Boulder Lecture - Mean Field Glassy Systems
Homework n° 1
An introduction to the replica method
(beware of the acrobatic math)

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These three exercises [1] are an introduction to the replica method, which is a very powerful technique used in disordered systems. This theoretical framework seems at first quite acrobatic–to say the least ; putting it on a firm mathematical basis has been and still is one of the main research problem in mathematical physics and probability theory.
The starting point of the replica method is one of the simple equalities :

$$\overline{\log Z} = \lim_{n\to 0} \frac{\overline{Z^n} - 1}{n} \tag{1}$$

$$\overline{\log Z} = \lim_{n\to 0} \frac{\ln \overline{Z^n}}{n} \tag{2}$$

where $\bar{\cdot}$ denotes the average over the quenched disorder. The trick is to compute the fraction on the right hand side for integer values of n and then analytically continue the results to get, using the $n \to 0$ limit, the left hand side.

1 Warm up : the easiest replica computation ever

Consider a "toy" disordered partition function $Z = e^{xN}$ where x plays the role of the quenched disorder ; x is equal to 1 with probability $1 - e^{-N}$ and to 3 with probability e^{-N}. We are interested in the thermodynamic limit where the size of the system, N, goes to infinity.

- Compute $(\log \overline{Z})/N$ and $(\overline{\log Z})/N$. Where does the difference among the two results come from ?
- Use now the replica trick (1) to get the result doing the thermodynamic limit first and the $n \to 0$ limit later (the reverse order must trivially give the correct result). Why does the $n \to 0$ limit help compared to the previous computation ?

In general the replica method is always used doing the thermodynamic limit first and the $n \to 0$ limit later, we will discuss this point later on.

2 Solution of the REM by the replica method

We are now going to obtain the thermodynamics of the REM, more precisely its free energy, by the replica method. This will be useful to understand how the method works in a simple but non trivial case. The starting point of the replica method is the computation of

$$\overline{Z^n} = \overline{\sum_{\mathcal{C}_1,\cdots,\mathcal{C}_n} e^{-\beta E_{\mathcal{C}_1}} \cdots e^{-\beta E_{\mathcal{C}_n}}} \tag{3}$$

In the following we shall assume that among all the possible elements of the sums, two contributions give an overwhelming contribution in the $N \to \infty$ limit.

1. This homework is partially based on exercises prepared for the course "Advanced Statistical Physics" by G. Biroli and G. Semerjian given at ENS Paris.

— The first one corresponds to all replica in different configurations, i.e. $\mathcal{C}_1 \neq \cdots \neq \mathcal{C}_n$. It is called *replica symmetric* and intuitively it could be a good guess at high temperature where the system samples a very large number of different configurations, and hence the probability that two replica fall into the same configuration is very low. Assuming that n is a fixed integer show that this contribution at exponential leading order reads

$$\overline{Z^n}\Big|_{RS} = e^{Nn\log 2 + Nn\beta^2/4}$$

— Use the replica trick to obtain the intensive free energy f. If everything went through correctly you should have found the high temperature expression of f derived during the lecture. As we showed, this is correct for $T \geq T_c$ only.

— Consider now the contribution to the sum (3) in which the replica are divided in n/m groups ; all replica belonging to the same group, say group α, are in the same configuration $\mathcal{C}_\alpha$, and all groups are in different configurations, i.e. $\mathcal{C}_1 \neq \cdots \neq \mathcal{C}_{n/m}$. This is called *one step replica symmetric breaking* and intuitively it could be a good guess at low temperature where the system samples only a few low-lying configurations, and hence the probability that two replica fall into the same configuration is high. Assuming that n, m are fixed integer with $n > 0$ and $1 \leq m \leq n$ show that this contribution at exponential leading order reads

$$\overline{Z^n}\Big|_{RSB} = e^{N\frac{n}{m}\log 2 + Nnm\beta^2/4}$$

— Use the replica trick to obtain that the contribution to the intensive free energy reads :

$$-\frac{1}{\beta}\left(\frac{1}{m}\log 2 + \frac{\beta^2}{4}m\right)$$

— Since the expression above depends on m we have to optimize over this parameter to get the dominant contribution. Given that we are working in the $n \to 0$ limit, we assume : (i) that m is a real number, (ii) following Parisi we generalise the inequalities $1 \leq m \leq n$ valid for $n > 1$ to the $n \to 0$ case by assuming that they get reversed, i.e. $0 \leq m \leq 1$. Furthermore, instead of minimizing the free energy with respect to m, as one would do usually, we will maximize it. Show that by following these recipes one finds back the expression of the free energy valid below T_c as well as the value of T_c.

The aim of this computation was to make you familiar with solutions by the replica method. Needless to say, if this is your first experience then the recipes one has to follow, e.g. taking $0 \leq m \leq 1$ and maximing instead on minimizing, may appear unjustified (and a bit crazy). There are rational and even rigorous justifications for these procedures, we will see some of them during the lectures.

3 An example of acrobatic math behind the replica method

The final outcome of the replica method is by now to a large extent rigorously proven in several cases, and hopefully you will master this technique at the end of this month. However, as much as it is useful to understand how the method works, it is also important to face some of its dark sides. Here is an example to make you feel a bit uneasy.

We consider a very simple disordered system : an Ising spin in a random quenched magnetic field h, where h is a Gaussian random variable of mean zero and variance one.

— Show that

$$\overline{\log Z} = \int \frac{dh}{\sqrt{2\pi}} e^{-h^2/2} \log\left(1 + e^{2\beta h}\right) \tag{4}$$

— In order to use the replica trick, you have to obtain $\overline{Z^n} - 1$. Show that

$$\overline{Z^n} - 1 = \sum_{k=1}^{n} e^{\frac{\beta^2}{2}(2k-n)^2} \frac{n!}{(n-k)!k!} + e^{\beta^2 n^2/2} - 1$$

— How to continue for real values of n the above expression? A good idea is to use the Gamma function $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$. Using the identity (that you can prove if you wish) $\Gamma(z+1) = z\Gamma(z)$ obtain that $\Gamma(k+1) = k!$ for positive integer k (we will use the standard definition $0! = 1$). This allows you to rewrite the previous equation as

$$\overline{Z^n} - 1 = \sum_{k=1}^{n} e^{\frac{\beta^2}{2}(2k-n)^2} \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} + e^{\beta^2 n^2/2} - 1$$

Moreover, since the Gamma function is infinite when the argument is a negative or zero integer you can extend the sum over k up to infinity, thus obtaining an expression where n can be made real.

$$\overline{Z^n} - 1 = \sum_{k=1}^{\infty} e^{\frac{\beta^2}{2}(2k-n)^2} \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} + e^{\beta^2 n^2/2} - 1$$

— We are now ready to perform the $n \to 0$ limit. In order to do that you have to establish (using the equality $\Gamma(z+1) = z\Gamma(z)$) a few properties of the Gamma function. Show that for $n \to 0$ and l a positive integer

$$\Gamma(l+n) \to (l-1)!$$

whereas for $n \to 0$ and l a negative (or zero) integer :

$$\Gamma(l+n) \to \frac{1}{n} \frac{(-1)^{-l}}{(-l)!}$$

— Using this result and the replica trick you can now recover (after some clever algebra) that

$$\overline{\log Z} = \int \frac{dh}{2\pi} e^{-h^2/2} \log\left(1 + e^{2\beta h}\right)$$

The aim of this exercise was making you acquainted with the acrobatic procedures one has to do in order to use the replica trick and also make you a bit uneasy in case you were too ready to accept the trick without afterthoughts. You could (should) certainly claim that there was some arbitrariness in the procedure–and you would be right!–but the final result is the correct one.

Boulder Lecture - Mean Field Glassy Systems
A Challenge in Random Matrix Theory
Dyson Brownian Motion and the Wigner Semi-Circle Law

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In the following we shall consider N by N symmetric real random matrices $\mathbf{M}$ such that the off-diagonal element M_{ij} and the diagonal ones M_{ii} are Gaussian independent random variables with zero mean and with variance :

$$\langle M_{ij}^2 \rangle = \frac{1}{N} \qquad \langle M_{ii}^2 \rangle = \frac{2}{N}$$

These are the GOE matrices we discussed during the lecture [1].

1 Dyson Brownian Motion and Eigenvalues Distribution

We shall find the distribution of their eigenvalues by analysing the following stochastic process on matrices :

— we add to the matrix $\mathbf{M}$ another Gaussian random matrix $\mathbf{g}$:

$$\tilde{\mathbf{M}}(t+dt) = \mathbf{M}(t) + \mathbf{g}$$

where $\mathbf{g}$ is another N by N symmetric real random matrix with off-diagonal and diagonal elements which are Gaussian independent random variables with zero mean and variance dt/N and $2dt/N$ respectively.

— we rescale the matrix $\tilde{\mathbf{M}}(t+dt)$ to keep the variance fixed :

$$\mathbf{M}(t+dt) = \frac{\tilde{\mathbf{M}}(t+dt)}{\sqrt{1+dt}}$$

Henceforth we shall study this process in the limit $dt \to 0$.

1. Show that the random matrix $\mathbf{M}(t+dt)$ is identical in law to $\mathbf{M}(t)$. Therefore this stochastic process leaves Gaussian random matrices invariant in law.
2. Show that in any orthogonal basis the statistical properties of the matrix $\mathbf{g}$ are the same : the off-diagonal and diagonal elements are Gaussian independent random variables with zero mean and variance dt/N and $2dt/N$ respectively.
3. Using second order perturbation theory (the same you learned in quantum mechanics) to show that the change $d\lambda_\alpha(t)$ of the eigenvalues under the stochastic process defined above reads :

$$d\lambda_\alpha(t) = -\frac{\lambda_\alpha(t)}{2}dt + g_{\alpha,\alpha} + \sum_{\beta(\neq\alpha)} \frac{g_{\alpha,\beta}^2}{\lambda_\alpha(t) - \lambda_\beta(t)} + o(dt)$$

4. Focus on the right hand-side and study the order in dt of the different terms (their averages and their fluctuations).
5. By comparing the fluctuations of the second term of the right hand-side to the fluctuations of the third one, justify why the latter can be replaced by its average.

1. This challenge is partially based on exercises prepared for the course "Advanced Statistical Physics" by G. Biroli and G. Semerjian given at ENS Paris.

6. Using the previous results show that the evolution of the eigenvalues under the stochastic process can be written in terms of the following stochastic equation :

$$\frac{d\lambda_\alpha(t)}{dt} = -\frac{\lambda_\alpha(t)}{2} + \frac{1}{N}\sum_{\beta(\neq\alpha)} \frac{1}{\lambda_\alpha(t)-\lambda_\beta(t)} + \eta_\alpha(t) \qquad \langle \eta_\alpha(t)\eta_\beta(t')\rangle = \frac{2}{N}\delta_{\alpha,\beta}\delta(t-t')$$

This is the famous Dyson Brownian Motion. We show an example of the eigenvalues trajectories in Fig.1.

7. Interpret the previous equation as a Langevin equation. What is the value of the temperature ? What is the expression of the potential $V(\{\lambda_\alpha\})$ of the "particles" $\lambda_\alpha(t)$?

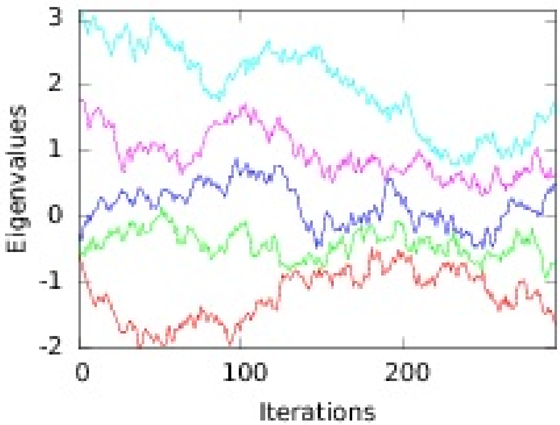

FIGURE 1 – Examples of Dyson Brownian motion for $N = 5$.

8. Since the stochastic process leaves Gaussian random matrices invariant in law, the probability density function of the λ_αs is invariant too. This means that this probability law is stationary in time. Using this fact and what you know about Langevin equation, give the interpretation of this probability density function and find its explicit form :

$$P(\{\lambda_\alpha\}) = \frac{1}{Z}\exp\left(-N\sum_{\alpha=1}^{N}\frac{\lambda_\alpha^2}{4} + \frac{1}{2}\sum_{\alpha\neq\beta}\log|\lambda_\alpha-\lambda_\beta|\right) \tag{1}$$

$$= \frac{1}{Z}\prod_{\alpha<\beta}|\lambda_\alpha-\lambda_\beta|\exp\left(-N\sum_{\alpha=1}^{N}\frac{\lambda_\alpha^2}{4}\right) \tag{2}$$

9. What is the physical effect corresponding to the limit $N \to \infty$? Discuss the physical role of the two contributions to the potential $V(\{\lambda_\alpha\})$.

10. Calling $\mathcal{N}(\lambda)$ the fraction of eigenvalues in the interval $[\lambda, \lambda + d\lambda]$ we define the density of eigenvalues $\rho(\lambda)$ as $\rho(\lambda)d\lambda = \mathcal{N}(\lambda)$ (where $1/N \ll d\lambda \ll 1$). Show that $P(\{\lambda_\alpha\})$ can be rewritten as a functional of $\rho(\lambda)$:

$$P(\{\lambda_\alpha\}) = C\exp\left(N^2\left[-\int d\lambda\rho(\lambda)\frac{\lambda^2}{4} + \frac{1}{2}\int d\lambda d\eta\rho(\lambda)\rho(\eta)\log|\lambda-\eta|\right]\right)$$

This expression requires some discussion. The term with the logarithm is tricky since to obtain it you had to neglect somehow a self interaction term $\lambda = \eta$ but actually the result can be rigorously justified. We won't do it here but just give the main idea behind, which is that the expression above has to be considered as a large deviation functional. In consequence, to justify it one has to show that the result you obtained for the term with the logarithm is correct up to sub-leading correction with respect to N^2, which can be proven to be indeed the case.

11. In order to finally obtain the probability of observing a given $\rho(\lambda)$ you have to integrate $P(\{\lambda_\alpha\})$ over all the positions of $\{\lambda_\alpha\}$ corresponding to the same $\rho(\lambda)$. This adds an entropy term to the argument of the exponential in the RHS above. However, since the entropy term is $-N\int d\lambda\rho(\lambda)\log\rho(\lambda)$ (this is the same you find for a gas of particles in liquid theory, as you could show), it gives a sub-leading contribution compared to N^2 and can be dropped. The conclusion is that to exponential order in N^2 one finds :

$$P[\rho(\lambda)] = C\exp\left(N^2\left[-\int d\lambda\rho(\lambda)\frac{\lambda^2}{4} + \frac{1}{2}\int d\lambda d\eta\rho(\lambda)\rho(\eta)\log|\lambda-\eta|\right]\right) \tag{3}$$

2 Wigner Semi-Circle Law

Because of the N^2 in the exponential of eq. (3) the density of eigenvalues converges in the large N limit to a non-fluctuating function which corresponds to the minimum of (3). Our next aim is to find this function : the famous Wigner semi-circle law.

1. Introduce a Lagrange multiplier Ω to enforce the normalization of $\rho(\lambda)$ and take the functional derivative of (3) to find the equation satisfied by $\rho(\lambda)$

$$-\frac{\lambda^2}{4} + \int d\eta\rho(\eta)\log|\eta-\lambda| = \Omega$$

2. In order to solve this equation, we regularize the logarithm as $\log|\eta-\lambda| = \frac{1}{2}\log(|\eta-\lambda|^2+\epsilon^2)$ with $\epsilon \ll 1$. We do this because in this way we can take the derivative with respect to λ of this equation and get :

$$\frac{\lambda}{2} = P\int d\eta\rho(\eta)\frac{1}{\lambda-\eta} \tag{4}$$

   where P means that we take the principal part of the integral.

3. Introduce now the resolvent $G(z) = \int d\eta\rho(\eta)\frac{1}{z-\eta}$ where z is a complex number with $Imz < 0$. Multiply eq. (4) by $\rho(\lambda)/(z-\lambda)$ and integrate over λ.
4. Show that the LHS of the resulting equation can be rewritten as $(-1+zG(z))/2$.
5. Using the identity

$$\frac{1}{z-\eta}\left[\frac{1}{\eta-\lambda} - \frac{1}{z-\lambda}\right] = \frac{1}{(\eta-\lambda)(z-\lambda)}$$

   show that the RHS of the resulting eq. can be rewritten as $G^2(z))/2$.

6. Solve the equation to get $G(z)$.
7. Recall, or show, that $\rho(\lambda) = \frac{1}{\pi}\lim_{h\to0^+} ImG(\lambda - ih)$.
8. Using this identity finally obtain that $\rho(\lambda) = \frac{\sqrt{4-\lambda^2}}{2\pi}$ which is the famous Wigner semi-circle law.