1. Su Schrieffer Heeger Model

Consider the 1D tight binding Hamiltonian

$$H[\Delta] = \sum_i t(1+\Delta)c^\dagger_{i,A}c_{i,B} + t(1-\Delta)c^\dagger_{i+1,A}c_{iB} + h.c. + u\Delta^2$$

Here Δ represents a dimerization distortion of the lattice.

(a) Compute the band structure for fixed Δ and show that there is an energy gap for all $\Delta \neq 0$. A useful way to do this is to write the Hamiltonian in momentum space as $H = \mathbf{h}(k)\cdot\vec{\sigma}$, where $\vec{\sigma}$ are Pauli matrices, and $\mathbf{h}(k)$ are functions you can determine. Show that at half filling the ground state energy $E_0(\Delta)$ is minimized for $\Delta = \pm\Delta_0$, where Δ_0 is non zero. The metallic state ($\Delta = 0$) thus exhibits the *Peierls instability* towards an insulating state with a doubly degenerate ground state.

(b) Take the continuum limit and show that for small Δ the states near the Fermi energy (at half filling) are described by the 1+1D Dirac Hamiltonian,

$$H = -iv_F\sigma^x\partial_x + m\sigma^z.$$

Identify the parameters v_F and m in terms of t, Δ and the lattice constant a.

(c) (*Jackiw-Rebbi problem*) Consider a spatially varying Δ, with a domain wall between the two ground states, so that $\Delta(x \to \pm\infty) = \pm\Delta_0$. By analyzing the continuum Hamiltonian in (b) show that there is an exact zero energy bound state of the Hamiltonian localized near the domain wall. Compute the wavefunction of that bound state for arbitrary $\Delta(x)$ satisfying the above constraint. (You may assume Δ is small and varies slowly, so that the continuum approximation is valid).

2. Generalized Haldane Model

Consider a tight binding model for electrons on a honeycomb lattice

$$H = t\sum_{<ij>} c^\dagger_i c_j + i\lambda_H \sum_{<<ij>>} \nu_{ij}c^\dagger_i c_j + \lambda_v\sum_i \xi_i c^\dagger_i c_i.$$

Here the first term is a sum over nearest neighbors of the honeycomb lattice. The second term is a sum over second neighbors with $\nu_{ij} = \pm 1$, depending on whether the electron turns right or left when going from the first to the second neighbor. This term represents a spatially varying magnetic field (with zero average) and violates time reversal symmetry. The third term is a staggered site potential, with $\xi_i = +1(-1)$ on the A(B) sublattice.

(a) Compute the band structure. Again, it is useful to write $H = \mathbf{h}(\mathbf{k})\cdot\vec{\sigma}$ and identify $\mathbf{h}(\mathbf{k})$.

(b) For small λ_B and λ_v take the continuum limit and show that the states near the two distinct corners of the Brillouin zone $\mathbf{k} = K_\pm$ are described by the 2+1D Dirac Hamiltonian,

$$H = -iv_F(\pm\sigma^x\partial_x + \sigma^y\partial_y) + m_\pm\sigma^z.$$

Identify the parameters v_F and $m_\pm$ as above. Sketch a phase diagram as a function of λ_H and λ_v with a line describing where the energy gap closes (ie $m = 0$), separating distinct phases.

(c) Consider a linear domain wall where m changes sign such that $m(x, y \to \pm\infty) = \pm m_0$. Show that there is a chiral fermion domain wall state with $E = vk_x$ inside the bulk energy gap. (I might have the sign wrong.)

(d) The chiral fermion mode you found in (c) shows that the domain wall is a boundary between an insulator and a quantum Hall state. Which side is the insulator? Use the properties of $\mathbf{h}(\mathbf{k})$ to determine the answer.

3. Kane Mele model of 2D quantum spin Hall insulator

Now consider the honeycomb lattice model with spin.

$$H = t \sum_{<ij>,\sigma} c^\dagger_{i,\sigma} c_{j,\sigma} + i\lambda_{so} \sum_{<<ij>>} \nu_{ij} s^z_{\sigma\sigma'} c^\dagger_{i,\sigma} c_{j,\sigma'} + \lambda_v \sum_i \xi_i c^\dagger_{i,\sigma} c_{i,\sigma}.$$

where s^z is a Pauli matrix describing spin.

(a) Show that this Hamiltonian is invariant under time reversal symmetry. The second term represents an intrinsic spin orbit interaction.

(b) Repeat the analysis of problem (2). Determine the structure of the edge states on a domain wall between a region where $\lambda_v = 0$ and $\lambda_v \gg \lambda_{so}$ (for $\lambda_{so} \neq 0$).

(c) Consider a Rashba term $i\lambda_R \sum_{<ij>} \vec{s}_{\sigma\sigma'} \cdot (\vec{d_{ij}} \times \hat{z}) c^\dagger_{i,\sigma} c_{j,\sigma'} + h.c.$ Show that this term violates the conservation of s^z, but respects time reversal symmetry. Determine the effect of this term on the domain wall states in (b).

4. Bernevig Qi Hughes Zhang model of 3D topological insulator

Consider a three dimensional four band tight binding described by the momentum space Hamiltonian

$$H = t\tau^x \left[\sigma^x \sin k_x a + \sigma^y \sin k_y a + \sigma^z \sin k_z a\right] + \tau^z \left[m + u(3 - \cos k_x a - \cos k_y a - \cos k_z a)\right]$$

Here $\vec{\sigma}$ are Pauli matrices describing the electron spin, while $\vec{\tau}$ are Pauli matrices describing an orbital degree of freedom. This model describes the inversion of two bands with opposite parity as m changes sign.

(a) Verify that this Hamiltonian respects time reversal symmetry and inversion symmetry and understand the dependence of the band structure on m.

(b) For small m show that the low energy continuum Hamiltonian has the form of a 3 dimensional Dirac hamiltonian.

(c) Show that a domain wall where m changes sign is associated with a 2+1D massless Dirac fermion surface state. Which sign of m is the topological insulator?

(d) Determine the Z_2 topological invariant characterizing this insulating band structure by evaluating the parity eigenvalues of the occupied states at the time reversal invariant momenta.

5. Kitaev model for a 1D topological superconductor

Consider a one dimensional tight binding model for a p wave superconductor,

$$H = \sum_i t(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) - \mu c_i^\dagger c_i + \Delta(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger)$$

(a) Determine the spectrum of quasiparticle excitations, noting the points as a function of Δ/t and μ/t where the energy gap vanishes. It is again useful to write the Bogoliubov Hamiltonian as $H(k) = \mathbf{h}(k) \cdot \vec{\tau}$, where $\vec{\tau}$ is a Pauli matrix in the Nambu particle-hole space. Note the similarity to problem (1).

(b) Show that a domain wall between states on opposite sides of the phase boundary you found in (a) is associated with a zero energy eigenstate of the Bogoliubov de Gennes Hamiltonian.

(c) (*Majorana chain*) Write $c_i = \gamma_{i,A} + i\gamma_{i,B}$, where $\gamma_{i,A/B}$ are Majorana fermion operators. Rewrite the Hamiltonian in these variables and show that for a special ratio Δ/t it becomes a nearest neighbor Majorana chain of the form

$$H = i \sum_i t_1 \gamma_{i,A}\gamma_{i,B} + t_2 \gamma_{i,B}\gamma_{i+1,A}$$

Determine $t_{1,2}$ in terms of the original parameters, and determine the region of the phase diagram that is a topological superconductor. Understand the presence of zero modes at the end by considering the strong coupling limits where either t_1 or t_2 go to zero.

(d) For a finite N site Majorana chain with $t_1 = 0$ and $t_2 = u$, describe the low energy eigenstates ($E \ll u$). It is useful to define a new fermion operator $c = \gamma_{1,A} + i\gamma_{N,B}$

6. Read Green model for a 2D p+ip topological superconductor

Consider a 2D tight binding model for spinless electrons described by the momentum space Bogoliubov de Gennes Hamiltonian

$$H = \tau^z \left[t(\cos k_x a + \cos k_y a) - \mu\right] + \Delta\left(\tau^x \sin k_x a + \tau^y \sin k_y a\right)$$

where $\vec{\tau}$ are Pauli Matrices in Nambu space.

(a) Verify that H respects the intrinsic particle-hole symmetry.

(b) Analyze the phase diagram and identify the topological superconductor phase. Show that domain walls between phases are associated with chiral Majorana fermion modes.