BOULDER 2003 SUMMER SCHOOL

A. GARG - LECTURE #2

TOPIC II: SPIN PATH INTEGRAL & SC LIMIT

- CLASSICAL DYNAMICS FOR SPIN (POISSON BRACKETS)
- QUANTIZE VIA PATH INTEGRAL
- LEAST ACTION/SEMICLASSICAL PATHS
- EXPLICIT BOUNDARY TERMS IN ACTION/HPF.
  - PATHS MUST BE COMPLEXIFIED
    $\hat{n}(t) \equiv \{\theta(t), \phi(t)\}$; θ, ϕ BOTH COMPLEX
  - SOLVES OVERDETERMINATION PROBLEM
  - YIELDS HAMILTON-JACOBI EQNS.
- GAUSSIAN FLUCTUATIONS
  - GLOBAL ANOMALY
  - STANDARD METHOD (JACOBI'S EQN.) FAILS
  - EXTRA TERM (SOLARI, 1987; KOCHETOV, 1995).
- DO FOR PARTICLE COHERENT STATE PATH INTEG
  - SAME PROBLEM, SAME SOLN., SIMPLER.
- BOHR-SOMMERFELD QUANTIZATION RULE.

0. Coherent states for a particle

Let a and a^+ be lowering and raising operators for a one-dimensional harmonic oscillator with frequency ω_R. The CR's and number eigenstates are defined as usual:

$$[a, a^+] = 1 , \tag{0.1}$$

ground state $|0\rangle$: $a|0\rangle = 0$. (0.2)

$$|n\rangle \equiv \frac{1}{\sqrt{n!}} (a^+)^n |0\rangle \quad , (n = 1, 2, 3, \ldots) . \tag{0.3}$$

$$a^+a |n\rangle = n |n\rangle . \tag{0.4}$$

Then, for any complex no. z, the coherent state is

$$|z\rangle \equiv e^{z a^+} |0\rangle \tag{0.5}$$

$$\langle n | z \rangle = z^n / \sqrt{n!} \tag{0.6}$$

$$\langle z' | z \rangle = e^{\bar{z}' z} \qquad \bar{z} = \text{complex conjugate of } z \tag{0.7}$$

Note. States not normalized. Also, not orthogonal.

$$\int \frac{d^2 z}{\pi} e^{-\bar{z} z} |z\rangle\langle z| = 1 \tag{0.8}$$

$\left(d^2 z = dx\,dy ; \quad \int \frac{d^2 z}{\pi} \bar{z}^n z^m e^{-\bar{z} z} = \delta_{nm} . \right)$.
$\langle z | x^{op} | z \rangle \sim (z + \bar{z})$; $\langle z | p^{op} | z \rangle \sim (z - \bar{z})/i$; so $z = \langle x \rangle + i \langle p \rangle$. (0.9)

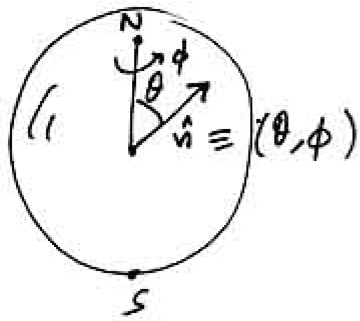
Note that we can now use these states as an overcomplete basis to analyze any 1d potential problem, including a harmonic osc. with a different freq. ω'. The oscillator used to define the coherent states is hence referred to as a <u>reference oscillator</u> – hence the R in ω_R. I do not know if a similar arbitrariness can be introduced into the spin coherent states.

1. Spin Coherent states

Let $|j,m\rangle$ be the usual J^2, J_z eigenstates

$$\left.\begin{aligned} J^2|j,m\rangle &= j(j+1)\,|j,m\rangle \\ J_z\,|j,m\rangle &= m\,|j,m\rangle \end{aligned}\right\} \quad (1.1)$$

We take the state $|j,j\rangle$ which has the maximal projection of $\vec{J}$ along $\hat{z}$, and rotate it so as to have maximal projection along a new direction $\hat{n}$



$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \quad (1.2)$$

$$|\hat{n}\rangle \equiv e^{-i\theta(-J_x\sin\phi + J_y\cos\phi)}\,|j,j\rangle . \quad (1.3)$$

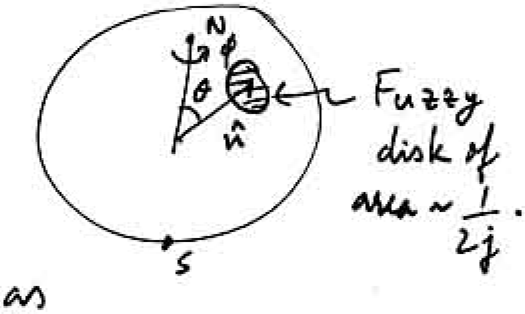
(You can use many other rotation operators to go from $|\hat{z}\rangle \equiv |j,j\rangle$ to $|\hat{n}\rangle$. This is a gauge choice in definition of $|\hat{n}\rangle$.)

The state $|\hat{n}\rangle$ has the properties

$$\vec{J}\cdot\hat{n}\,|\hat{n}\rangle = j\,|n\rangle , \quad (1.4)$$

$$\langle\hat{n}|(\vec{J}\times\hat{n})^2|\hat{n}\rangle = j ; \quad (1.5)$$

i.e. $|\vec{J}_\perp| \sim \sqrt{j}$. As j gets large, this is v. small compared to j. In fact, it cannot be made any smaller, and so $|\hat{n}\rangle$ is as close as one can get to a classical state in which all components of $\vec{J}$ are definite – have zero uncertainty. One can think of $|\hat{n}\rangle$ as occupying a disk of area $\sim (2j)^{-1}$ on the unit sphere.



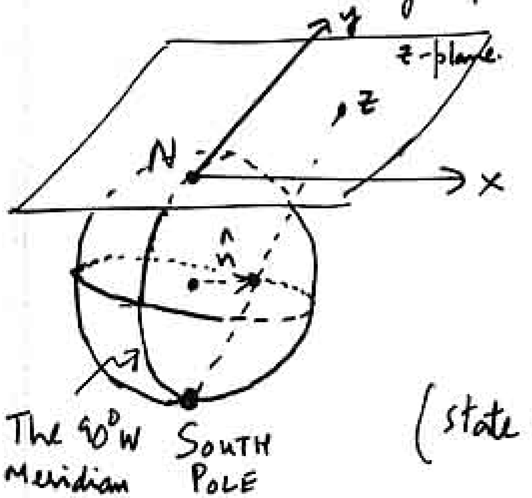
Further properties of $|\hat{n}\rangle$:

$$\langle \hat{n} | \vec{J}^{op} | \hat{n} \rangle = j\,\hat{n} \qquad \text{(very useful)}. \tag{1.6}$$

$$\underset{\uparrow \text{ spherical harmonic tensor operator}}{\langle \hat{n} | \mathcal{Y}_{\ell m}(\vec{J}^{op}) | \hat{n} \rangle} = \frac{\ell!}{2^\ell j^\ell}\binom{2j}{\ell}\, j^\ell \underbrace{Y_{\ell m}(\hat{n})}_{\text{spherical harmonic}} . \tag{1.7}$$

Proof is left as an exercise for the reader. [(1.6) is easy, (1.7) is not.]

In stead of $\hat{n}$ & the polar coordinates (θ,ϕ), I like to use the sterographic representation.



$$z = \tan\frac{\theta}{2}\, e^{-i\phi} \tag{1.8}$$

$$|z\rangle \equiv e^{zJ_-}|j,j\rangle \tag{1.9}$$

$$\langle z'|z\rangle = (1+\bar{z}'z)^{2j} \tag{1.10}$$

(state not orthogonal, or normalized.)

Finally, we have the resolution of the unity

$$\mathbb{1} = \frac{2j+1}{\pi}\int \frac{d^2z}{(1+\bar{z}z)^{2j+2}}\,|z\rangle\langle z| \tag{1.11}$$

$$= \frac{2j+1}{\pi}\int \sin\theta\, d\theta\, d\phi\; |\hat{n}\rangle\langle\hat{n}| \tag{1.12}$$

Advantage of $|z\rangle$ is that matrix elements like $\langle z'| J_x^{\,n} J_y^{\,k} |z\rangle$ etc. will be holomorphic in z, antiholomorphic in z'.

See Assa Auerbach's book, and Dan Arovas's lecture notes for many other neat ways of writing & working with these states. (Warning: Dan uses $\hat{\Omega}$ for my $\hat{n}$. I like Ω for areas.)

2. The discrete coherent-state path integral

Suppose we are given a 1d problem with Hamiltonian $\mathcal{H}$. The most general quantum mechanical object is the propagator

$$\langle \psi_f | e^{-i\mathcal{H}t/\hbar} | \psi_i \rangle \quad , \qquad (2.1)$$

where $|\psi_i\rangle$ & $|\psi_f\rangle$ are some initial and final states. To analyze the behaviour in general, we want to use some complete basis set for the $|\psi\rangle$'s, e.g., the position eigenstates $|x\rangle$. This leads to

$$\langle x_f | \, e^{-i\mathcal{H}T} \, | x_i \rangle , \qquad (\hbar = 1 \text{ hence}) \qquad (2.2)$$

which can be written via the Feynman path integral

$$\langle x_f | e^{-i\mathcal{H}T} | x_i \rangle = \int_{x(0)=x_i}^{x(T)=x_f} [dx] \, \exp\left\{ +i \int_0^T dt \left(\tfrac{1}{2} m \dot{x}^2 - V(x) \right) \right\} . \qquad (2.3)$$

However, the states $|x\rangle$ are not very classical ($\Delta p = \infty$). The more classical object would appear to be the coherent-state propagator

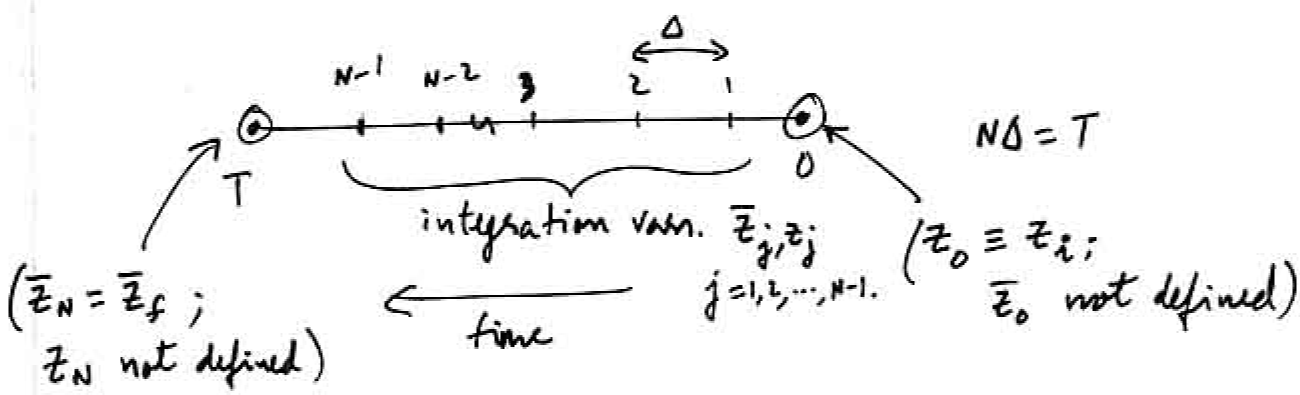
$$K(\bar{z}_f, z_i; T) = \langle z_f | \, e^{-i\mathcal{H}T} \, | z_i \rangle . \qquad (2.4)$$

This can also be written (at least formally) as a path integral. For particles, this may seem an idle exercise, since we have (2.3) and know its semiclassical limit. (See Eq. 7.13). For spin, coherent states are the only way I know to get path integrals. All the problems that afflict the spin coherent state path integral also afflict the particle coherent state path integral. The two are analyzed in exactly the same way, but in the spin case, you have to carry around additional factors of $2j$, $2j+1$, and the $(1+\bar{z}z)^{-2j}$ measure factors, which makes it a right royal pain. So I will do only the particle case – this has all the features. You can do the spin case for yourself in exact parallel, or read

H.G. Solari, J. Math. Phys. 28, 1097 (1987),
or E.A. Kochetov, " 36, 4667 (1995),
or M. Stone, K.-S. Park, AG, JMP 41, 8025 (2000).

OK. Enough chit-chat. Shut up and calculate. Insert $N-1$ factors of unity using (0.8):

$$K = \prod_{j=1}^{N-1} \int \frac{d^2 z_j}{\pi} e^{-\bar{z}_j z_j} \langle \bar{z}_{j+1} | e^{-i\mathcal{H}\Delta} | z_j \rangle \times \langle \bar{z}_1 | e^{-i\mathcal{H}\Delta} | z_0 \rangle \quad (2.5)$$



Note: $(N-1)$ insertions of unity, but N matrix elements, one for each "link", or time slice.

$$\langle \bar{z}_{j+1} | e^{-i\mathcal{H}\Delta} | z_j \rangle \approx \langle z_{j+1} | z_j \rangle \Big[1 - i\Delta \underbrace{\frac{\langle z_{j+1} | \mathcal{H} | z_j \rangle}{\langle j+1 | j \rangle}}_{\equiv H_s(\bar{z}_{j+1}, z_j)} + \mathcal{O}(\Delta^2) \Big] \quad (2.6)$$

$$\approx e^{+\bar{z}_{j+1} z_j} e^{-i\Delta H_s(\bar{z}_{j+1}, z_j)} + \mathcal{O}(\Delta^2) \quad (2.7)$$

Write action for discrete path:

$$
\begin{aligned}
iS^{disc} = \ & \bar{z}_N z_{N-1} - \bar{z}_{N-1} z_{N-1} + \left(\bar{z}_{N-1} z_{N-2} - \bar{z}_{N-2} z_{N-2}\right) \\
& + \cdots + \left(\bar{z}_2 z_1 - \bar{z}_1 z_1\right) + \bar{z}_1 z_0 \\
& - i\Delta \sum_{j=1}^{N-1} H_s\left(\bar{z}_{j+1}, z_j\right),
\end{aligned}
\tag{2.8}
$$

$$
\text{And } K = \lim_{\substack{\Delta \to 0 \\ (N\Delta = T)}} \int \frac{d^2 z_1\, d^2 z_2 \cdots d^2 z_{N-1}}{\pi^{N-1}}\, e^{iS_{disc}}. \tag{2.9}
$$

$$
\underset{\sim}{\text{Note}}:\ iS_{disc} = \text{fn. of} \left\{ \begin{matrix} \overset{z_i}{\underset{\parallel}{z_0}}, & z_1, & z_2, & \ldots, & \overset{\text{Not!}}{\cancel{z_N}} \\ \underset{\text{NOT!}}{\cancel{\bar{z}_0}}, & \bar{z}_1, & \bar{z}_2, & \ldots, & \underset{\bar{z}_f}{\underset{\parallel}{\bar{z}_N}} \end{matrix} \right\} \tag{2.10}
$$

When we give in and regard the path as smooth in the $\Delta \to 0$ limit, (2.10) will imply the boundary conditions

$$
z(0) = z_i, \quad \bar{z}(T) = \bar{z}_f. \tag{2.11}
$$

As it stands, however, (2.9) is exact (or at least as exact as any self-respecting physicist ever wants): We do some zillion-fold multiple integral, repeat with ever increasing values of N, and take the limit $\cancel{\Delta}\ N \to \infty$. Needless to say, nobody knows how to do that except for the harmonic oscillator. However, when we look at the semiclassical approximation, it will turn out that we have to come back to S_{disc}, as the continuous form will prove treacherous.

3. Formal continuous path integral

We take the $\Delta \to 0$ limit in (2.8) - (2.10), hope that only continuous & smooth paths matter, and write

$$K = \int [d^2 z] \quad \exp\, i S[\bar{z}(t), z(t)] \quad , \qquad (3.1)$$

with
$$i S[\bar{z}(t), z(t)] = \frac{1}{2}\left(\bar{z}_f\, z(T) + \bar{z}(0) z_i \right)$$
$$+ \int_0^T \left\{ \frac{z\dot{\bar{z}} - \bar{z}\dot{z}}{2} - i\, H_s(t) \right\} dt, \qquad (3.2)$$

$$H_s(t) = \langle \bar{z}(t) |\, \mathcal{H}\, | z(t) \rangle / \langle \bar{z}(t) | z(t) \rangle , \qquad (3.3)$$

and the path obeys

$$z(0) = z_i, \quad \bar{z}(T) = \bar{z}_f \; . \qquad (3.4)$$

Note that S contains explicit boundary dependent terms. You can write, by integration by parts,

$$\int_0^T \frac{z\dot{\bar{z}} - \bar{z}\dot{z}}{2}\, dt = \int_0^T z\dot{\bar{z}} - \frac{1}{2}\left(\bar{z}_f\, z(T) - \bar{z}(0) z_i \right), \qquad (3.5)$$

and write the boundary term as $\bar{z}(0) z_i$, but this is less symmetrical.

Now in the semiclassical limit, "$\hbar \to 0$", we assume that there is some analog of the stationary phase approx. for ordinary integrals. Instead of a <u>point</u> of stationary phase, we now now have a <u>path</u> of stationary phase/action, i.e. we need a "least action principle."

<u>Note</u>: S is really the Hamilton Principal Function (or the action at fixed time.)

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• Vary $z(t)$, keeping $\delta z(0) = 0$. ($z(0) = z_i$ is fixed.)

$$i\delta S = \frac{1}{2}\bar{z}_f\,\delta z(T) + \int_0^T \left(\frac{\delta z\,\dot{\bar{z}} - \bar{z}\,\delta\dot{z}}{2} - i\,\frac{\partial H_s}{\partial z}\,\delta z \right) dt \qquad (3.6)$$

integrate by parts

$$= \cancel{\frac{1}{2}\bar{z}_f\,\delta z(T)} - \frac{1}{2}\Big(\cancel{\underbrace{\bar{z}(T)}_{\bar{z}_f}\,\delta z(T)} - \bar{z}(0)\,\cancel{\delta z(0)}^{\,0} \Big)$$

$$+ \int_0^T \left(\dot{\bar{z}} - i\,\frac{\partial H_s}{\partial z} \right) \delta z\, dt \qquad (3.7)$$

The integrand must vanish for all t. This gives us one Euler-Lagrange eqn. Varying $\bar{z}(t)$ gives us the other. They are:

$$\frac{dz}{dt} = -i\,\frac{\partial H_s}{\partial \bar{z}} \quad ; \quad \frac{d\bar{z}}{dt} = i\,\frac{\partial H_s}{\partial z} \qquad (3.8)$$

These are exactly like Hamilton's eqns. If you write $z = x + ip$, $\bar{z} = x - ip$, you will recover

$$\frac{dx}{dt} = \frac{\partial H_s}{\partial p}, \qquad \frac{\partial p}{\partial t} = -\frac{\partial H_s}{\partial x} \qquad (3.9)$$

• Solving (3.8) gives us the "classical" (or semiclassical) path. But to solve (3.8) we must also give boundary conditions. What to give? Eq. (3.4) says

$$z(0) = z_i, \qquad \bar{z}(T) = \bar{z}_f \qquad (3.10)$$

This only fixes $x + ip$ at $t = 0$ & $x - ip$ at $t = T$. If we also specify $\bar{z}(0)$ (say equal to z_i^*) & $\bar{z}(T)$ (say equal to $\bar{z}_f^*$), we fix x_i, p_i, and x_f, p_f. However (3.8) is equivalent to a single 2nd order differential eqn. If we demand four boundary conditions, the problem will, generally speaking, have no solutions. It is overdetermined.

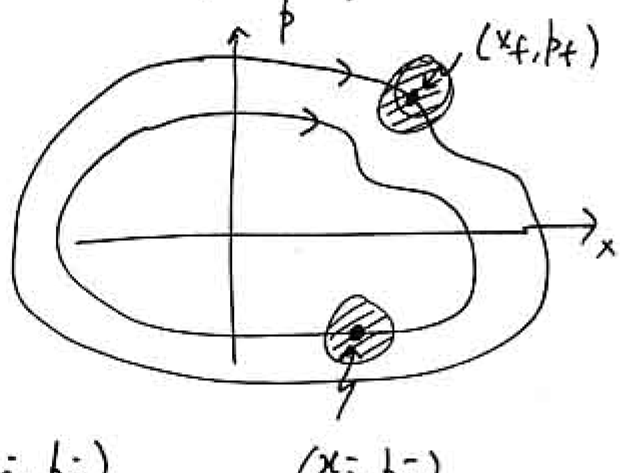
Thus, the conclusion is that if we want to have a non-trivial solution to (3.8), we must allow $\bar{z}(0)$ & $z(T)$ to float. In particular,

$$\bar{z}(0) \neq z_i^* \quad , \quad z(T) \neq (\bar{z}_f)^* \qquad (3.11)$$

(Here $*$ denotes true complex conjugate, and the bar is some kind of formal conjugate.) In fact, we cannot assume that $z(t)$ and $\bar{z}(t)$ will be true complex conjugates. In other words, if we write

$$\left.\begin{aligned} x(t) &= \tfrac{1}{2}\left(z(t) + \bar{z}(t)\right) \\ p(t) &= \tfrac{1}{2i}\left(z(t) - \bar{z}(t)\right) \end{aligned}\right\} \quad (3.12)$$

$x(t)$ & $p(t)$ will both be complex quantities. In fancy lingo, we must "complexify" the path and the phase space.



- The reason for this is easy to understand. From (3.9) we get

$$\frac{dH_s}{dt} = \frac{\partial H_s}{\partial x}\dot{x} + \frac{\partial H_s}{\partial p}\dot{p} = 0 .$$

If we try to go from some (x_i, p_i) to some arbitrary (x_f, p_f), the points will generally not lie on the same orbit in phase, and there will be no solution with real $x(t)$ & $p(t)$. One has to go to a 4-dim'l complex phase space.

Exactly the same thing happens in the case of spin. One can think of z & $\bar{z}$ (or $\cos\theta$ & ϕ) as conjugate phase-space variables. A solution to the semiclassical eqns. of motion can be found only by allowing $\bar{z}(t) \neq [z(t)]^*$, i.e., by letting $\theta(t)$ & $\phi(t)$ both be complex, i.e., by going from the 2-dim'l surface of the real sphere to the complex unit sphere, which has dimension 4.

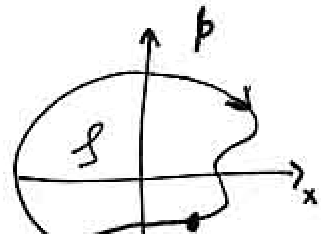
Another way to think of this problem is to go back to S_{disc}, and try and find its minimum value by setting

$$\frac{\partial S_{disc}}{\partial z_j} = \frac{\partial S_{disc}}{\partial \bar{z}_j} = 0 \quad ; \quad j = 1, 2, \cdots, N-1. \qquad (3.13)$$

Note that we can't vary $z_i \equiv z_0$ or $\bar{z}_f \equiv \bar{z}_N$. The solutions to the $2\times(N-1)$ eqns. (3.13) will in general be independently complex numbers for z_j & $\bar{z}_j$.

(Another point to note is that the kinetic term in (3.2) is a geometrical phase. If we consider a closed path.

$$S_{kinetic} = -\frac{i}{2}\oint (z\, d\bar{z} - \bar{z}\, dz)$$
$$= i\int_{S} dz \wedge d\bar{z} = \text{Area}(S).$$



Also note this is $\oint p\, dx$.
This is, therefore, a geometrical phase, although not as interesting as the one for spin.)

- Once one has gone to a 4-d phase space, the semiclassical action (HPF) has all the requisite properties. Thus, suppose the soln. to the semiclassical eqns. of motion (3.8) is denoted $z_s(t)$, $\bar{z}_s(t)$. Then, define

$$S_s(\bar{z}_f, z_i, T) = S(\bar{z}_s(t), z_s(t)) \tag{3.14}$$

The action S_s obeys the Hamilton-Jacobi eqns. (not hard to show - vary the end points in (3.2))

$$\frac{\partial S_s}{\partial z_i} = -i\,\bar{z}(0) \quad , \quad \frac{\partial S_s}{\partial \bar{z}_f} = -i\,z(T) \tag{3.15}$$

$$\left.\begin{aligned} \partial S_s/\partial T &= -\,H_s(\bar{z}_f, z(T)) \\ &= H_s(\bar{z}(0), z_i) \end{aligned}\right\} \tag{3.16}$$

The usual eqns. are as follows. Consider a problem where we have bc's $x(0) = x_i$, $X(T) = x_f$, and let the soln. be $x_{cl}(t)$.

$$S_{cl}(x_f, x_i, T) \equiv \int_0^T \left(\frac{1}{2} m\, \dot{x}_{cl}^2 - V(x_{cl}(t))\right) dt \tag{3.17}$$

Then, $$\frac{\partial S_{cl}}{\partial x_i} = p(0) \ ; \quad \frac{\partial S_{cl}}{\partial x_f} = -p(T) \text{ etc.} \tag{3.18}$$

The emergence of the Hamilton-Jacobi eqns. is another important reason why the action must contain explicit boundary terms.

4. Fluctuation determinant - continuous form

Having found the path(s) that stationarize the action (3.1), we must now integrate over the small departures from the classical path.

Write

$$z(t) = z_s(t) + \eta(t) \,, \qquad \bar{z}(t) = \bar{z}_s(t) + \bar{\eta}(t), \tag{4.1}$$

$$\eta(0) = \bar{\eta}(T) = 0 \,, \tag{4.2}$$

$$S[z(t), \bar{z}(t)] = S_c + \cancel{\delta S}^{\,0} + \delta^2 S + \cdots \tag{4.3}$$

$\delta^2 S$ is quadratic in η & $\bar{\eta}$. Cubic & higher order terms are neglected. Then

$$K \simeq e^{iS_s} \underbrace{\int [d^2\eta] \, e^{i\delta^2 S}}_{K_{\text{reduced}}} \tag{4.4}$$

K_{red} is a Gaussian integral, which should reduce to finding a determinant.

$$i\delta^2 S = \int_0^T dt \left\{ \frac{\eta\dot{\bar{\eta}} - \bar{\eta}\dot{\eta}}{2} - \frac{i}{2}\left(\frac{\partial^2 H_s}{\partial z_s^2}\eta^2 + 2\frac{\partial^2 H_s}{\partial z_s \partial \bar{z}_s}\eta\bar{\eta} + \frac{\partial^2 H_s}{\partial \bar{z}_s^2}\bar{\eta}^2 \right) \right\} \tag{4.5}$$

$$= -\frac{i}{2}\int_0^T dt \, \begin{pmatrix} \bar{\eta}(t) & \eta(t) \end{pmatrix} \begin{pmatrix} -i\partial_t + A(t) & B(t) \\ \bar{B}(t) & i\partial_t + A(t) \end{pmatrix} \begin{pmatrix} \eta(t) \\ \bar{\eta}(t) \end{pmatrix} . \tag{4.6}$$

Here, $$A(t) = \frac{\partial^2 H_s}{\partial \bar{z}_s \partial z_s} \,, \quad B(t) = \frac{\partial^2 H_s}{\partial \bar{z}_s^2} \,, \quad \bar{B}(t) = \frac{\partial^2 H_s}{\partial z_s^2} \tag{4.7}$$

$\neq [B(t)]^* \leftarrow$ Note.

Call the 2×2 matrix differential operator in (4.6) $\mathcal{D}$:

$$\mathcal{D} = \begin{pmatrix} -i\partial_t + A(t) & B(t) \\ \bar{B}(t) & i\partial_t + A(t) \end{pmatrix} \qquad (4.8)$$

If we think of this as the limit of the discrete form as $\Delta\to 0$, then this is an infinite order matrix. As a warm up, therefore, let us do some finite order matrices.

<u>Digression on finite order Gaussian integrals</u>

Ex. 1 $$\int \frac{d^2\eta}{\pi}\, e^{-\bar{\eta}\eta a} = \iint \frac{dx\,dy}{\pi}\, e^{-a(x^2+y^2)} \qquad \left[\begin{array}{l} \eta = x+iy \\ \bar\eta = x-iy \\ \bar\eta\eta = x^2+y^2 \end{array}\right]$$

$$= \frac{1}{\pi}\left(\sqrt{\frac{2\pi}{2a}}\right)^2 = \frac{1}{a} \qquad (4.9)$$

Ex. 2 Now consider

$$Q = (\bar\eta \quad \eta)\underbrace{\begin{pmatrix} a & b \\ b & a \end{pmatrix}}_{\leftarrow M \rightarrow}\begin{pmatrix} \eta \\ \bar\eta \end{pmatrix} \qquad \left|\ \begin{pmatrix} \eta \\ \bar\eta \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \right.$$

etc.

$$= (x \quad y)\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= (x \quad y)\begin{pmatrix} 2(a+b) & \\ & 2(a-b) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= 2(a+b)x^2 + 2(a-b)y^2. \qquad (4.10)$$

$$\therefore \int \frac{d^2\eta}{\pi}\, e^{-\frac{Q}{2}} = \frac{1}{\pi}\sqrt{\frac{\pi}{a+b}}\sqrt{\frac{\pi}{a-b}} = (\mathrm{Det}\, M)^{-1/2} \qquad (4.11)$$

<u>Ex 3</u> $$Q' = (\bar{\eta}\ \eta)\underbrace{\begin{pmatrix} a & b \\ b & a \end{pmatrix}}_{\leftarrow M \rightarrow}\begin{pmatrix} \bar{\eta} \\ \eta \end{pmatrix}$$

Note difference from Ex. 2

But $$\begin{pmatrix} \bar{\eta} \\ \eta \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\leftarrow \sigma_x \rightarrow}\begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix}$$

So $$\int \frac{d^2\eta}{\pi} e^{-\frac{1}{2}Q'} = (\mathrm{Det}\, M.\ \underbrace{\mathrm{Det}\, \sigma_x}_{-1})^{-1/2} = (-\mathrm{Det}\, M)^{1/2} \quad (4.12)$$

This small difference in how one writes the quadratic form leads to this additional minus sign, which can be annoying if one forgets this point.

For larger matrices the same pattern will hold; e.g.

$$\int \frac{d^2\eta_1}{\pi}\frac{d^2\eta_2}{\pi}\left[e^{-\frac{1}{2}(\bar{\eta}_1\ \eta_1\ \bar{\eta}_2\ \eta_2)\underbrace{M}_{4\times 4}\begin{pmatrix} \bar{\eta}_1 \\ \eta_1 \\ \bar{\eta}_2 \\ \eta_2 \end{pmatrix}}\right] = \left((-1)^2\mathrm{Det}\, M\right)^{1/2} \quad (4.13)$$

<u>End of digression</u>.

OK, so we need to work out the determinant of the operator in (4.8). As it will turn out, this det is tricky. Indeed, for any operator, defining the det always requires care. We could try and define it as the product of all its eigenvalues, but that does not work here. The standard method, which works for the Feynman propagator, is based on Jacobi's accessory equation (see comments & references at end of this section.) Applied to this problem, this method would say that you solve the differential eqn.

$$\begin{pmatrix} -i\partial_t + A(t) & B(t) \\ \bar{B}(t) & i\partial_t + A(t) \end{pmatrix} \begin{pmatrix} \psi_0(t) \\ \bar{\psi}_0(t) \end{pmatrix} = 0$$

with b.c.'s $\psi_0(0) = 0, \ \bar{\psi}_0(0) = 1$.

Then $\boxed{\det \mathcal{D} = \left(\bar{\psi}_0(T)\right)^{-1/2}}$ (Naive Jacobi method) (4.14)

It is remarkable that the det should turn out to be the soln. of a differential eqn. but I will give some justification for that later. I won't try and explain why something like (4.14) should work, as it turns out that (4.14) is wrong! for the spin problem. This is explained in Sec. 5.

Comments: The Jacobi eqn. is discussed in

S. Coleman, Aspects of Symmetry. Cambridge U. Press, 1985; Chap. 7, Sec. 6.6, Appendix 1.

L.S. Schulman, Techniques & Applications of Path Integrals, Wiley, 1981; Chapters 6, 12 & 13.

Coleman's derivation is elegant, perhaps too elegant. Schulman's discussion (you have to search and piece it together) is easier to bite into.

5. The Global Anomaly.

Let's assume that the solution (4.14) is correct. We now note that in

$$K_{red} = \int d^2\eta \; e^{-\frac{i}{2}\int_0^T dt \; (\bar\eta \;\; \eta)\, \mathcal{D} \begin{pmatrix} \eta \\ \bar\eta \end{pmatrix}} \qquad (5.1)$$

there is a symmetry. Let

$$\begin{aligned} \eta(t) &\to \eta(t)\, e^{i\theta(t)} \\ \bar\eta(t) &\to \bar\eta(t)\, e^{-i\theta(t)} \end{aligned} \qquad (5.2)$$

where $\theta(t)$ is arbitrary. The path integral (5.1) should be unchanged under this transformation, as

$$d\eta(t)\, d\bar\eta(t) \;\to\; d\eta(t)\, d\bar\eta(t) \qquad \text{(just a rotation)}$$

But

$$(\bar\eta \;\; \eta)\,\mathcal{D}\begin{pmatrix} \eta \\ \bar\eta \end{pmatrix} \to (\bar\eta \;\; \bar\eta)\,\tilde{\mathcal{D}}\begin{pmatrix} \eta \\ \bar\eta \end{pmatrix}, \qquad (5.3)$$

with

$$\tilde{\mathcal{D}} = \begin{pmatrix} -i\partial_t + A + \partial_t\theta & B \\ \bar B & i\partial_t + A + \partial_t\theta \end{pmatrix} \qquad (5.4)$$

However the solution to the new Jacobi eqn. is different. If we write

$$\tilde{\mathcal{D}} \begin{pmatrix} \tilde\Psi_0 \\ \tilde{\bar\Psi}_0 \end{pmatrix} = 0 \quad , \text{ then } \; \tilde{\bar\Psi}_0(T) = e^{i[\theta(T) - \theta(0)]}\,\bar\Psi_0(T) \qquad (5.5)$$

Thus the det of $\mathcal{D}$ has an anomaly in it, which can only be resolved by going back to the discrete form. Stone, Park & Garg, JMP (2000) has one argument for fixing the problem, but since we haven't really discussed the Jacobi eqn., I am going to give Solari's original argument, where he works out the discrete det by hand.

6. Fluctuation determinant - discrete form

Let us return to the action for a discrete path, Eqs. (2.8) – (2.10), and find its stationary point, and do the fluctuations around it.

First, let us write out the stationarity conditions, just for completeness.

$$\left.\begin{aligned} 0 = i\frac{\partial S^{disc}}{\partial z_j} &= (\bar{z}_{j+1} - \bar{z}_j) - i\Delta \frac{\partial}{\partial z_j} H_s(\bar{z}_{j+1}, z_j) \\ 0 = i\frac{\partial S^{disc}}{\partial \bar{z}_j} &= -(z_j - z_{j-1}) - i\Delta \frac{\partial}{\partial \bar{z}_j} H_s(\bar{z}_j, z_{j-1}) \end{aligned}\right\} \quad (6.1)$$

These are $2\times(N-1)$ eqns. in as many variables.

Now let us work out the second variation. The only non-zero second derivatives are

$$i\frac{\partial^2 S^{disc}}{\partial z_j^2} = -i\Delta \frac{\partial^2}{\partial z_j^2} H_s(\bar{z}_{j+1}, z_j) \equiv -D_{jj} \quad (6.2)$$

$$i\frac{\partial^2 S^{disc}}{\partial \bar{z}_j^2} = -i\Delta \frac{\partial^2}{\partial \bar{z}_j^2} H_s(\bar{z}_j, z_{j-1}) \equiv -D_{\bar{j}\bar{j}} \quad (6.3)$$

$$i\frac{\partial^2 S^{disc}}{\partial \bar{z}_j \partial z_j} = -1 \qquad \equiv -D_{j\bar{j}} = -D_{\bar{j}j} \quad (6.4)$$

$$i\frac{\partial^2 S}{\partial \bar{z}_{j+1}\partial z_j} = 1 - i\Delta \frac{\partial^2}{\partial \bar{z}_{j+1}\partial z_j} H_s(\bar{z}_{j+1}, z_j) \equiv -D_{\overline{j+1},j} \quad (6.5)$$

This is defined only for $j = 1, 2, \cdots, \underline{N-2}$.

The notation for the subscripts in $D_{..}$ is self-evident.

$$K_{red} = \prod_{k=1}^{N-1} \int \frac{d^2\eta_k}{\pi} \, e^{-\frac{1}{2}(\bar\eta_1 \, \eta_1 \cdots \bar\eta_{N-1} \, \eta_{N-1}) \, \mathcal{D}^{(N-1)} \begin{pmatrix} \bar\eta_1 \\ \eta_1 \\ \vdots \\ \eta_{N-1} \end{pmatrix}} \qquad (6.6)$$

$$= \left[(-1)^{N-1} \left(\det \mathcal{D}^{(N-1)} \right) \right]^{-1/2} \qquad (6.7)$$

using (4.12) & (4.13), generalized to bigger matrices.

The matrix $\mathcal{D}^{(N-1)}$ has the following block form

$$\mathcal{D}^{(N-1)} = \left[\begin{array}{cc|cc|ccc|cc} D_{\bar 1 \bar 1} & D_{\bar 1 1} & & & & & & & \\ D_{1 \bar 1} & D_{11} & D_{1 \bar 2} & & & & & & \\ \hline & D_{\bar 2 1} & D_{\bar 2 \bar 2} & D_{\bar 2 2} & & & & & \\ & & D_{2 \bar 2} & D_{22} & D_{2 \bar 3} & & & & \\ \hline & & & D_{\bar 3 2} & \ddots & & & & \\ & & & & & & & & \\ & & & & & & D_{N-2, \overline{N-1}} & & \\ \hline & & & & & D_{\overline{N-1}, N-2} & D_{\overline{N-1}, \overline{N-1}} & D_{\overline{N-1}, N-1} & \\ & & & & & & D_{N-1, \overline{N-1}} & D_{N-1, N-1} & \end{array} \right] \qquad (6.8)$$

Start at the top left, and consider the square submatrix

$$\left.\begin{array}{lll} \text{(i) ending at } D_{\bar k \bar k} & - \text{ call its det } & L_k \\ \text{(ii) ending at } D_{kk} & - \quad " \quad " \quad " & M_k \end{array}\right\} \qquad (6.9)$$

matrix for M_k is order $2k \times 2k$

" " L_k is " $(2k-1) \times (2k-1)$.

Now unzip the dets starting from lower right.

$$\left.\begin{aligned} M_k &= D_{kk} L_k - D_{\bar{k}k} D_{k\bar{k}} M_{k-1} \\ L_k &= D_{\bar{k}\bar{k}} M_{k-1} - D_{k-1,\bar{k}} D_{\bar{k},k-1} L_{k-1} \end{aligned}\right\} \quad (6.10)$$

These equations will hold for $k = 1$, provided we set

$$L_0 = 0, \quad M_0 = 1. \qquad (6.11)$$

and $$\det\left(\mathcal{D}^{(N-1)}\right) = M_{N-1} \qquad (6.12)$$

Now define

$$M_k = (-1)^k a(k\Delta) \; ; \quad L_k = (-1)^k b(k\Delta) \qquad (6.13)$$

Then the recursion relations (6.10) along with the definitions (6.4) - (6.5) can be written as

$$a(k\Delta) - a((k-1)\Delta) = i\Delta \frac{\partial^2 H_s}{\partial z_k^2} b(k\Delta) \qquad (6.14a)$$

$$b(k\Delta) - b((k-1)\Delta) + 2i\Delta \frac{\partial^2 H_s}{\partial \bar{z}_k \partial z_{k-1}} b((k-1)\Delta) = -i\Delta \frac{\partial^2 H_s}{\partial \bar{z}_k^2} a((k-1)\Delta) \qquad (6.14b)$$

Now let $\Delta \to 0$. These become differential eqns:

$$\left.\begin{aligned} i\dot{a} + \bar{B}(t)\, b &= 0 \\ -i\dot{b} + 2A(t)\, b + B(t)\, a &= 0 . \end{aligned}\right\} \quad (6.15)$$

and the b.c.'s are $a(0) = 1$, $b(0) = 0$. $\qquad (6.16)$

These go into the Jacobi eqn. if we write

$$\left.\begin{aligned} b(t) &= \Psi_0(t)\, e^{-i\int_0^t A(t')\,dt'} \\ a(t) &= \bar{\Psi}_0(t)\, e^{-i\int_0^t A(t')\,dt'} \end{aligned}\right\} \quad (6.17)$$

We get

$$-i\,\dot{\psi}_0 + A(t)\psi_0 + B\,\bar{\psi}_0 = 0$$
$$i\,\dot{\bar{\psi}}_0 + A(t)\,\bar{\psi}_0 + \bar{B}\,\psi_0 = 0$$

i.e.
$$\begin{pmatrix} -i\partial_t + A & B \\ \bar{B} & i\partial_t + A \end{pmatrix} \begin{pmatrix} \psi_0 \\ \bar{\psi}_0 \end{pmatrix} = 0 \qquad (6.18)$$

and the b.c.'s are also correct: $\psi_0(0) = 0$, $\bar{\psi}_0(0) = 1$. (6.19)

Finally,

$$K_{red} = \left[(-1)^{N-1} M_{N-1} \right]^{-1/2} = \left[a((N-1)\Delta) \right]^{-1/2}$$
$$\xrightarrow[\Delta\to 0]{} \left[a(T) \right]^{-1/2} = \left[\bar{\psi}_0(T) \right]^{-1/2} e^{\frac{i}{2}\int_0^T A(t)\,dt} \qquad (6.20)$$

Extra term (Solari-Kochetov)

The extra term is missing from the answer (4.14) that we get use the naive Jacobi eqn. solution. There are several attempts to do the Gaussian fluctuations in the literature which get the wrong answer, as they miss this subtlety.
(See, e.g. Garg & Kim, PRB 45, 921 (1992).)

Stone, Park, & Garg, JMP (2000) shows that with the S-K term, the propagator is self-consistent under composition, i.e.

$$K_{sc}(\bar{z}_3, z_1, T) = \int \frac{d^2 z_2}{(1+\bar{z}z)^{2j+2}} \frac{2j+1}{\pi} K_{sc}(\bar{z}_3, z_2; T-\tau) K_{sc}(\bar{z}_2, z_1; \tau). \qquad (6.21)$$

Finally, Jacobi's eqn. also allows us to relate $\bar{\psi}_0(T)$ to derivatives of the action itself. We do this next.

7. The Jacobi accessory equation

Solve the eqns. of motion (3.8) as an initial value problem.

$$\text{Init. val's } z_0, \bar{z}_0 \neq z_0^* : \text{ trajectory} \equiv z(t), \bar{z}(t) \tag{7.1}$$

Now change the initial values slightly, to

$$\text{Init. vals. } z_0, \bar{z}_0 + \epsilon : \text{ trajectory} \equiv z'(t), \bar{z}'(t) \tag{7.2}$$

Write
$$\left.\begin{aligned} z'(t) &= z(t) + \epsilon\, u(t) \\ \bar{z}'(t) &= \bar{z}(t) + \epsilon\, v(t) \end{aligned}\right\} \tag{7.3}$$

Clearly,
$$u(0) = 0, \quad v(0) = 1 \tag{7.4}$$

The equations obeyed by u & v are found by varying the eqns. of motion themselves:

$$\left.\begin{aligned} \dot{u} &= -i\left(u \underbrace{\frac{\partial^2 H_s}{\partial z \partial \bar{z}}}_{=A} + v \underbrace{\frac{\partial^2 H_s}{\partial \bar{z}^2}}_{=B}\right) \\ \dot{v} &= i\left(u \underbrace{\frac{\partial^2 H_s}{\partial z^2}}_{=\bar{B}} + v \underbrace{\frac{\partial^2 H_s}{\partial \bar{z} \partial z}}_{=A}\right) \end{aligned}\right\} \tag{7.5}$$

But this is exactly Jacobi's eqn., with

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \psi_0(t) \\ \bar{\psi}_0(t) \end{pmatrix} \quad \text{\& the b.c.'s are also correct.} \tag{7.6}$$

If we now think of $\bar{z}(t)$ as a function of $\bar{z}_0$ & z_0, we have also shown that

$$\bar{\psi}_0(t) = \frac{\partial \bar{z}(t)}{\partial \bar{z}_0} \Rightarrow \bar{\psi}_0(T) = \frac{\partial \bar{z}(T)}{\partial \bar{z}_0}. \tag{7.7}$$

We now use the Hamilton-Jacobi eqns. (3.15).

$$\bar{z}_0 = i \frac{\partial S_s}{\partial z_i}(\bar{z}_f, z_i; T) \qquad z_i = z_0 . \qquad (7.8)$$

Keeping $z_0 = z_i$ fixed, vary $\bar{z}_0$. Then

$$1 = i \frac{\partial^2 S_s}{\partial \bar{z}_f \partial z_i} \frac{\partial \bar{z}_f}{\partial \bar{z}_0} . \qquad (7.9)$$

$$\therefore [\Psi_0(T)]^{-1/2} = \left(\frac{i\partial^2 S_s}{\partial \bar{z}_f \partial z_i}\right)^{1/2} \qquad (7.10)$$

Likewise, one can also show that

$$\Psi_0(T) = \partial z_i / \partial z(T) . \qquad (7.11)$$

and this also leads to (7.10).

Thus our final result for the semi-classical propagator is

$$\boxed{K_{sc}(\bar{z}_f, z_i; T) = \left(i \frac{\partial^2 S_s}{\partial \bar{z}_f \partial z_i}\right)^{1/2} e^{i S_s + \frac{i}{2}\int_0^T \frac{\partial^2 H_s}{\partial \bar{z} \partial z} dt}} \qquad (7.12)$$

Solari - Kochetov term.

Compare with the answer for the Feynman path integral:

$$K_{sc}(x_f, x_i; T) = \left(\frac{i}{2\pi} \frac{\partial^2 S_c(x_f, x_i; T)}{\partial x_f \partial x_i}\right)^{1/2} e^{i S_c} \qquad (7.13)$$

For the spin-case one gets something very like (7.12).

$$\frac{\partial^2 H_s}{\partial \bar{z} \partial z} \propto \left(\frac{\partial^2 H_s}{\partial x^2} + \frac{\partial^2 H_s}{\partial p^2}\right) \text{"="} \nabla^2 H_s . \qquad (7.14)$$

In the spin case, you get the angular part of the Laplacian.

8. Bohr-Sommerfeld quantization

As one example of the use of the semiclassical propagator, we can derive the B-S quantization rules. For particles, we get

$$\oint_{H_s = E} p\,dq + \frac{1}{2}\int\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial p^2}\right)H_s\,dt = \left(n+\tfrac{1}{2}\right)(2\pi\hbar) \quad (8.1)$$

Note that what appears here is the orbit $H_s = E$, where

$$H_s = \langle z|\mathcal{H}|z\rangle . \quad (8.2)$$

so H_s is not quite the classical Hamiltonian H_c corresponding to $\mathcal{H}$. Using the Wigner-Moyal formalism, one can show that (8.1) is equivalent to the usual rule,

$$\oint_{H_c = E} p\,dq = \left(n+\tfrac{1}{2}\right)(2\pi\hbar) . \quad (8.3)$$

For spin, we get something something more involved. Written in terms of H_s, the semi-classical Hamiltonian, it looks very much like (8.1). In terms of the classical Hamiltonian H_c (which must be related to the qm. Hamiltonian $\mathcal{H}$ by a unique prescription), we get

$$\left(j+\tfrac{1}{2}\right)\oint_{H_c=E}(1-\cos\theta)\,d\phi + \frac{1}{2j}\int_0^{T(E)}\left(\vec{J}\cdot\vec{\nabla}_J'\,H_c\right)dt = (2n+1)\pi . \quad (8.4)$$

(See Garg & Stone cond-mat/0304125 for details. This paper is actually rather short on them. Perhaps a future longer paper will rectify the defect.)