Weak interactions and disorder in 0D Conductors

1. Single-electron states in a weakly-disordered, finite-size (0D) conductor.

   1.1. How electrons are confined
   1.2. Localized states: correlation of their energies
   1.3. The idea of Random Matrix Theory (RMT), Gaussian Ensembles
   1.4. Wave functions of localized states in RMT
   1.5. Crossovers between ensembles of different symmetry classes

2. Interactions in disordered 0D conductors

   2.1. Universal Hamiltonian
   2.2. Off-diagonal matrix elements of interaction Hamiltonian

3. Electron transport through a 0D conductor
   3.1. Coulomb blockade effect in transport
   3.2. Breit-Wigner formula for tunneling
   3.3. Mesoscopic fluctuations of the tunneling conductance

4. Electron relaxation in a 0D conductor

1.1. Why electrons are confined?

(a) Grains of metal: work function

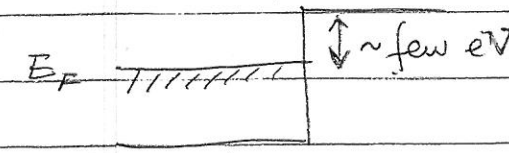


Fig. 1

Consider a metallic grain of linear size L (and volume $V = L^d$). Assuming no symmetries, the level spacing in the grain is $\delta E \equiv \frac{1}{\nu_d} = \frac{1}{V\nu} = \frac{1}{L^d \nu}$. Here ν is the bulk density of states (i.e., number of states per unit volume per unite energy).

Compare δE with the electrostatic energy for charge e: $E_c \sim e^2/L$ (works for any d)

$$\frac{\delta E}{E_c} \sim \frac{1}{L^{d-1}} \cdot \frac{1}{e^2 \nu} \sim \left(\frac{\lambda_F}{L}\right)^{d-1} \cdot \frac{1}{r_s} \ll 1.$$

Here $r_s = e^2/\hbar v_F$ is the gas parameter, $r_s \gtrsim 1$.
Fermi wavelength (for a metal!) $\lambda_F \sim 0.3\,\text{nm}$; $L \gtrsim 10^2\,\text{nm}$

(b) Quantum dots in semiconductor heterostructures

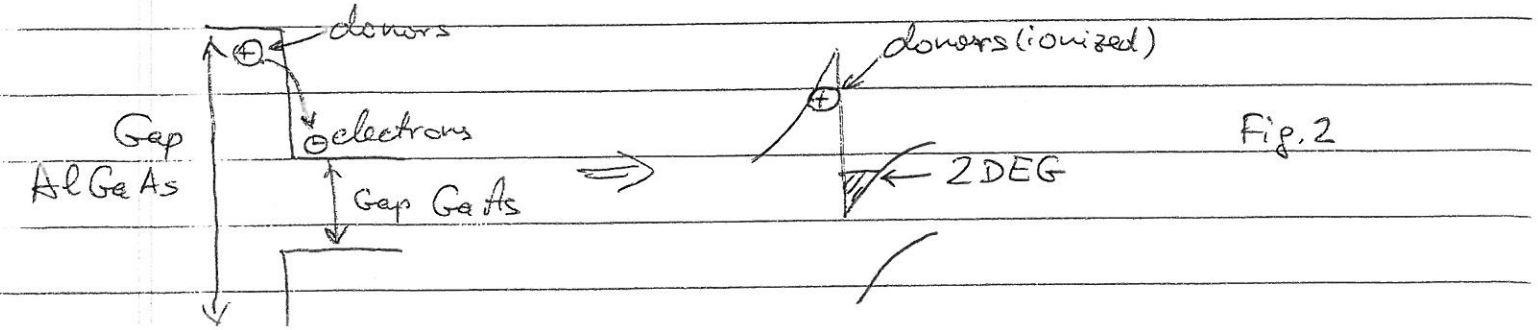


Fig. 2

Ref to experiments: L. Kouwenhoven & C.M. Marcus, Phys. World 11, 35 (1998) + refs. therein

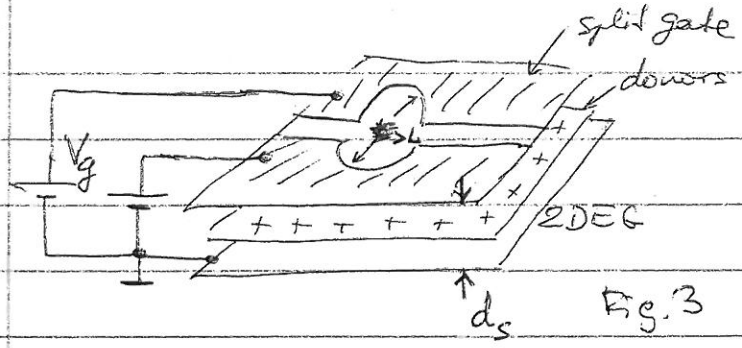


Fig. 3

Gate (negatively biased) depletes 2DEG

Field created by the gate: $E_g = \frac{V_g}{d_s}$. Poisson: $E_g = \frac{4\pi e \delta n}{\varkappa}$

$$\delta n = \frac{\varkappa V_g}{4\pi e d_s}$$

$V_g \leftarrow \sim 1\,V$

$d_s \leftarrow 0.1 \div 1\,\mu m$

$\delta n \leftarrow 10^{11} \div 10^{12}\,cm^{-2}$

Depletion is reached at $\delta n = n_s$ (also $\sim 10^{11} \div 10^{12}\,cm^{-2}$)

$\Longrightarrow$ Geometry of the gate(s) controls the size and the shape of dot(s). Lack of spatial symmetries → chaotic "billiard" for electrons.

For roughly equally spaced levels:

$$\delta E \sim \frac{\hbar^2}{m^* L^2} \sim \frac{\mathcal{E}_F}{N}$$

$\mathcal{E}_F \leftarrow$ Fermi energy

$N \leftarrow$ number of electrons in a dot

$L \leftarrow$ dot size

At $L \lesssim d_s \Rightarrow E_c \sim e^2/\varkappa L$; $\boxed{\delta E / E_c \sim 1/r_s \sqrt{N}}$

Typically $N \sim 1 \div 10^3$

1.2. Description of localized states

$$\hat{\mathcal{H}}_F = \int d\vec{r} \left[\frac{1}{2m} \nabla\psi^+ \nabla\psi + \mathcal{U}(\vec{r})\,\psi^+\psi \right] \qquad (1)$$

$\mathcal{U}(\vec{r})$ is a confining potential (+ disorder!)

Single-particle eigenstates:

$$\left(-\frac{\nabla^2}{2m} + U(\vec{r})\right)\varphi_\alpha(\vec{r}) = \varepsilon_\alpha \varphi_\alpha(\vec{r}), \qquad \int d\vec{r}\,|\varphi_\alpha(r)|^2 = 1 \qquad (2)$$

$$\hat{\mathcal{H}}_F = \sum_\alpha \varepsilon_\alpha d_\alpha^+ d_\alpha, \qquad \hat{d}_\alpha = \int d\vec{r}\,\hat{\psi}(\vec{r})\varphi_\alpha^*(\vec{r})$$

$U(\vec{r})$ depends on a <u>random</u> realization $\Rightarrow$ we are interested in statistical properties of ε_i, φ_i, rather than their specific form for a given realization.

We concentrate first on the properties of energy spectrum. Introduce the retarded/advanced Green funct.,

$$G^{R/A}(\varepsilon, \vec{r}_1, \vec{r}_2) = \sum_\alpha \frac{\varphi_\alpha^*(r_2)\varphi_\alpha(r_1)}{\varepsilon - \varepsilon_\alpha \pm i0} \qquad (3)$$

(cf. the momentum representation for free-space case in A. Andreev notes).

The single-particle density of states is

$$\nu(\varepsilon) \underset{\text{Def}}{=} \sum_\alpha \delta(\varepsilon - \varepsilon_\alpha) = \frac{1}{2\pi i}\int d\vec{r}\left\{G^A(\varepsilon, \vec{r}, \vec{r}) - G^R(\varepsilon, \vec{r}, \vec{r})\right\} \qquad (4)$$

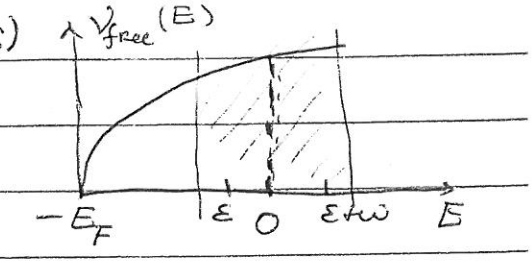
We define the average level spacing δE by

$$\langle \nu(\varepsilon)\rangle = 1/\delta E \qquad (5)$$

$\langle \ldots \rangle$ is average over disorder (e.g. location of impurities) and ε is far away from the bottom of the band.

The correlation between the levels energies is quantified by the pair correlation function,

$$R^2(\omega) = (\delta E)^2 \langle \nu(\varepsilon)\,\nu(\varepsilon+\omega)\rangle - 1 \qquad (6)$$



We are interested in a narrow compared ~~to~~ E_F band of energies $\Rightarrow$ replace $\nu_{free}(E)$ by a constant DOS. This is why $R^2(\omega)$ is independent of ε (same true for εE), and we may think of $\langle \nu(\varepsilon)\,\nu(\varepsilon+\omega)\rangle$ as of $\lim_{E_F > E_0 \to \infty} \frac{1}{E_0} \int_{E_0/2}^{E_0/2} d\varepsilon \, \langle \nu(\varepsilon)\,\nu(\varepsilon+\omega)\rangle$

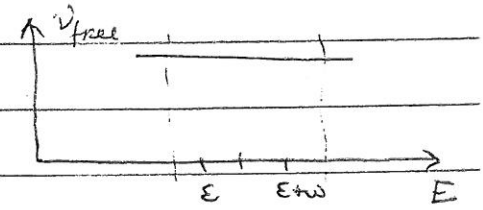


Fig. 4

B. L. Altshuler, B. I. Shklovskii, Sov. Phys. JETP 64, 127 (1986)

Using that simplification after we substitute Eq. (4) into $\Rightarrow$ Eq. (6), we find

$$R^2(\omega) = \frac{(\delta E)^2}{(2\pi)^2} \cdot 2\,\mathrm{Re} \int d\vec{r}_1\, d\vec{r}_2 \, \langle G^R(\varepsilon+\omega, r_1, r_1)\, G^A(\varepsilon, r_2, r_2)\rangle - 1 \qquad (7)$$

We are interested in sample sizes $L \gg \lambda_F$, so typically $|\vec{r}_1 - \vec{r}_2| \gg \lambda_F$ in the integrand here. Its reducible part is $\propto \nu^2 = 1/(\delta E)^2$ cancels -1 in Eq. (7). The irreducible part is expressed in terms of diffuson and cooperon.

Indeed, here is a graph representation of $\underbrace{G^R}_{\text{bare}}(\varepsilon, r_1, r_1)$ & and ~~$G(\varepsilon, r_1$~~ $G^R(\varepsilon+\omega, r_1, r_1) \cdot G^A(\varepsilon, r_2, r_2)$:

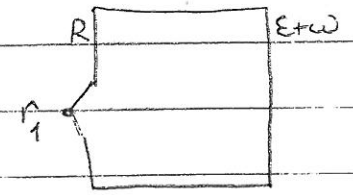


Fig 5a

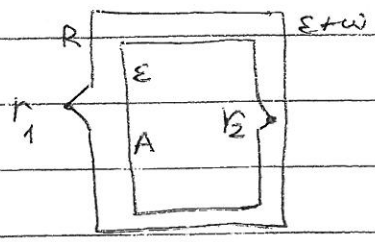


Fig. 5b

The independent "dressing" of $G^R(\varepsilon+\omega, r_a, r_b)$ and $G^A(\varepsilon, r_a, r_b)$ results in $\langle G^R(\varepsilon+\omega, r_a, r_b)\rangle$ and $\langle G^A(\varepsilon, r_a, r_b)\rangle$ which decay as ~~exp{-r/2ℓ}~~ $(1/|r_a - r_b|)\cdot \exp\{-|r_a - r_b|/2\ell\}$, (at $r \gg \ell$) see lectures of A. Andreev. ~~Therefore~~ This decay is not important in Fig. 5a, as $r_a = r_b = r_1$. The "joint dressing" of G^R and G^A in Fig. 5b, however, produces a long-ranged contribution to $\ll G^R(\varepsilon+\omega, r_1, r_1) G^A(\varepsilon, r_2, r_2) \gg_c$ (the irreducible part of the averaged product). Consider the following two diagrams:

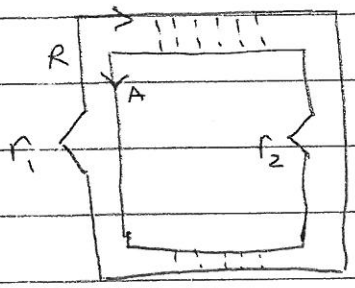


Fig. 6a

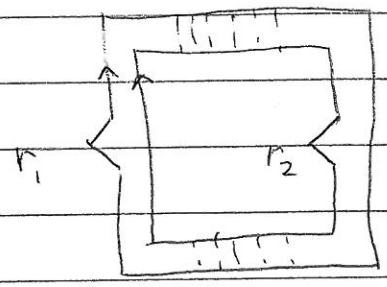
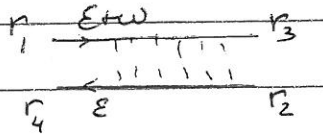
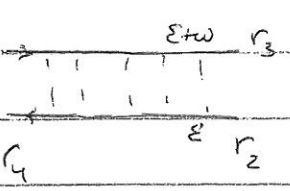


Fig. 6b

The object 

corresponds to a diffuson (~~4~~ times fast-decaying functions of $|\vec{r}_1 - \vec{r}_4|$, $|\vec{r}_2 - \vec{r}_3|$).



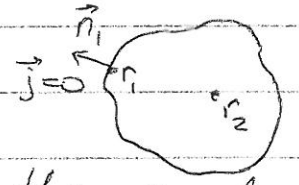
$$\sim \frac{e^{-|\vec{r}_1-\vec{r}_4|/2\ell}\cdot e^{-|\vec{r}_3-\vec{r}_2|/2\ell}}{|\vec{r}_1-\vec{r}_4|\cdot|\vec{r}_3-\vec{r}_2|} D\left(\frac{(\vec{r}_3+\vec{r}_2)}{2}, \frac{(\vec{r}_1+\vec{r}_4)}{2}, \omega\right) \qquad (8)$$

Here $D(\vec{r}_1,\vec{r}_2;\omega)$ is the real-space counterpart of $D(q,\omega)$ in A. Andreev lectures. By inspecting $D(q,\omega)$ one may see that $D(\vec{r}_1,\vec{r}_2;\omega)$ satisfies (true to the name) the diffusion equation

$$\left(-i\omega + D\left(-i\nabla_{r_1}\right)^2\right) D(\vec{r}_1,\vec{r}_2,\omega) = \delta(\vec{r}_1-\vec{r}_2) \qquad (9)$$

In free space $D(\vec{r}_1,\vec{r}_2,\omega) = D(\vec{r}_1-\vec{r}_2,\omega)$ is a Fourier transform of $\sim 1/(Dq^2-i\omega)$. For a confined geometry, one has to supplement Eq (9) by a zero-current condition

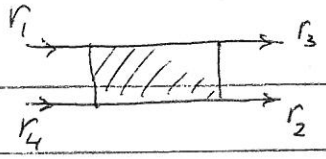
$$\left.\frac{\partial D}{\partial n_1}\right|_{\text{boundary}} = 0 \qquad (10)$$



at the boundary of the conductor. We may expand $D(\vec{r}_1,\vec{r}_2,\omega)$ in eigenfunctions $f_n(\vec{r})$ satisfying the equation

$$-D\nabla^2 f_n = \gamma_n f_n, \quad \left.\frac{\partial f_n}{\partial \vec{n}}\right|_{\text{Boundary}} = 0, \quad \|f\| = 1 \qquad (11)$$

Regardless the shape of the conductor, $\gamma_0 = 0$ and $f_0 = 1/V^{1/2}$ is a solution of Eq. (11) – this is the zero mode of the diffuson. The next mode has eigenenergy $\gamma_1 \sim D/L^2$. We may associate it with the Thouless energy, $\boxed{\gamma_1 \equiv E_T}$.

The object 

is considered similarly and is reduced to a cooperon. In the absence of magnetic field (which creates flux through electron orbits), $D(\vec{r}_1, \vec{r}_2, \omega) = C(\vec{r}_1, \vec{r}_2, \omega)$, see lectures of A. Andreev. In the presence of the field, equation for $C(\vec{r}_1, \vec{r}_2, \omega)$ differs from Eq. (9) by a replacement

$$-i\vec{\nabla}_r \to -i\vec{\nabla}_r - \frac{2e}{c}\vec{A}(\vec{r}); \qquad \text{curl}\,\vec{A} = \vec{B}, \tag{12}$$

and proper generalization of the current operator in Eq. (10). The presence of the vector-potential term lifts the energy of the $n=0$ mode,

$$\gamma_0(B) \sim E_T \cdot \left(\Phi/\Phi_0\right)^2 \quad \text{at flux } \Phi \ll \Phi_0 \tag{13}$$

Note that $\gamma_0(B) \sim \delta E$ defines a flux

$$\Phi_c \sim \left(\frac{\delta E}{E_T}\right)^{1/2} \Phi_0 = \Phi_0/\sqrt{g} \tag{14}$$

Where g is the dimensionless conductance of the sample, $g \gg 1$ for a "good" conductor with $k_F l \gg 1$. The correlation flux Φ_c determines the characteristic value of flux needed to shift levels by amount $\sim \delta E$, and thus change $R^2(\omega)$ at $\omega \sim \delta E$. Further increase of Φ modifies $R^2(\omega)$ in broader range of $\omega \lesssim E_T(\Phi/\Phi_0)^2$ (true for $\Phi \lesssim \Phi_0$).

Returning now to the irreducible correlator

we may replace (at $|\vec{r}_1 - \vec{r}_2| \gg l$)

$$\langle\langle G^R(\varepsilon+\omega, r_1, r_1)\, G^A(\varepsilon, r_2, r_2)\rangle\rangle = \begin{cases} D^2(\vec{r}_1, \vec{r}_2, \omega), & \Phi \gg \Phi_c,\ \omega \lesssim E_T \\ D^2(\vec{r}_1, \vec{r}_2, \omega) + C^2(\vec{r}_1, \vec{r}_2, \omega), & \Phi = 0 \end{cases} \quad (15)$$

see Eqs. 6(c,b). Using now Eqs. (7), (9), (11), and (15) we find

$$R^2(\omega) = \frac{(\delta E)^2}{\beta\pi^2} \mathrm{Re} \sum_n \frac{1}{(i\omega+\gamma_n)^2} = -\frac{(\delta E)^2}{\beta^2\pi^2\omega^2} + \frac{(\delta E)^2}{\beta\pi^2} \mathrm{Re} \sum_{\gamma_n \neq 0} \frac{1}{(i\omega+\gamma_n)^2} \quad (16)$$

level repulsion — $\sim (\delta E)^2/E_T^2$

(note the (–) sign!) ⇐ valid at $\omega \gtrsim \delta E$

Here $\beta = 1$ corresponds to $\Phi = 0$ case → time-reversal symmetry observed, Orthogonal ensemble, and $\beta = 2$ to broken TR symmetry ($B \neq 0$), Unitary ensemble of random states.

Note that $R^2(\omega)$ is universal at $\omega \ll E_T$.

1.3. The idea of RMT; 1.4. Wave functions in RMT; 1.5. Crossovers between ensembles.

The universal part of Eq. (16) and its generalisation to arbitrarily small $\omega/\delta E$ can be obtained from RMT which considers a gaussian ensemble of random matrices h_{ij}, with the distribution

$$P\{\hat{h}\} \sim \exp\{-A\,\mathrm{Tr}\{\hat{h}^2\}\} \quad (17)$$

such that

$$\langle h_{ij} h_{ij'} \rangle = \frac{M}{\pi^2} (\delta E)^2 \delta_{ij'} \delta_{ji'}, \quad \beta = 2 \text{ (for definiteness)} \quad (18)$$

Here the matrix ($M \times M$) size M is large (formally one needs $M \to \infty$ limit; explicit value of M does not matter as long as $M > E_T/\delta E = g \gg 1$).

$\beta = 2 \to$ Gaussian unitary ensemble (GUE)

$\beta = 1 \to$ Gaussian orthogonal ensemble (GOE)

(additional term $\sim \delta_{ii'} \delta_{jj'}$ in Eq. (18))

The free-fermion Hamiltonian in RMT takes form

$$\hat{\mathcal{H}}_F = \sum_{ij} h_{ij} \psi_i^+ \psi_j .$$

There is also a way to describe crossover between GOE and GUE (see, e.g. ABG review, Section 2.2 for further refs.)

[Excercise: take RM (2×2) with real matrix elements h_{ij} ($i,j = 1,2$) and $\langle h_{ij}^2 \rangle = 2(\delta E)^2/\pi^2$. Find the probability density of having level splitting $\varepsilon \ll \delta E$.]

Eigenvectors of the random matrix are also random. The joint distribution of the components C_i^α of a normalized eigenvector is

$$W(\{C_i^\alpha\}) = \text{const} \cdot \delta\left(1 - \sum_{i=1}^{M} |C_i^\alpha|^2\right), \qquad (19)$$

$\langle C_i^\alpha C_j^\beta \rangle \sim \delta_{\alpha\beta}$, $\langle C_i^\alpha \varepsilon^\beta \rangle = 0$ ($\vec{C}^\alpha$ distribution of is independent of eigenvalues distribution).

$$W(\eta) = \begin{cases} \dfrac{1}{\sqrt{2\pi\eta}} e^{-\eta/2} & \text{GOE} \\ e^{-\eta} & \text{GUE} \end{cases}, \quad \eta = M C_i^2 . \qquad (20)$$

(Porter, Thomas Phys. Rev. 104, 483 (1956)).

A. Altland, Brouwer, L. G. – Phys. Rep. 358, No 5–6 (2002) – a review (ABG)

Returning to real space, $\langle C_i^\alpha C_j^\beta \rangle \sim \delta_{\alpha\beta}$ means that $\langle \varphi_\alpha^*(r) \varphi_\beta(r') \rangle \propto \delta_{\alpha\beta}$ (GOE or GUE) and in addition $\langle \varphi_\alpha(r) \varphi_\beta(r') \rangle \propto \delta_{\alpha\beta}$ for GOE. In other words, in the leading in $1/g$ order, $\langle \varphi_\alpha^*(r) \varphi_\beta(r') \rangle = 0$ for $\alpha \neq \beta$.

To determine the result of averaging for $\alpha = \beta$, we note

$$G^A(\varepsilon, \vec{r}_1, \vec{r}_2) - G^R(\varepsilon, \vec{r}_1, \vec{r}_2) = 2\pi i \sum_\beta \varphi_\beta(r_1) \varphi_\beta^*(r_2) \delta(\varepsilon - \varepsilon_\alpha) \quad (21)$$

(cf. Eq. (3)), integrate it over an interval $(\varepsilon_\alpha + \delta/2, \varepsilon_\alpha - \delta/2)$ around some ε_α with $\delta \ll 1$ and average over disorder potential. The result (details: see ABG, Section 2.3) is

$$\langle \varphi_\alpha(\vec{r}_1) \varphi_\beta^*(\vec{r}_2) \rangle = \delta_{\alpha\beta} \frac{1}{V} F(k_F |\vec{r}_1 - \vec{r}_2|), \quad |\vec{r}_1 - \vec{r}_2| \ll \ell, L \quad (21a)$$

$$F(k_F r) = \langle e^{i\vec{k}\vec{r}} \rangle_{|\vec{k}| = k_F} \quad (21b)$$

This is obtained from $\langle G(\varepsilon \approx \varepsilon_F, r_1, r_2) \rangle$ or from the Berry ansatz (1977): $\varphi(\vec{r}) \sim \sum_k C_{\vec{k}} e^{i\vec{k}\vec{r}}$, random $C_{\vec{k}}$ with $|\vec{k}| = k_F$ and $\langle C_{\vec{k}_1} C_{\vec{k}_2}^* \rangle \sim \delta_{\vec{k}_1 \vec{k}_2}$. Note that $F(0) = 1$ and $F(x \gg 1)$ decays as $1/x$ in 3D and $1/x^{1/2}$ in 2D.

Distribution of a local value, $|\varphi_\alpha(\vec{r})|^2$ (away from boundary by $\gtrsim \lambda_F$) is given by Eq. (20). Crossover between the distributions - see Eq. (14) for the needed flux of B field.

Higher-order averages in the leading in $1/g$ approximation:

$$\langle \varphi_\alpha(\vec{r}_1)\varphi_\beta(\vec{r}_2)\varphi^*_\gamma(\vec{r}_3)\varphi^*_\delta(\vec{r}_4)\rangle^{(0)} = \delta_{\alpha\delta}\delta_{\beta\gamma}F_{14}F_{23} + \delta_{\alpha\gamma}\delta_{\beta\delta}F_{13}F_{24},$$

$$F_{ij} \equiv F(k_F|\vec{r}_i - \vec{r}_j|) \qquad (22)$$

in the case of GUE. (In GOE, there is also a term $\propto \delta_{\alpha\beta}\delta_{\gamma\delta}F_{12}F_{34}$).

Final remarks about the single-electron states.

① The Hamiltonian

$$\mathcal{H}_F = \sum_\alpha \varepsilon_\alpha d^+_\alpha d_\alpha, \qquad \text{cf. (2)}$$

ε_α described by an RMT

② Most of the results above are insensitive to the ratio $L/\ell \gtrsim 1$. If scattering occurs only at the boundary (and not in the volume), while electron motion remains chaotic, the details of Eqs. (9) and (11) change. That leads to a change in the estimate of Thouless energy; $E_T \sim v_F/L$, see ABG for further refs.

③ GOE, GUE are applicable in the absence of SO interaction. If SO is present ⇒ GSE. Application of $\vec{B}$ field (orbital, or Zeeman) → crossover to GUE for spinors. Interesting consequences for g-factors [D.G. Salinas et al PRB 60, 6137 (1999); P.W. Brouwer et al PRL 85, 369 (2000), K.A. Matveev et al PRL 85, 369 (2000)].

2.1. Interactions: universal Hamiltonian

In the basis of wave functions φ_α, the ~~not~~ two-particle (density-density) interaction takes form

$$\mathcal{H}_{int} = \frac{1}{2} \sum_{\sigma_1,\sigma_2} \sum_{\alpha\beta\gamma\delta} h_{\alpha\beta\gamma\delta}\, d^+_{\alpha\sigma_1} d^+_{\beta\sigma_2} d_{\gamma\sigma_2} d_{\delta\sigma_1} \qquad (23)$$

$$h_{\alpha\beta\gamma\delta} = \int d\vec{r}_1\, d\vec{r}_2\, \varphi_\alpha(r_1)\varphi_\beta(r_2)\varphi^*_\gamma(r_2)\varphi^*_\delta(r_1)\, V(\vec{r}_1-\vec{r}_2) \qquad (24)$$

(no SO interaction!)

We will see that

$$\mathcal{H}_{int} = \mathcal{H}^{(0)}_{int} + \mathcal{H}^{(1/g)}_{int} \qquad (25)$$

where $\mathcal{H}^{(0)}_{int}$ is universal, independent of the geometry, describing the low-energy ($|\varepsilon| \lesssim E_T$) properties; $\mathcal{H}^{(1/g)}$ scales as some (positive) power of $\frac{1}{g}$ (i.e., small) and fluctuates.

The part $\mathcal{H}^{(0)}_{int}$ may include only invariants of the RMT distribution Eq. (17), to be compatible with RMT. No SO $\Rightarrow$ the invariants are

$$\hat{N} = \sum_{\alpha\sigma} d^+_{\alpha\sigma} d_{\alpha\sigma}; \quad \vec{S} = \frac{1}{2} \sum_{\alpha\sigma_1\sigma_2} d^+_{\alpha\sigma_1} \vec{\sigma}_{\sigma_1\sigma_2} d_{\alpha\sigma_2}; \qquad (26a)$$

$$\hat{T} = \sum_\alpha d_{\alpha\uparrow} d_{\alpha\downarrow} \qquad (26b)$$

(cf. also Chalker lectures notes).

Gauge invariance : $\mathcal{H}_{int}^{(0)}$ depends only on T^+T;
SU(2) invariance : $\mathcal{H}_{int}^{(0)}$ ——— " ——— on $(\vec{S})^2$
$\mathcal{H}_{int} \propto \psi^4$: $\mathcal{H}_{int}^{(0)}$ is a linear combination,

$$\mathcal{H}_{int}^{(0)} = E_c \hat{N}^2 - E_s (\vec{S})^2 + \lambda T^+ T \qquad (27)$$

Coefficients E_c, E_s, and λ are found by averaging Eq. (24) with the help of Eq. (22).

For a "toy model" of a short-range interaction,

$$V(\vec{r}) = v \cdot \delta E \cdot V \, \delta(\vec{r}) \qquad (28)$$

(constant $v \ll 1$) and GUE we find

$$h_{\alpha\beta\gamma\delta} = v \cdot \delta E \cdot (\delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\alpha\gamma}\delta_{\beta\delta}) \qquad (29)$$

Using identity

$$2\delta_{\sigma_1\sigma_4}\delta_{\sigma_2\sigma_3} = \delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4} + \vec{\sigma}_{\sigma_1\sigma_3} \cdot \vec{\sigma}_{\sigma_2\sigma_4}$$

in Eq. (23) we find

$$E_c = \frac{1}{4} v \cdot \delta E, \quad E_s = V \cdot \delta E, \quad \lambda = 0 \qquad (30)$$

Note that $\lambda = 0$ is a direct consequence of GUE. The smallness (at $v \ll 1$) of E_c is a consequence of a short-ranged (and weak interaction)

Now we briefly discuss: (1) how E_c and E_s are modified in the case of Coulomb interaction, and (2) what happens to λ in GOE ensemble (details: (1) in Section 2.3.2 of ABG; (2) in Appendix B of that review).

(1) We assume the screening radius r_D ($\sim 1/\sqrt{e^2 \nu_{3D}}$ in 3D)

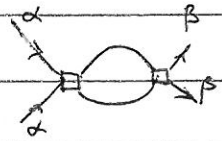
is small, $r_D \ll L$. In this case E_c is an electrostatic energy of the 0D conductor (size L) charged by a single electron. It can be expressed in terms of capacitance, $\boxed{E_c \equiv e^2/2C \sim e^2/\varkappa L.}$ The exchange energy is evaluated with the help of Eq. (26) for $F(k_F r)$ and screened Coulomb potential,

$$E_s = \int dr_1 dr_2 \, V_{scr}(\vec{r}_1 - \vec{r}_2) F^2(k_F|\vec{r}_1 - \vec{r}_2|) \simeq \frac{1}{\pi} \cdot \delta E \cdot r_s \ln \frac{1}{r_s} \quad (31)$$

Note that $E_s > 0$ favoring FM state, but $E_s \ll \delta E$ so there is no instability at small r_s.

(2) Effective λ in Eq. (27) depends on the sign of the Bare interaction constant. Indeed, consider correction to the low-energy ($|\epsilon| \lesssim E_T$) Hamiltonian originating from the energy scale $E_T \lesssim |E| < E^* \leftarrow$ high-energy cut-off,

$$\delta \mathcal{H}_{\alpha\alpha\beta\beta} \simeq -\sum_{E^* > |\epsilon_{\gamma,\delta}| > E_T} \frac{\mathcal{H}_{\alpha\alpha\gamma\delta}\mathcal{H}_{\delta\gamma\beta\beta}}{|\epsilon_\gamma + \epsilon_\delta|} \theta(\epsilon_\gamma \epsilon_\delta) \quad (32)$$



$$\simeq -\frac{1}{\delta E} \cdot \langle \mathcal{H}_{\alpha\alpha\beta\beta} \rangle^2 \ln\left(\frac{E^*}{E_T}\right) \quad (32a)$$

The leading-log series can be summed (e.g. by RG)

$$\frac{d}{d(\ln n_{imp})}\left(\frac{\lambda}{\delta E}\right) = -\left(\frac{\lambda}{\delta E}\right)^2 \Rightarrow \lambda(E_T) = \frac{\lambda(E^*)}{1 + \frac{\lambda(E^*)}{\delta E} \ln \frac{E^*}{E_T}} \quad (33)$$

For $\lambda(E^*) > 0$ and large $\ln(E^*/E_T)$, the effective constant $\lambda(E_T)/\delta E$ is small (neglect in the rest). If $\lambda(E^*) < 0 \Rightarrow$ BCS instability (superconducting grain)

The divergence in Eq. (33) at $\lambda(E^*)<0$ occurs at $E_{BCS} \sim E^* \exp\left\{-\frac{\delta E}{|\lambda(E^*)|}\right\}$. As long as $E_{BCS} \gg \delta E$, the superconducting transition is only weakly broadened, and T_c is independent of the size of the conductor. Due to the ~~finite~~ fixed number of electrons, and due to their BCS pairing, there are oscillations of the ground-state energy with the number parity of electrons in a conductor. This and related issues are reviewed in: J. von Delft, D.C. Ralph, Phys. Rep. 345, 61 (2001) – exp. and theory; see also K.A. Matveev et.al. Mod. Phys. Lett. 8, 1007 (1994). In the limit $E_{BCS} \lesssim \delta E$, the BCS transition is totally smeared out; however, the even-odd effect in the ground state energy persists, see Matveev, Larkin, PRL 78, 3749 (1997).

useful refs:

In the rest of these lectures, we set $\lambda \to 0$, $E_s \sim \delta E \cdot r_s$, and $E_c \sim e^2/\varepsilon L$ in Eq. (27).

2.2. Interactions: off-diagonal matrix elements

As it directly follows from Eq. (22), average matrix elements $\langle h_{\alpha\beta\gamma\delta} \rangle$ with the indices not equal each other pair-wise are zero. Being on average zero, $h_{\alpha\beta\gamma\delta}$ exhibit mesoscopic fluctuations, $\langle |h_{\alpha\beta\gamma\delta}|^2 \rangle \neq 0$. A detailed evaluation is tedious, we demonstrate here only the idea (for details, see refs. [66,67] in ABG).

Here are the steps. (1) Using Eq. (24), write $h_{\alpha\beta\gamma\delta} \cdot h^*_{\alpha\beta\gamma\delta}$ in the form of 4-fold $\int dr_1 dr_2 dr_3 dr_4 \ldots$. (2) The integrand contains a product of 8 wavefunctions with pair-wise equal indices, like $\varphi_\alpha(r_1)\varphi^*_\alpha(r_3)$. Form $\frac{1}{\delta}\int_{\varepsilon_\alpha-\delta/2}^{\varepsilon_\alpha+\delta/2} d\varepsilon \left(G^A - G^R\right)$ out of these pairs, using Eq. (21). There will be 4 factors $(G^A - G^R)$ with different pairs of coordinates. (3) Now try averaging $\langle \underbrace{(G^A-G^R)\cdot\ldots\cdot(G^A-G^R)}_{\text{4 factors}} \rangle$ over disorder. The reducible part, $\underbrace{\langle G^A-G^R\rangle\cdot\ldots\cdot\langle G^A-G^R\rangle}_{\text{4 averages}}$ will be short-ranged, yielding negligible (in the limit $k_F L \to \infty$) result. (The value of $\langle |h_{\alpha\beta\gamma\delta}|^2\rangle$ we are looking for must scale as $(\delta E)^2$, at fixed $g = E_T/\delta E$.) The contribution having the right scaling with δE, comes from $\langle (G^R-G^A)*(G^R-G^A)\rangle \cdot \langle (G^R-G^A)*(G^R-G^A)\rangle$. A careful consideration shows that

$$\langle (G^R-G^A)_{E_1, r_1, r_2} \; (G^R-G^A)_{E_2, r_3, r_4} \rangle \propto \left(F_{14}F_{23} + F_{13}F_{24}\right) \cdot \operatorname{Re} D(r_1, r_2; E_2 - E_1) \tag{34}$$

(4) Finally, using $D(r_1, r_2, \omega) \sim \sum_n \frac{f_n(r_1) f_n(r_2)}{\gamma_n + i\omega}$, c.f. Eq. (11) for notations, and performing $\int dr_1 \ldots dr_4$ we arrive at

$$\boxed{\langle |\delta h_{\alpha\beta\gamma\delta}|^2 \rangle = c \cdot v^2 \cdot \left(\frac{\delta E}{g}\right)^2}, \quad c = \frac{2}{\pi^2} \sum_{\gamma_n \neq 0} \left(\gamma_1/\gamma_n\right)^2. \tag{35}$$

Here v defined in Eq. (28). Note the $\langle |h_{\alpha\beta\gamma\delta}|^2\rangle \propto 1/g^2$ dependence

3. Electron Transport through a 0D conductor

Typical setting: tunnel junctions to source and drain (L,R) and capacitive coupling to a gate

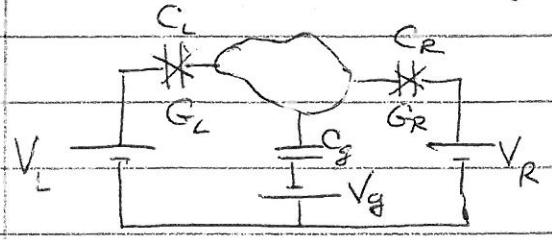


Fig. 7.

$$C \equiv C_\Sigma = C_L + C_R + C_g$$

Electrostatic energy:

$$E(\hat{N}) = E_c\,(\hat{N} - N)^2 \qquad (35)$$

$$N = V_g / e C_g, \quad E_c = e^2/2C_\Sigma$$

Typical measurements: linear conductance $G(V_g, T)$; non-linear differential conductance dI/dV_{LR} vs. $V_{LR} \equiv V_L - V_R$,

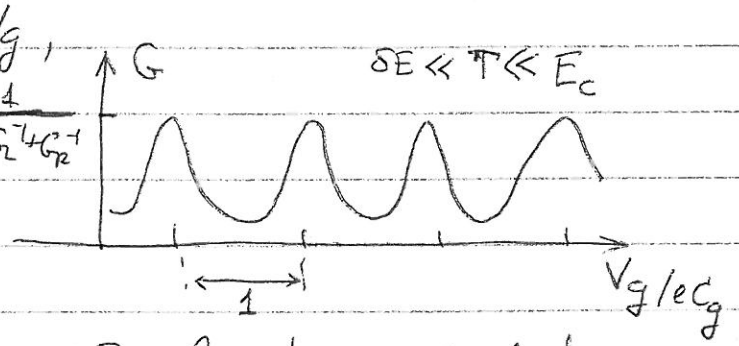


Fig. 8a – Linear conductance

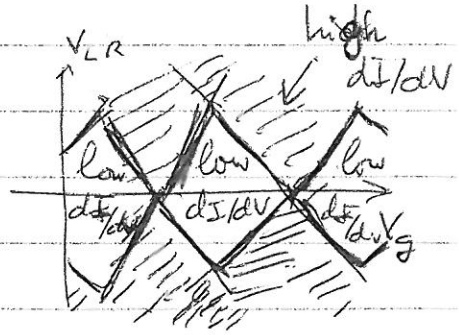


Fig. 8b – "Coulomb diamonds"

Key point: at N = half-integer there is no energy penalty for changing N by 1, while at N = integer the penalty is $\sim E_c$. That leads to oscillatory $G(V_g)$ at $T \ll E_c$, $G_{valley}/G_{peak} \sim e^{-E_c/T}$

3.1 Coulomb blockade: rate equations at $\delta E \ll T \ll E_c$

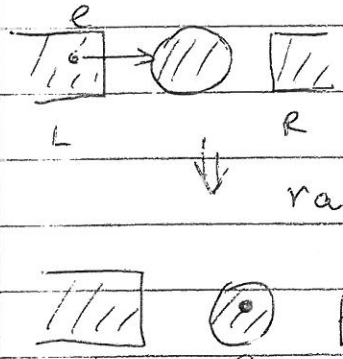


Fig. 9

Probability of no extra electron on a dot: W_0

$$W_0 + W_1 = 1 \qquad (37)$$

$$(T \ll E_c)$$

$$\frac{1}{\tau_{0\to 1}} = W_0 \frac{G_L}{e^2/\pi\hbar} \int dE_L\, dE_{dot}\; n(E_L)\left(1 - n(E_{dot})\right) \qquad (38)$$

$$\times\, \delta(E_L + eV_L + E_0 - E_{dot} - E_1)$$

$$= W_0 \frac{G_L}{e^2/\pi\hbar} f(E_1 - E_0 - eV_L),$$

where

$$f(x) = \frac{x}{e^{x/T} - 1}, \qquad x = E_1 - E_0 - eV_L \qquad (39)$$

(E_1, E_0: electrostatic energies at $V_L = 0$)

In equilibrium:

$$W_0 = \frac{f(E_0 - E_1)}{f(E_0 - E_1) + f(E_1 - E_0)}, \qquad W_1 = \frac{f(E_1 - E_0)}{f(E_1 - E_0) + f(E_0 - E_1)} \qquad (40)$$

Balance eq.:

$$G_L\left[W_0 f(E_1 - E_0 - eV_L) - W_1 f(E_0 - E_1 + eV_L)\right] \qquad (41)$$

$$= G_R\left[W_1 f(E_0 - E_1 - eV_R) - W_0 f(E_1 - E_0 - eV_R)\right]$$

Linear conductance ($V = V_L - V_R \to 0$)

$$G = \frac{1}{R_\infty} \frac{f(\varepsilon) f'(\varepsilon) + f(-\varepsilon) f'(\varepsilon)}{f(\varepsilon) + f(-\varepsilon)}, \qquad (42)$$

$\varepsilon = E_1 - E_0$ depends on V_g, $R_\infty = \frac{1}{G_L} + \frac{1}{G_R}$. Using Eq. (40) we find

$$G(N,T) = \frac{1}{2R_\infty} \cdot \frac{2E_c (N - N^*)/T}{\sinh\left(2E_c (N - N^*)/T\right)}, \qquad (43)$$

$N^* = n + 1/2$. All G_{peak} are the same here. Assumed: independent tunneling through G_L, G_R; equilibration between tunneling events.

3.2. Low-temp. conductance ($T \ll \delta E$), resonant tunneling

At $T \ll \delta E$, only the closest to the chemical potential level in the dot contributes significantly to transport. We consider here spinless fermions (instead of the true $s = 1/2$ electrons) to simplify the description. Then only two states of the dot are important ("empty level" and ~~full~~ "occupied level"). We denote the energy of occupied level (measured from the value for empty one) as ε_n. A (tunnel) Hamiltonian one may employ to analyze conductance has the form

$$\mathcal{H} = \sum_k (\xi_k - eV_L) a_k^+ a_k + \sum_p (\xi_p - eV_R) a_p^+ a_p + \varepsilon_n a_n^+ a_n + \sum_k \left(t_{kn} a_k^+ a_n + \text{h.c.}\right) + \sum_p \left(t_{pn} a_p^+ a_n + \text{h.c.}\right) \qquad (44)$$

Here the first 2 terms describe the leads (see Fig. 7), and

the last two describe tunneling into the dot. To evaluate conductance, we need the L → R tunneling amplitude $|\ |^2 \equiv \pi_{LR}$ (as it is clear from Eq. (44), tunneling is elastic in our model)

The geometric series is easy to sum up to obtain

$$T_{LR}(\varepsilon) = \frac{4\Gamma_L^{(n)}\Gamma_R^{(n)}}{\left(\Gamma_L^{(n)}+\Gamma_R^{(n)}\right)^2 + (\varepsilon-\varepsilon_n)^2}, \quad \Gamma^{(n)}_{L,R} = \frac{2\pi}{\hbar}\int d(k,p)\,|t_{(k,p)n}|^2 \times \delta(\xi_{k,p}-\varepsilon_0) \qquad (45)$$

$\Gamma_{L,R}$ are level width with respect to electron escape to the L, R lead. $T_{LR}(\varepsilon)$ is the familiar Breit-Wigner formula. The linear conductance is obtained from $T_{LR}(\varepsilon)$ by integration with $(-\partial f_F/\partial\varepsilon)$:

$$G(\varepsilon_0, \Gamma_L^{(n)}, \Gamma_R^{(n)}, T) = \frac{e^2}{2\pi\hbar}\int d\varepsilon \left(-\frac{\partial f_F}{\partial\varepsilon}\right) \frac{4\Gamma_L^{(n)}\Gamma_R^{(n)}}{\left(\Gamma_L^{(n)}+\Gamma_R^{(n)}\right)^2 + (\varepsilon-\varepsilon_0)^2}, \qquad (46)$$

$$f_F(\varepsilon) = \frac{1}{e^{\varepsilon/T}+1}.$$

At $T \gg \Gamma_L+\Gamma_R$, the detailed shape of resonance, Eq. (45) is not important, so possibly the made simplifications (neglected spin degeneracy) is not crucial. The peak conductance associated with level n is

$$G^{(n)}_{max} \sim \frac{e^2}{h}\,\frac{\Gamma_L^{(n)}\Gamma_R^{(n)}}{\Gamma_L^{(n)}+\Gamma_R^{(n)}}\cdot\frac{1}{T} \equiv \frac{e^2}{h}\cdot\frac{1}{T}\cdot\Gamma_n \qquad (47)$$

3.3. Mesoscopic fluctuations of $G_{max}^{(n)}$.

We note now that $|t_{kn}|^2 \propto |\psi_n(\vec{r}=\vec{R}_L)|^2$, the probability density for an electron to be near the contact (L). [In fact, the relation between $|t_{kn}|^2$ and $|\psi_n(\vec{R}_L)|^2$ is a bit more complicated, but it does not change the outcome – see, e.g., Prada, Sols, Eur. Phys. J. B40, 379 (2004), Houzet et al. PRB 72, 104507 (2005).] In the case of chaotic or disordered 0D conductor,

$$P(\Gamma_n) = \int d\Gamma_L^{(n)}\, d\Gamma_R^{(n)}\; W(\Gamma_L^{(n)})\, W(\Gamma_R^{(n)})\, \delta\left(\frac{\Gamma_L^{(n)}\Gamma_R^{(n)}}{\Gamma_L^{(n)}+\Gamma_R^{(n)}} - \Gamma_n\right), \tag{48}$$

Where $W(\Gamma_{L,R}^{(n)})$ are Porter-Thomas distributions, Eq. (20), properly normalized.

In the case of symmetric (on average) setting, one finds (Jalabert, Stone, Alhassid, 1992)

$$G_{max} = \frac{e^2}{h}\,\frac{\pi\overline{\Gamma}}{2T}\,\alpha, \qquad P(\alpha) = \begin{cases} \sqrt{2/\pi\alpha}\; e^{-2\alpha} & (GOE) \\ 4\alpha\left[K_0(2\alpha)+K_1(2\alpha)\right] e^{-2\alpha} & (GUE) \end{cases} \tag{49}$$

Crossover GOE → GUE: Falko, Efetov 1996. Further refs. to theory and experiment – see ABG review.

3.4. Remarks RE Conductance through a dot

(a) In even valleys, at $T \to 0$, conductance $G_{valley} \propto G_L G_R \cdot \frac{\delta E}{E_c}$, fluctuates

(b) $S = 1/2 \Rightarrow$ Rabe eqs. instead of Breit–Wigner for G_{max},

does not change much the result Eq. (49), see ABG review

© In odd valleys, $S=1/2$ leads to Kondo effect and high conductance, despite Coulomb blockade. The E_s term, see Eqs (27), (31), allows for higher spins of a quantum dot, affecting conductance. See

⟹ L.G. Pustilnik (2005)

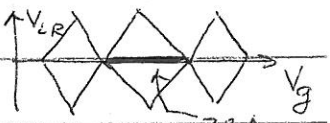


Fig 10

4. Electron relaxation in disordered conductors: 0D and higher dimensions.

We may use Eq (35) to estimate the electron relaxation rate $1/\tau(\varepsilon)$,

$$\frac{1}{\tau_{\varepsilon_\alpha}} = \frac{2\pi}{\hbar}\left\langle \sum_{\beta\gamma\varepsilon} \int_0^{\varepsilon} d\omega\, \delta(\varepsilon_\alpha - \varepsilon_\beta - \omega)\, \delta(\varepsilon_\gamma - \varepsilon_\varepsilon - \omega)\, \theta(\varepsilon_\delta + \omega) |h_{\alpha\beta\gamma\varepsilon}|^2 \right\rangle \tag{50}$$

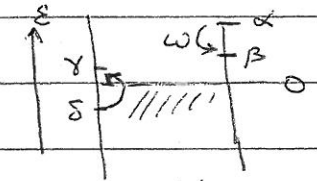


Fig. 11

One may think of "open" 0D conductor with some broadened (Γ) levels due to tunneling into reservoir. Then let us replace $\sum_{\beta\gamma\varepsilon} \ldots$ by $\int d\varepsilon_\beta d\varepsilon_\gamma d\varepsilon_\varepsilon / (\delta E)^3 \ldots$ Then Eqs. (50) and (35) yield

⟹
$$\frac{1}{\tau_\varepsilon} \sim \frac{\varepsilon^2}{\delta E} \cdot \frac{1}{g^2} \cdot v^2, \quad \varepsilon \lesssim E_T. \tag{51}$$

There are two important observations to make:

Lecture notes Low-Temperature Transport Through a Quantum Dot – L.G. Pustilnik, Les Houches, Session LXXXI, 2004. Eds.: M. Bouchiat, Y. Gefen, S. Guéron, G. Montambaux, J. Dalibard – Nanophysics: Coherence and Transport, Elsevier 2005; p. 427

Course 7, S. Guéron, and

Europhys. Lett. 28, 115 (1994)
B.L. Altshuler, A.G. Aronov, in: Electron-electron Int. in Disordered Systems, ed. by A.L. Efros & M. Pollak, N-Holl. 1985

① $1/\tau_\varepsilon$ depends on disorder (g in denominator of Eq. (51)): although $1/\tau_\varepsilon \propto \varepsilon^2$ like in a clean system (FL), but coefficient depends on D_α at $\varepsilon \lesssim E_T$.

② At $\varepsilon \sim E_T$, the level width $1/\tau_\varepsilon \sim \varepsilon E$. That may (interaction-induced

indicate a possibility to determine level-width self-consistently, even in the limit $\Gamma \to 0$. Further discussion of that direction is related to the many-body localization problem (probably D. Huse may discuss that.)

Note that Eq. (34) can be applied to evaluate $1/\tau_\varepsilon$ in any dimension (3D, film, wire with width $w \gg \lambda_F$), not only 0D. Considering higher dimensions, it is instructive compare "clean" and "dirty" cases (see review of Altshuler & Aronov, 1985). In a clean system,

$$\frac{1}{\tau_\varepsilon} \propto \int_0^{\varepsilon} d\omega \int_{-\omega}^{0} d\varepsilon' \int_0^{\infty} q^2 dq \underbrace{\int dx \int dx'}_{-1 \quad 0 \quad -1} \delta(\omega - q v_F x)\, \delta(\omega - q v_F x') \cdot |V_{\vec{q}}|^2 \qquad (52)$$

Here $V_{\vec{q}}$ is the interaction Fourier component, $x = \cos(\hat{\vec{p}, \vec{q}})$, $x' = \cos(\hat{p', q})$. Integrating over x, x' and accounting for the fact that $\int dq \ldots$ is converging at upper limit ($q \lesssim 2k_F$), independent of ε ($\ll E_F$), we find:

$$\frac{1}{\tau_\varepsilon} \propto \int_0^{\varepsilon} d\omega \int_{-\omega}^{0} d\varepsilon' \int_0^{} q^2 dq\, |V_q|^2 \frac{1}{(q v_F)^2} \propto \varepsilon^2 \qquad (53)$$

Here the factor $1/q v_F$ has the meaning of

interaction time for a ballistic electron motion. It is this factor that gets replaced by Re $D(q,\omega)$ in the case of diffusive motion:

$$\frac{1}{\tau_\varepsilon} \propto \int_0^{\varepsilon} d\omega \int_{-\omega}^{0} d\varepsilon' \int d\mathbf{q}^{d} |V_q|^2 \left\{ \mathrm{Re} \frac{1}{i\omega + Dq^2} \right\}^2 \qquad (54)$$

As the result,

$$\frac{1}{\tau_\varepsilon} \propto \begin{cases} \varepsilon^{1/2} & d=1 \leftarrow \text{diffusive wire} \\ \varepsilon & d=2 \leftarrow \text{film} \\ \varepsilon^{3/2} & d=3 \end{cases} \qquad (55)$$

(A. Schmid 74, Altshuler, Aronov 1979, 1981)

The $1/\tau_\varepsilon \propto \varepsilon^{1/2}$ in diffusive wires (free of magnetic impurities!) was tested in SACLAY, see e.g. B. Huard et. al. Solid St. Comm. 131, 599 (2004).

[Turns out magnetic impurities drastically affect $1/\tau_\varepsilon$ - Kaminski, LG PRL 86, 2400 (2001) - theo; experiments - in addition to cited above, see also, e.g., B. Huard et al PRL 95, 036802 (2005).]