\vec{D}$ fields (as well as the energy) are left unchanged by the transformation.

Figure 2: Same as Fig. (1) but for a vortex with winding number -1.

Figure 3: Bound pair of vortices with opposite winding numbers. These objects are bound together by the logarithmic confining potential below T_{KT}.

Figure 4: Filled circles: Superfluid density of a ^{4}He film measured with a torsional pendulum. Dashed line is the theoretical curve for a static theory [12] that ignores the finite oscillation frequency of the pendulum. Solid line is a four parameter fit to the dynamical theory of Ambegaokar et al. [13] Data of Bishop et al. [10]

Figure 5: Resistance of a film above the Kosterlitz-Thouless temperature fit to a model in which the resistance is proportional to the vortex density ξ^{-2} with ξ given by Eq.(30). Data of Hebard and Fiory [14].

Figure 6: Non-linear IV characteristics of 2D superconducting films at different temperatures. Note that at the critical temperature, $V \sim I^3$ as expected. Data of Epstein et al. [15].

Figure 7: Exponent a in the nonlinear IV relation $V \sim I^a$ as a function of temperature. Data of Epstein et al. [15].

References

[1] J. M. Kosterlitz and D. J. Thouless, J. Phys. C**6**, 1181 (1973).

[2] *Percolation, Localization, and Superconductivity*, Edited by Allen M. Goldman and Stuart A. Wolf, NATO ASI Series B Vol. **109** (Plenum, New York, 1983). See especially the articles by Minnhagen and Mooij.

[3] David R. Nelson in *Fundamental Problems in Statistical Mechanics V*, Edited by E. G. D. Cohen (North Holland, 1980).

[4] B. I. Halperin in *Proceedings of the Kyoto Summer Institute on Low Dimensional Systems*, (1979).

[5] Petter Minnhagen, 'The two-dimensional Coulomb gas, vortex unbinding, and superconducting-superfluid films,' *Rev. Mod. Phys.* **59**, 1001 (1987).

[6] J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, *Phys. Rev. B* **16**, 1217 (1977).

[7] H.J.F. Knops and L. W. J. den Ouden, 'Momentum Space Renormalization for the Sine-Gordon Model,' *Physica* **103A**, 597 (1980).

[8] Stu Samuel, 'Grand partition function in field theory with applications to sine-Gordon field theory,' *Phys. Rev. D* **18**, 1916 (1978).

[9] P. M. Chaikin and T. C. Lubensky, *Principles of condensed matter physics* (Cambridge Press, 1995).

[10] D. J. Bishop and J. D. Reppy, *Phys. Rev. Lett.* **40**, 783 (1978).

[11] J. R. Schrieffer, *Theory of Superconductivity* (Addison-Wesley, 1964).

[12] D. R. Nelson and J. M. Kosterlitz, *Phys. Rev. Lett.* **39**, 1201 (1977).

[13] V. Ambegaokar, B. I. Halperin, D. R. Nelson and E. D. Siggia, *Phys. Rev. Lett.* **40**, 783 (1978); *Phys. Rev. B* **21**, 1806 (1980).

[14] A. F. Hebard and A. T. Fiory, *Phys. Rev. Lett.* **50**, 1603 (1983).

[15] K. Epstein, A. M. Goldman and A. M. Kadin, *Proceedings LT16, Physica* **109 & 110 B+C**, 2087 (1982).