Boulder Lecture - Mean Field Glassy Systems

Homework n° 2

To break or Not To Break

On the meaning of replica symmetry breaking

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The following exercise is a way to understand the physical origin of the one step replica symmetry breaking solution, in particular of the one you considered in the first homework for the low temperature phase of the REM. It is based on a method called "m coupled replicas" introduced by R. Monasson.

Let us first focus on a system without quenched disorder that we expect to display a rough free-energy landscape as the one of the REM or the p-spin spherical model that we studied during the lectures. We assume that the partition function can be written as :

$$Z = \sum_{\alpha} e^{-\beta f_\alpha N} \tag{1}$$

where N is the size of the system, α the index of the "states" and f_α the intensive free energy of state α. The problem we want to solve is finding a general method to obtain the configurational entropy and the free energy of the states that dominate the sum above.
The starting point of the method is to consider m replicas of the same system coupled via a small attraction $\epsilon = 0^+$. The specific form of the coupling is not important as long as it thermodynamically favours a high overlap (suitably defined depending on the system) between the replicas. We furthermore assume that because of the attraction the m replicas fall into the same state [1]. One therefore obtains for the partition function of the system of m coupled replicas :

$$Z_m = \sum_{\alpha} e^{-\beta f_\alpha m N} = \int df e^{N[S(f)-\beta m f]}$$

where we assumed that the system is below T_d and a well defined curve of the configurational entropy $S(f)$ exists.

1. Show that $\frac{d}{dm}(\log Z_m)/N = -\beta f^*(m)$ where $f^*(m)$ is the value that maximizes $S(f) - \beta m f$.
2. Show that $\frac{d}{dm}\left[\frac{1}{m}(\log Z_m)/N\right] = -\frac{1}{m^2}S(f^*(m))$.
3. Using these results, how can one get the configurational entropy and the free energy of the states that give the leading contribution to the partition function of the original system (1) by knowing $(\log Z_m)/N$?
4. How one can get the curve $S(f)$ from the knowledge of $(\log Z_m)/N$ (hint : study graphically the equation that fixes $f^*(m)$) ?

Let us consider now a disordered system and let us apply to it the method above. We have now to compute $\overline{\log Z_m}$. We can use the replica method to do it :

$$\overline{\log Z_m} = \lim_{n' \to 0} \frac{1}{n'} \log \overline{(Z_m)^{n'}}$$

In the usual replica method that we have used for example to get the free energy of the REM we computed instead

$$\overline{\log Z} = \lim_{n \to 0} \frac{1}{n} \log \overline{Z^n}$$

1. Depending on the temperature this might be the most favorable solution or a metastable one.

and, to obtain the low temperature solution, we focused on a case, called 1RSB, in which the n replicas are divided in m groups and are in the same state within the group and in different states among different groups. We shall now establish a connection between the two approaches.

1. Show that the contribution obtained using the 1RSB solution to $\overline{\log Z}$ reads

$$\overline{\log Z}_{1RSB} = \frac{1}{m}\overline{\log Z_m}$$

   if we assume a replica symmetric solution for the n' replicas.
2. Use this result to write the 1RSB contribution to the free energy, $-\frac{1}{\beta}\overline{\log Z}_{1RSB}$, in terms of $f^*(m)$ and $S(f^*(m))$.
3. Using this result, justify physically the recipe I gave you in the first homework to fix $m = 1$, i.e. that the high temperature solution corresponds to $m = 1$, whereas the low temperature solution corresponds to the value of $m < 1$ at which the derivative with respect to m of the 1RSB contribution to the free energy is zero. Explain why the transition point can be obtained as the temperature at which the derivative with respect to m of the 1RSB contribution is zero at $m = 1$.

The aim of this exercise was to make you understand some of the physics behind the 1RSB solution, which actually encodes and allows to compute the properties of the rough energy landscape we discussed in the lectures. The two main messages are : (1) very different systems are solved by a 1RSB solution, which is therefore the origin of the universality class of rough energy landscapes described during the lectures ; (2) the replica method is generically easier and hence preferred to compute (indirectly) properties of the energy and free energy landscapes. For example, the replica method was used to established the structure of the solutions space in random K-SAT and the free energy landscape for super-cooled liquids in large dimensions.
The method we developed in this homework was introduced in the paper, that I suggest you to read, R. Monasson, *Structural Glass Transition and the Entropy of the Metastable States*, PRL **75** 2847 (1995).