Lagrangian Methods Assignment 1 – Due Sep 19, 2002

Blobs in 1 dimension: recall that $\phi_\delta(x) = \frac{1}{\delta}\phi(x/\delta)$

1. Consider blobs given by functions of the form

$$\phi(x) = \frac{A_p}{(x^2+1)^{p/2}}.$$

   For each integer $p = 3, 4, 5, 6, 7, 8, 9, 10$, find the appropriate constant A_p so that $\int_{-\infty}^{\infty} \phi(x)dx = 1$. Comment on the patterns you find.

   For all blobs, compute the second moment $M_2(\phi)$.

   Choose three of these blobs and, for each one, plot $\phi_\delta(x)$ for $\delta = 1, 1/2$ and $1/8$ on the same graph.

2. Suppose now that you look to design blobs of the form

$$\phi(x) = \frac{A_p - B_p\, x^2}{(x^2+1)^{p/2}}.$$

   that satisfy $M_0(\phi) = 1$ and $M_2(\phi) = 0$. What is the smallest possible integer p for which you can do this? Denote your answer by p_o.

   For each integer p between p_o and 10, find appropriate constants A_p and B_p.

   For these blobs, compute the fourth moment $M_4(\phi)$.

   Choose three of these blobs and, for each one, plot $\phi_\delta(x)$ for $\delta = 1, 1/2$ and $1/8$ on the same graph.

3. Consider the blob given by

$$\phi(x) = \frac{A}{\sqrt{\pi}} e^{-x^2}$$

   Find the constant A so that $M_0(\phi) = 1$ and compute the second moment $M_2(\phi)$.

   Plot $\phi_\delta(x)$ for $\delta = 1, 1/2$ and $1/8$ on the same graph.

4. Now consider the blob

$$\phi(x) = \frac{A_0 + A_1 x^2}{\sqrt{\pi}} e^{-x^2}.$$

Find A_0 and A_1 such that $M_0(\phi) = 1$ and $M_2(\phi) = 0$ and plot $\phi_\delta(x)$ for $\delta = 1, 1/2$ and $1/8$ on the same graph.

5. Take the blob from problem 3 and call it $\phi_1(x)$. We know that $M_2(\phi_1) \neq 0$. Define new blobs using the recursion

$$\phi_{n+1}(x) = -\frac{1}{4n}\left(\phi_n''(x) + \frac{2}{x}\phi_n'(x)\right) \qquad \text{for } n \geq 1$$

Use this formula to find $\phi_2(x)$ and compare with the blob you found in problem 4.

Find ϕ_n for $n = 3, 4, 5$ and verify that each new blob in the sequence satisfies more (even) moment conditions. Can you prove this?

Plot $\phi_1(x)$ through $\phi_5(x)$ on the same graph.

6. Blobs with compact support: consider the blobs

$$\phi_1(x) = \begin{cases} A\left(1 + \cos(\pi x)\right), & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

and

$$\phi_2(x) = \begin{cases} A(x^2 - 1)^4, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

Find A so that $M_0(\phi) = 1$ for each blob.

Blobs in 2 dimensions: In 2D the scaling for radially symmetric blobs is $\phi_\delta(r) = \frac{1}{\delta^2}\phi(r/\delta)$. We require that

$$M_0(\phi) = 2\pi \int_0^\infty r\phi(r)dr = 1.$$

For higher moment conditions we require

$$M_k(\phi) = C_k \int_0^\infty r^{k+1}\phi(r)dr = 0.$$

7. Find the constants A_p for $p = 3, 4, 5, 6, 7$ so that the blob

$$\phi(r) = \frac{A_p}{(r^2 + 1)^{p/2}}$$

satisfies $M_0(\phi) = 1$. Plot $\phi(r)$ vs. r.

8. Find the constant A so that the blob

$$\phi(r) = Ae^{-r^2}$$

satisfies $M_0(\phi) = 1$.

9. Find the constants A and B so that the blob

$$\phi(r) = (A + Br^2)e^{-r^2}$$

satisfies $M_0(\phi) = 1$ and $M_2(\phi) = 0$.

10. In two dimensions, the recursion for higher-order blobs of this form is

$$\phi_{n+1}(x) = -\frac{1}{4\pi\phi_n(0)}\left(\phi_n''(r) + \frac{3}{r}\phi_n'(r)\right) \qquad \text{for } n \geq 1$$

Verify that each new blob in the sequence satisfies another moment condition.

Plot a few of these blobs on the same graph.