Lagrangian Methods Assignment 1 – Due Sep 19, 2002

**Blobs in 1 dimension:** recall that $\phi_\delta(x) = \frac{1}{\delta} \phi(x/\delta)$

1. Consider blobs given by functions of the form

$$\phi(x) = \frac{A_p}{(x^2 + 1)^{p/2}}.$$

For each integer $p = 3, 4, 5, 6, 7, 8, 9, 10$, find the appropriate constant $A_p$ so that $\int_{-\infty}^{\infty} \phi(x) dx = 1$. Comment on the patterns you find.

For all blobs, compute the second moment $M_2(\phi)$.

Choose three of these blobs and, for each one, plot $\phi_\delta(x)$ for $\delta = 1, 1/2$ and $1/8$ on the same graph.

2. Suppose now that you look to design blobs of the form

$$\phi(x) = \frac{A_p - B_p x^2}{(x^2 + 1)^{p/2}}.$$

that satisfy $M_0(\phi) = 1$ and $M_2(\phi) = 0$. What is the smallest possible integer $p$ for which you can do this? Denote your answer by $p_o$.

For each integer $p$ between $p_o$ and 10, find appropriate constants $A_p$ and $B_p$.

For these blobs, compute the fourth moment $M_4(\phi)$.

Choose three of these blobs and, for each one, plot $\phi_\delta(x)$ for $\delta = 1, 1/2$ and $1/8$ on the same graph.

3. Consider the blob given by

$$\phi(x) = \frac{A}{\sqrt{\pi}} e^{-x^2}$$

Find the constant $A$ so that $M_0(\phi) = 1$ and compute the second moment $M_2(\phi)$.

Plot $\phi_\delta(x)$ for $\delta = 1, 1/2$ and $1/8$ on the same graph.
4. Now consider the blob
\[ \phi(x) = \frac{A_0 + A_1 x^2}{\sqrt{\pi}} e^{-x^2}. \]
Find \( A_0 \) and \( A_1 \) such that \( M_0(\phi) = 1 \) and \( M_2(\phi) = 0 \) and plot \( \phi_\delta(x) \) for \( \delta = 1, 1/2 \) and \( 1/8 \) on the same graph.

5. Take the blob from problem 3 and call it \( \phi_1(x) \). We know that \( M_2(\phi_1) \neq 0 \). Define new blobs using the recursion
\[ \phi_{n+1}(x) = -\frac{1}{4n} \left( \phi_n''(x) + \frac{2}{x} \phi_n'(x) \right) \quad \text{for } n \geq 1 \]
Use this formula to find \( \phi_2(x) \) and compare with the blob you found in problem 4.
Find \( \phi_n \) for \( n = 3, 4, 5 \) and verify that each new blob in the sequence satisfies more (even) moment conditions. Can you prove this?
Plot \( \phi_1(x) \) through \( \phi_5(x) \) on the same graph.

6. Blobs with compact support: consider the blobs
\[ \phi_1(x) = \begin{cases} A (1 + \cos(\pi x)) , & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} \]
and
\[ \phi_2(x) = \begin{cases} A(x^2 - 1)^4 , & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} \]
Find \( A \) so that \( M_0(\phi) = 1 \) for each blob.

Blobs in 2 dimensions: In 2D the scaling for radially symmetric blobs is \( \phi_\delta(r) = \frac{1}{\delta^2} \phi(r/\delta) \). We require that
\[ M_0(\phi) = 2\pi \int_0^\infty r \phi(r) dr = 1. \]
For higher moment conditions we require
\[ M_k(\phi) = C_k \int_0^\infty r^{k+1} \phi(r) dr = 0. \]
7. Find the constants $A_p$ for $p = 3, 4, 5, 6, 7$ so that the blob
\[
\phi(r) = \frac{A_p}{(r^2 + 1)^{p/2}}
\]
satisfies $M_0(\phi) = 1$. Plot $\phi(r)$ vs. $r$.

8. Find the constant $A$ so that the blob
\[
\phi(r) = Ae^{-r^2}
\]
satisfies $M_0(\phi) = 1$.

9. Find the constants $A$ and $B$ so that the blob
\[
\phi(r) = (A + Br^2)e^{-r^2}
\]
satisfies $M_0(\phi) = 1$ and $M_2(\phi) = 0$.

10. In two dimensions, the recursion for higher-order blobs of this form is
\[
\phi_{n+1}(x) = -\frac{1}{4\pi\phi_n(0)} \left( \phi_n''(r) + \frac{3}{r} \phi_n'(r) \right) \quad \text{for } n \geq 1
\]
Verify that each new blob in the sequence satisfies another moment condition.
Plot a few of these blobs on the same graph.