Path integrals and stationary-phase approximations

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The general formalism for path integrals expressed in terms of an arbitrary continuous representation (generalized coherent states) is applied to give c-number formulations for the canonical algebra and for the spin algebra, and is used to derive meaningful stationary-phase approximations for these two cases. Some clarification of a recent discussion by Jevicki and Papanicolaou for the spin case is given.

I. INTRODUCTION

Transition matrix elements expressed in the form of path integrals provide a natural scheme to obtain stationary-phase approximations for propagators.[1] Conventionally such analyses have considered transitions between initial and final coordinate eigenstates and have expressed the path integral either in the pure Lagrangian form or in the traditional phase-space form. However, there is an alternative formulation of the path integral which leads directly to transitions between initial and final states that are coherent states.[2,3] Moreover, entirely analogous formulations exist for other kinematical systems, such as spin systems, that lead to strictly c-number path integrals for a wide variety of problems.[4,5] Such an alternative path-integral formulation also leads to an unconventional form for the stationary-phase approximation. In this paper we apply the general formalism developed elsewhere to formulate and analyze path integrals[4] to the special cases of the canonical algebra and the spin algebra. The latter system has relevance for static colored quarks coupled to Yang-Mills fields, and it is pedagogically useful to discuss this case carefully. To set the stage properly we first make some general remarks.

II. PATH INTEGRALS FOR GENERAL SYSTEMS

Elsewhere we have derived an expression for the path integral in terms of an arbitrary continuous representation (generalized coherent states) suitable for analyzing stationary-phase approximations.[4] In particular, this expression is formally given by

$$\langle l'',t''|l',t'\rangle \equiv \langle l''|e^{-i(t''-t')\mathcal{H}}|l'\rangle$$

$$=\lim_{\epsilon\to 0}\mathfrak{N}\int \exp\left\{i\int_{t'}^{t''}[i\langle l|\dot{l}\rangle+\tfrac{1}{2}i\epsilon\langle\dot{l}|(1-|l\rangle\langle l|)|\dot{l}\rangle-H(l)]dt\right\}\mathfrak{D}l\,. \tag{1}$$

Here $l=l(t)$ is an L-component path connecting $l'\equiv l(t')$ and $l''\equiv l(t'')$; $|l\rangle$ denotes one vector of an overcomplete family of states (generalized coherent states) which admits a resolution of unity in the form

$$1=\int |l\rangle\langle l|\,\delta l\,, \tag{2}$$

where δl is a suitable measure, $|\dot{l}\rangle\equiv d|l\rangle/dt$, $\mathfrak{D}l\equiv\prod_t \delta l(t)$, $\mathfrak{N}$ is a normalization fixed by the composition law

$$\langle l''',t'''|l',t'\rangle=\int\langle l''',t'''|l'',t''\rangle\langle l'',t''|l',t'\rangle\delta l''\,, \tag{3}$$

and the classical Hamiltonian H is related to the quantum Hamiltonian $\mathcal{H}$ by

$$H(l)\equiv\langle l|\mathcal{H}|l\rangle\,. \tag{4}$$

The first and third term in the Lagrangian lead to a conventional first-order action; the unusual additional term—which incidentally implies a characteristic geometry on the Hilbert space hypersurface of vectors $\{|l\rangle\}$—has a special role to play in stationary-phase approximations. Note that the first-order extremal equations of the conventional action are generally incompatible with the required boundary conditions $l'\equiv l(t')$ and $l''\equiv l(t'')$. On the other hand, the additional term in the action changes the extremal equations to second-order ones consistent with the boundary conditions. The action evaluated for the extremal solution (or solutions) leads to the dominant stationary-phase approximation. The derivation[4] of this form of the path integral shows that the extremal solution is correctly given by this procedure and, furthermore, indicates the validity of the dominant stationary-phase approximation provided the extra (ϵ) term in the Lagrangian makes no explicit contribution to the extremal action in the limit $\epsilon\to 0$; this is actually the case for the examples of the canonical and spin algebras that we consider. On the other hand, if the extra term did not make a vanishing contribution, then recourse to the proper lattice-space formulation for the path integral would always provide a correct prescription to obtain the dominant stationary-phase approximation.[4] As usual, the first-order correction to the dominant approximation due to fluctuations is obtained most simply by applying the unitarity condition

$$\langle l'''|l'\rangle=\int\langle l'',t''|l''',t'\rangle^{*}\times\langle l'',t''|l',t'\rangle\delta l'' \tag{5}$$

to the needed degree of accuracy.

The examples that follow serve to illustrate and explain the general scheme. They should suggest a variety of other systems with a few or an infinite number of degrees of freedom for which parallel methods are applicable.

III. PATH INTEGRALS FOR COHERENT STATES

In the present example the states $\{|l\rangle\}$ are taken as the set of states of the form

$$|p,q\rangle\equiv\exp[i(pQ-qP)]|0\rangle\,, \tag{6}$$

for all real p and q, where P and Q form an irreducible Heisenberg canonical operator pair, with $[Q,P]=i$, and $|0\rangle$ denotes the (unit frequency) harmonic-oscillator ground state, a state for which $(Q+iP)|0\rangle=0$. Of course, the states in question are just the familiar coherent states

$$|p,q\rangle\equiv|z\rangle\equiv e^{-|z|^2/2}\sum_{n=0}^{\infty}(n!)^{-1/2}z^n|n\rangle\,, \tag{7}$$

where $z\equiv(q+ip)/\sqrt{2}$ and $|n\rangle$ denotes the nth excited harmonic-oscillator eigenstate.[6,7] It has been established elsewhere,[8,9] in this case, that a resolution of unity is based on $\delta l=dp\,dq/2\pi$, and in particular

$$1=\int|p,q\rangle\langle p,q|(dp\,dq/2\pi) \tag{8}$$

when integrated over all phase space.

A. Path integral

In terms of these states the formal expression for the path integral reads[4]

$$\langle p'',q'',t''|p',q',t'\rangle\equiv\langle p'',q''|e^{-i(t''-t')\mathcal{H}}|p',q'\rangle$$
$$=\lim_{\epsilon\to 0}\mathfrak{N}\int\exp\left\{i\int_{t'}^{t''}[\tfrac{1}{2}(p\dot q-\dot p q)+\tfrac{1}{4}i\epsilon(\dot p^2+\dot q^2)-H(p,q)]dt\right\}\mathfrak{D}p\,\mathfrak{D}q\,, \tag{9}$$

where $\mathfrak{D}p\mathfrak{D}q\equiv\prod_t dp(t)dq(t)$, $H(p,q)\equiv\langle p,q|\mathcal{H}|p,q\rangle$, and $\mathfrak{N}$ is a normalization fixed by the composition law (3). Observe that the real part of the action in (9) has the usual phase-space form, while the imaginary part, the term involving the ϵ factor, has the canonical form dp^2+dq^2 characteristic of a flat two-dimensional space. Even under arbitrary transformations of the p,q variables, which include conventional canonical transformations, the flat-space nature implied by this term remains unchanged. (The appearance of a flat metric can be traced to the fact that the Heisenberg group is a factor representation of an Abelian group.)

B. Stationary-phase approximation

The stationary-phase approximation to the path integral in (9) leads[4] to the two extremal equations

$$\dot q-\partial H(p,q)/\partial p=\tfrac{1}{2}i\epsilon\ddot p\,, \tag{10a}$$

$$\dot p+\partial H(p,q)/\partial q=-\tfrac{1}{2}i\epsilon\ddot q\,. \tag{10b}$$

For very small ϵ, the solution to such equations generally separates into three temporal regions: Near one end, $t-t'=O(\epsilon)$, away from either end, $t-t'=O(1)$, $t''-t=O(1)$, and near the other end, $t''-t=O(\epsilon)$.

Vanishing Hamiltonian

As a first example, assume for simplicity that $H\equiv 0$, for which the solution to (10) for small ϵ reads

$$q(t)=\bar q+(q'-\bar q)e^{-2(t-t')/\epsilon}+(q''-\bar q)e^{-2(t''-t)/\epsilon}\,, \tag{11a}$$

$$p(t)=\bar p+(p'-\bar p)e^{-2(t-t')/\epsilon}+(p''-\bar p)e^{-2(t''-t)/\epsilon}\,, \tag{11b}$$

where

$$\bar{q} \equiv \tfrac{1}{2}(q''+q') - \tfrac{1}{2}i(p''-p')\,, \tag{12a}$$

$$\bar{p} \equiv \tfrac{1}{2}(p''+p') + \tfrac{1}{2}i(q''-q')\,. \tag{12b}$$

A useful rewriting of Eq. (12) is given by

$$q'+ip' = \bar{q}+i\bar{p}\,, \tag{13a}$$

$$q''-ip'' = \bar{q}-i\bar{p}\,. \tag{13b}$$

The solution for $q(t)$ (say) clearly illustrates the three temporal regions: an initial, rapid change from q' to $\bar{q}$, a long, intermediate duration as $\bar{q}$, and a final, rapid change from $\bar{q}$ to q''. The intermediate value $\bar{q}$ is in general complex which reflects the fact that in performing the stationary-phase (or saddle-point) approximation the integration variables generally become complexified in the search for extremals of the action.

When the action is evaluated for this solution (with $H=0$ still) the result for the path integral is given by

$$\exp\{\tfrac{1}{2}i(q''p'-p''q') - \tfrac{1}{4}[(p''-p')^2+(q''-q')^2]\}. \tag{14}$$

The ϵ term makes no contribution to this result thanks to the boundary conditions, and, moreover, this expression exactly equals $\langle p'',q''|p',q'\rangle$. In other words, the dominant stationary-phase approximation (without fluctuations) for vanishing Hamiltonian yields the correct answer for the coherent-state path-integral (propagator).

Nonvanishing Hamiltonian

When the Hamiltonian is nonvanishing, the solution to the equations of motion (10) for small ϵ reads

$$q(t) = \bar{q}(t) + (q'-\bar{q}')e^{-2(t-t')/\epsilon} + (q''-\bar{q}'')e^{-2(t''-t)/\epsilon}\,, \tag{15a}$$

$$p(t) = \bar{p}(t) + (p'-\bar{p}')e^{-2(t-t')/\epsilon} + (p''-\bar{p}'')e^{-2(t''-t)/\epsilon}\,, \tag{15b}$$

where

$$\bar{q}' \equiv \bar{q}(t')\,,\quad \bar{p}' \equiv \bar{p}(t')\,, \tag{16a}$$

$$\bar{q}'' \equiv \bar{q}(t'')\,,\quad \bar{p}'' \equiv \bar{p}(t'')\,, \tag{16b}$$

and $\bar{q}(t)$ and $\bar{p}(t)$ are (generally) complex solutions of the usual classical equations

$$\dot{q} - \partial H(p,q)/\partial p = 0\,, \tag{17a}$$

$$\dot{p} + \partial H(p,q)/\partial q = 0\,. \tag{17b}$$

The boundary conditions appropriate to these classical equations are given by

$$q'+ip' = \bar{q}'+i\bar{p}'\,, \tag{18a}$$

$$q''-ip'' = \bar{q}''-i\bar{p}''\,. \tag{18b}$$

In particular, for the initial conditions one can set

$$\bar{q}' = q'+w\,, \tag{19a}$$

$$\bar{p}' = p'+iw\,, \tag{19b}$$

where w is a complex parameter which is adjusted to fit the complex final condition $\bar{q}''-i\bar{p}''=q''-ip''$, and where $\bar{p}'',\bar{q}''$ is the time-evolved solution of the classical equations with initial conditions $\bar{p}',\bar{q}'$.

When the action is evaluated for this extremal solution the ϵ term again vanishes due to the boundary conditions and the dominant stationary-phase approximation to the propagator (10) is given by

$$\langle p'',q'',t''|p',q',t'\rangle \approx \exp\left\{\tfrac{1}{2}i(q''\bar{p}''-p''\bar{q}''+\bar{q}'p'-\bar{p}'q') + i\int_{t'}^{t''}[\tfrac{1}{2}(\bar{p}\dot{\bar{q}}-\dot{\bar{p}}\bar{q}) - H(\bar{p},\bar{q})]dt\right\}. \tag{20}$$

If H is quadratic then this expression actually gives the correct result, but otherwise not.

Discussion

In general, the extremal solution $p(t), q(t)$ is complex, but it need not be. In particular, if the solution of the classical equations starting with p', q' already evolves to p'', q'', then the solution remains real. Periodic solutions fulfill $p''=p'$, $q''=q'$, and they may either be complex or real. Real, periodic solutions are relevant for evaluating the energy spectrum through the trace of the evolution operator. It is noteworthy in this formulation that periodic solutions require equal momenta (p) as well as equal coordinates (q), rather than just equal coordinates as conventionally assumed with path integrals.

We recall that in the studies of bound-state spectra of various model problems by Dashen, Hasslacher, and Neveu,[10] additional stationary-phase arguments were used that ensured equal momenta along with equal coordinates, both of which were necessary to obtain their principal results.[11] The present work and that of Ref. 4 lead to the requirement of both equal momenta and equal coordinates in a direct and natural fashion.

IV. PATH INTEGRALS FOR SPIN STATES

In the example discussed in this section the states $\{|l\rangle\}$ are taken as the set of states of the form

$$|\theta,\phi\rangle \equiv e^{-i\phi\hat{S}_3}e^{-i\theta\hat{S}_2}|0\rangle \tag{21}$$

for all θ,ϕ, with $0\leq\theta<\pi$ and $0\leq\phi<2\pi$, where $\hat{S}_j$, $j=1,2,3$, form an irreducible representation of the spin algebra,

$$[\hat{S}_j, \hat{S}_k] = i\epsilon_{jkl}\hat{S}_l\,, \tag{22a}$$

$$\hat{\vec{S}}^2 \equiv \sum \hat{S}_j{}^2 = s(s+1)\,, \tag{22b}$$

with $s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and where $|0\rangle$ denotes the eigenstate of $\hat{S}_3$ with maximal eigenvalue

$$\hat{S}_3|0\rangle = s|0\rangle\,. \tag{23}$$

Clearly the states $|\theta, \phi\rangle$ implicitly depend on s, but we suppress that particular dependence.

For $s = \frac{1}{2}$ and $\hat{\vec{S}}$ the usual spin-$\frac{1}{2}$ representation (half the standard Pauli matrices),

$$|\theta, \phi\rangle = \begin{pmatrix} e^{-i\phi/2}\cos\frac{1}{2}\theta \\ e^{i\phi/2}\sin\frac{1}{2}\theta \end{pmatrix}\,; \tag{24a}$$

for $s = 1$ and $\hat{\vec{S}}$ the usual spin-1 representation,

$$|\theta, \phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta\cos\phi - i\sin\phi \\ \cos\theta\sin\phi + i\cos\phi \\ -\sin\theta \end{pmatrix}\,. \tag{24b}$$

A similar construction applies for still higher spins as well. Moreover, a resolution of unity holds for each s in the form[12]

$$1 = \int |\theta, \phi\rangle\langle\theta, \phi|[(2s+1)/4\pi]\sin\theta\, d\theta\, d\phi \tag{25}$$

integrated over the unit sphere, where 1 denotes the $(2s+1)\times(2s+1)$ unit matrix.

A. Path integral

In terms of such states the formal expression for the path integral reads

$$\begin{aligned}\langle\theta'', \phi'', t''|\theta', \phi', t'\rangle &\equiv \langle\theta'', \phi''|e^{-i(t''-t')\mathcal{H}}|\theta', \phi'\rangle \\ &= \lim_{\epsilon\to 0}\mathfrak{N}\int \exp\left\{i\int_{t'}^{t''}[s\cos\theta\dot{\phi} + \tfrac{1}{4}is\epsilon(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - H(\theta, \phi)]dt\right\}\mathfrak{D}\Omega\,,\end{aligned} \tag{26}$$

where

$$\mathfrak{D}\Omega \equiv \prod_t d\Omega(t) \equiv \prod_t [\sin\theta(t)d\theta(t)d\phi(t)]\,,$$

$$H(\theta, \phi) \equiv \langle\theta, \phi|\mathcal{H}|\theta, \phi\rangle\,,$$

and $\mathfrak{N}$ is a normalization fixed by the composition law (3).

An alternative expression for the path integral (26) also holds. It may be determined that

$$\vec{S} \equiv \vec{S}(\theta, \phi) \equiv \langle\theta, \phi|\hat{\vec{S}}|\theta, \phi\rangle \tag{27}$$

is given by the triplet

$$S_1 = s\sin\theta\cos\phi\,, \tag{28a}$$

$$S_2 = s\sin\theta\sin\phi\,, \tag{28b}$$

$$S_3 = s\cos\theta\,. \tag{28c}$$

Furthermore, for $\mathcal{H} \equiv \mathcal{H}(\hat{\vec{S}})$ it is also possible to redefine the lattice-space path integral in such a way that the expression which enters the path integral as the classical Hamiltonian is given instead by

$$H'(\theta, \phi) \equiv \mathcal{H}(\vec{S}(\theta, \phi))\,. \tag{29}$$

This expression possibly differs from the previous one by terms of order $\hbar$.

Note also that the term in the path integral (26) involving the ϵ factor has the canonical form $d\theta^2 + \sin^2\theta d\phi^2$ characteristic of the two-dimensional surface of the unit sphere. This geometry directly reflects the fact that we are dealing with states defined with the aid of the rotation group, and it clearly is a coordinate-invariant property.

Canonical formulation

An important and characteristic feature of the path integral (26) for spin stems from the particular form of the kinematic term in the action, namely

$$\int s\cos\theta\dot{\phi}\,dt\,. \tag{30}$$

It is evident from this expression that $s\cos\theta$ plays the role of momentum conjugate to ϕ, i.e.,

$$p \equiv s\cos\theta\,, \tag{31}$$

and the Hamiltonian $H(\theta, \phi)$ may be unambiguously reexpressed in terms of p and ϕ. Indeed, since (up to an irrelevant sign)

$$s\sin\theta d\theta d\phi = s d\cos\theta d\phi = dp d\phi\,, \tag{32}$$

the resultant path integral expressed in p and ϕ variables has just the standard form of a phase-space integral for a single degree of freedom. Note that Dirac brackets[13] or related techniques are not required in our discussion of classical (or quantum) spin formulations, unlike the treatment elsewhere.[14] In particular, the classical variables $S_j = S_j(\theta, \phi)$, $j = 1, 2, 3$, defined in (28), exactly fulfill the Poisson brackets appropriate to spin variables,

$$\{S_j, S_k\}_{\text{Pb}} = \epsilon_{jkl} \hat{S}_l , \tag{33}$$

where

$$\{A, B\}_{\text{Pb}} \equiv \frac{\partial A}{\partial \phi}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial \phi} . \tag{34}$$

This relation shows the particular interplay of $p = s\cos\theta$ and ϕ as proper conjugate variables.

Spin variable formulation

Another way to rewrite the classical action and the path integral for spin is in terms of the variables $\vec{S}$, which is a form sometimes encountered in the literature. It is straightforward to see that

$$\frac{S_3(S_1\dot{S}_2 - \dot{S}_1 S_2)}{s^2 - S_3^2} = s\cos\theta\dot{\phi} \tag{35}$$

and that

$$\dot{\vec{S}}^2 \equiv \dot{S}_1^2 + \dot{S}_2^2 + \dot{S}_3^2 = s^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) . \tag{36}$$

These relations mean that the path integral (26) may alternatively be formulated as

$$\langle \vec{S}'', t'' | \vec{S}', t' \rangle \equiv \langle \theta'', \phi'', t'' | \theta', \phi', t' \rangle$$
$$= \lim_{\epsilon \to 0} \mathfrak{N} \int \exp\left\{ i \int_{t'}^{t''} \left[\frac{S_3(S_1\dot{S}_2 - \dot{S}_1 S_2)}{s^2 - S_3^2} + \tfrac{1}{4} i s^{-1} \epsilon \dot{\vec{S}}^2 - H(\vec{S}) \right] dt \right\} \delta(\vec{S}^2 - s^2) \mathfrak{D}\vec{S} , \tag{37}$$

where the notation is as before and where the δ-functional ensures that $\vec{S}^2(t) = s^2$ for all t. The first-order action that appears here is the one studied by Doering and by Gilbert.[14] The objection to this form raised by Jevicki and Papanicolaou[15] and the preference they express for an alternative first-order Lagrangian that differs by a total derivative is difficult to understand (see below), especially when concentrating on diagonal matrix elements as they do.

Alternative for spin 1/2

Yet another approach suitable for spin $\frac{1}{2}$ is evidently provided by states of the form

$$|\chi\rangle \equiv \begin{pmatrix} (1 - |\chi|^2)^{1/2} \\ \chi \end{pmatrix} , \tag{38}$$

where the complex variable χ obeys $0 \leq |\chi| \leq 1$. Up to an overall phase, this is just the same vector introduced in (24a) for spin $\frac{1}{2}$ as is clear from the identification

$$\chi \equiv \sin\tfrac{1}{2}\theta e^{i\phi} . \tag{39}$$

It follows that

$$i\langle \chi | \dot{\chi} \rangle = \tfrac{1}{2} i \chi^* \overleftrightarrow{\partial}_t \chi = \tfrac{1}{2}\cos\theta\dot{\phi} - \tfrac{1}{2}\dot{\phi} , \tag{40}$$

which differs from the previous kinematic expression $\frac{1}{2}\cos\theta\dot{\phi}$ for $s = \frac{1}{2}$, owing to the aforementioned difference in phase factor, but such a distinction is of no physical importance whatsoever. Additionally it follows that

$$1 = \int |\chi\rangle\langle \chi| (2/\pi) d\chi_r d\chi_i , \tag{41}$$

where $\chi \equiv \chi_r + i\chi_i$ and the integration is over the entire unit disc, $0 \leq |\chi| \leq 1$. Consequently the formal path integral for spin $\frac{1}{2}$ (omitting the ϵ factor) becomes

$$\langle \chi'', t'' | \chi', t' \rangle \equiv e^{-i\phi''/2} \langle \theta'', \phi'', t'' | \theta', \phi', t' \rangle e^{i\phi'/2}$$
$$= \mathfrak{N} \int \exp\left\{ i \int_{t'}^{t''} [\tfrac{1}{2} i \chi^* \overleftrightarrow{\partial}_t \chi - H(\chi)] dt \right\} \mathfrak{D}\chi_r \mathfrak{D}\chi_i , \tag{42}$$

where $H(\chi) \equiv \langle \chi | \mathcal{H} | \chi \rangle$. Coincidentally, this formulation is identical to the one proposed by the author in 1960 for a c-number path integral for a single Fermi degree of freedom,[16] which is of course precisely that suitable for a single spin-$\frac{1}{2}$ system as well. (The ϵ factor can easily be worked out in this language, and is essential to make sense of the stationary-phase approximation, but it is omitted here for convenience.)

The difference in overall phase between the two states $|\chi\rangle$ and $|\theta, \phi\rangle$ is responsible for the difference in the kinematical term in the classical action for the two cases. If we reexpress the χ-form in terms of the spin variables $\vec{S}$, it follows that

$$\tfrac{1}{2} i \chi^* \overleftrightarrow{\partial}_t \chi = \frac{S_2\dot{S}_1 - S_1\dot{S}_2}{\frac{1}{2} + S_3} , \tag{43}$$

which coincides with the kinematical expression (for $s = \frac{1}{2}$) advocated by Jevicki and Papanicolaou.[15] It is now easy to see the origin of their particular form of the Lagrangian. They based their derivation of the path integral for spin variables on a reduction of states within the context of a coherent-state formulation. Since coherent states conventionally are defined with a real "first" term, i.e., $\langle 0|z\rangle$ [cf., Eq. (7)], then the overall phase is like that for the $|\chi\rangle$ states (since $\langle 0|\chi\rangle$ is also real) rather than like the $|\theta,\phi\rangle$ states as we have defined them.

Relation to Euler angles

It is also worth noting that the definition of the $|\theta,\phi\rangle$ states in (21) is clearly connected to the description of rotations in terms of Euler angles. In one such parametrization one can introduce three-parameter vectors of the form

$$|\theta,\phi,\psi\rangle \equiv e^{-i\phi\hat{S}_3}e^{-i\theta\hat{S}_2}e^{-i\psi\hat{S}_3}|0\rangle . \tag{44}$$

But if $\hat{S}_3|0\rangle = s|0\rangle$, as we have assumed, then

$$|\theta,\phi,\psi\rangle = e^{-is\psi}|\theta,\phi\rangle , \tag{45}$$

and ψ enters simply as an overall phase factor, devoid of any physical significance. On the other hand, in terms of the full set of Euler angles the condition that $|0\rangle$ be an eigenvector of $\hat{S}_3$ may be relaxed and a seemingly covariant formulation may be given. This formulation entails one too many coordinates and some form of gauge condition needs to be imposed (our implicit gauge condition was $\psi\equiv 0$). Moreover, the resultant action is only superficially covariant since there is always a direction implicit in any acceptable choice of vector $|0\rangle$, namely the direction of the vector $\vec{v}\equiv\langle 0|\hat{\vec{S}}|0\rangle$, a vector which must not vanish if the kinematical term is to make any sense at all.[17]

Our further analysis of the spin model is based on the initial formulation as expressed in terms of the variables θ and ϕ.

B. Stationary-phase approximation

The stationary-phase approximation to the path integral for spin variables (26) leads to the two extremal equations

$$s\sin\theta\dot{\phi} + \partial H(\theta,\phi)/\partial\theta = -\tfrac{1}{2}is\epsilon(\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2) , \tag{46a}$$

$$s\sin\theta\dot{\theta} - \partial H(\theta,\phi)/\partial\phi = \tfrac{1}{2}is\epsilon(\sin^2\theta\ddot{\phi} + 2\sin\theta\cos\theta\dot{\theta}\dot{\phi}) . \tag{46b}$$

As before, for very small ϵ, the solution to such equations is rapidly varying near the boundaries t' and t'', and slowly varying in between.

Vanishing Hamiltonian

As a first example, assume for simplicity that $H\equiv 0$, for which the solution to (46) for small ϵ reads

$$\cos\theta(t) = \bar{c} + (\cos\theta' - \bar{c})e^{-2(t-t')/\epsilon} + (\cos\theta'' - \bar{c})e^{-2(t''-t)/\epsilon} , \tag{47}$$

while

$$\phi(t) = \phi' + \tfrac{1}{2}i\ln\left(\frac{1+\cos\theta'}{1-\cos\theta'}\right) - \tfrac{1}{2}i\ln\left(\frac{1+\bar{c}+(\cos\theta'-\bar{c})e^{-2(t-t')/\epsilon}}{1-\bar{c}-(\cos\theta'-\bar{c})e^{-2(t-t')/\epsilon}}\right) \tag{48a}$$

for t near t' or intermediate times, and

$$\phi(t) = \phi'' - \tfrac{1}{2}i\ln\left(\frac{1+\cos\theta''}{1-\cos\theta''}\right) + \tfrac{1}{2}i\ln\left(\frac{1+\bar{c}+(\cos\theta''-\bar{c})e^{-2(t''-t)/\epsilon}}{1-\bar{c}-(\cos\theta''-\bar{c})e^{-2(t''-t)/\epsilon}}\right) \tag{48b}$$

for t near t'' or intermediate times. Equality of these two expressions for $\phi(t)$ at intermediate times requires that

$$\frac{1-\bar{c}}{1+\bar{c}} = \tan\tfrac{1}{2}\theta''\tan\tfrac{1}{2}\theta' e^{-i(\phi''-\phi')} , \tag{49}$$

a relation that fixes the (generally) complex constant $\bar{c}$.

When the action is evaluated for this solution (with $H=0$ still) the dominant stationary-phase approximation is given by

$$[\cos\tfrac{1}{2}\theta''\cos\tfrac{1}{2}\theta' e^{i(\phi''-\phi')/2} + \sin\tfrac{1}{2}\theta''\sin\tfrac{1}{2}\theta' e^{-i(\phi''-\phi')/2}]^{2s} . \tag{50}$$

Thanks to the boundary condition the ϵ term again makes no contribution to this result validating this expression. Moreover, this expression equals[18] $\langle\phi'',\phi''|\theta',\phi'\rangle$ for all s, $s=\frac{1}{2}, 1, \frac{3}{2}, \ldots$. In other words, the dominant stationary-phase approximation (without fluctuations) for vanishing Hamiltonian yields the correct answer for the path-integral (propagator) expressed in terms of spin continuous representations for *all* s, $s=\frac{1}{2}, 1, \frac{3}{2}, \ldots$. Since the integrals involved are all non-Gaussian this is rather a remarkable result.

Nonvanishing Hamiltonian

When the Hamiltonian is nonvanishing, the solution to the equations of motion (46) for small ϵ again has three characteristic temporal regions. In particular, the solution is given by

$$\cos\theta(t) = \cos\bar{\theta}(t) + (\cos\theta' - \cos\bar{\theta}')e^{-2(t-t')/\epsilon} + (\cos\theta'' - \cos\bar{\theta}'')e^{-2(t''-t)/\epsilon} , \tag{51}$$

while

$$\phi(t)=\overline{\phi}(t)-\overline{\phi}'+\phi'+\tfrac{1}{2}i\ln\left(\frac{1+\cos\theta'}{1-\cos\theta'}\right)-\tfrac{1}{2}i\ln\left(\frac{1+\cos\overline{\theta}'+(\cos\theta'-\cos\overline{\theta}')e^{-2(t-t')/\epsilon}}{1-\cos\overline{\theta}'-(\cos\theta'-\cos\overline{\theta}')e^{-2(t-t')/\epsilon}}\right) \quad (52a)$$

for t near t' or intermediate times, and

$$\phi(t)=\overline{\phi}(t)-\overline{\phi}''+\phi''-\tfrac{1}{2}i\ln\left(\frac{1+\cos\theta''}{1-\cos\theta''}\right)+\tfrac{1}{2}i\ln\left(\frac{1+\cos\overline{\theta}''+(\cos\theta''-\cos\overline{\theta}'')e^{-2(t''-t)/\epsilon}}{1-\cos\overline{\theta}''-(\cos\theta''-\cos\overline{\theta}'')e^{-2(t''-t)/\epsilon}}\right) \quad (52b)$$

for t near t'' or intermediate times. In these expressions

$$\overline{\theta}'\equiv\overline{\theta}(t'),\quad \overline{\phi}'\equiv\overline{\phi}(t'), \quad (53a)$$

$$\overline{\theta}''\equiv\overline{\theta}(t''),\quad \overline{\phi}''\equiv\overline{\phi}(t''), \quad (53b)$$

where $\overline{\theta}(t)$ and $\overline{\phi}(t)$ are (generally) complex solutions of the classical equations

$$s\sin\theta\dot{\phi}+\partial H(\theta,\phi)/\partial\theta=0, \quad (54a)$$

$$s\sin\theta\dot{\theta}-\partial H(\theta,\phi)/\partial\phi=0. \quad (54b)$$

The appropriate boundary conditions for these classical equations are found by insisting that $\phi(t)$ changes from ϕ' to $\overline{\phi}'$ during the rapid temporal change near t', and that $\phi(t)$ changes from $\overline{\phi}''$ to ϕ'' during the rapid temporal change near t''. This requirement leads to the boundary conditions

$$\tan\tfrac{1}{2}\theta' e^{i\phi'}=\tan\tfrac{1}{2}\overline{\theta}' e^{i\overline{\phi}'}, \quad (55a)$$

$$\tan\tfrac{1}{2}\theta'' e^{-i\phi''}=\tan\tfrac{1}{2}\overline{\theta}'' e^{-i\overline{\phi}''}. \quad (55b)$$

In particular, for the initial conditions one can set

$$\tan\tfrac{1}{2}\overline{\theta}'=e^{w}\tan\tfrac{1}{2}\theta', \quad (56a)$$

$$\overline{\phi}'=\phi'+iw, \quad (56b)$$

where w is a complex parameter which is adjusted to fit the complex final condition

$$\tan\tfrac{1}{2}\overline{\theta}'' e^{-i\overline{\phi}''}=\tan\tfrac{1}{2}\theta'' e^{-i\phi''}, \quad (57)$$

and where $\overline{\theta}'',\overline{\phi}''$ is the time evolved solution of the classical equations with initial conditions $\overline{\theta}'$, $\overline{\phi}'$.

When the action is evaluated for this extremal solution the ϵ term again vanishes due to the boundary conditions and the dominant stationary-phase approximation to the propagator (26) is given by

$$\langle\theta'',\phi'',t''|\theta',\phi',t'\rangle\approx\left(\frac{\sin\theta''\sin\theta'}{\sin\overline{\theta}''\sin\overline{\theta}'}\right)^{s}\exp\left\{i\int_{t'}^{t''}[s\cos\overline{\theta}\,\dot{\overline{\phi}}-H(\overline{\theta},\overline{\phi})]dt\right\}. \quad (58)$$

For a general, nonvanishing Hamiltonian, of course, this approximation cannot be expected to provide the correct result by itself.

Discussion

In general, the extremal solution $\theta(t)$, $\phi(t)$ is complex, but it need not be. In particular, if the solution of the classical equations starting with θ', ϕ' already evolves to θ'', ϕ'', then the solution remains real. Periodic solutions fulfill $\theta''=\theta'$, $\phi''=\phi'$, and they may either be complex or real. Real, periodic solutions are relevant for evaluating the energy spectrum through the trace of the evolution operator. Once again, it is noteworthy in this formulation that periodic solutions require equal momenta (θ) as well as equal coordinates (ϕ), rather than just equal coordinates as conventionally assumed with path integrals.

We remark that in the study of the spectra of the one-dimensional continuous Heisenberg spin chain by Jevicki and Papanicolaou[15] periodic solutions having both equal momenta and equal coordinates were assumed in order to obtain their principal results. The present work provides a clear demonstration of the necessity of that assumption, as well as clarifying the entire question of path integrals for spin variables and the associated stationary-phase approximation.

[1]See, e.g., R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965); M. C. Gutzwiller, J. Math. Phys. 8, 1979 (1967); 10, 1004 (1969); 11, 1791 (1970); 12, 343 (1971); R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4114, (1974); 10, 4130 (1974); 10, 4138 (1974).

[2]J. R. Klauder, Ann. Phys. (N. Y.) 11, 123 (1960), particularly Eq. (12).

[3]S. S. Schweber, J. Math. Phys. 3, 831 (1962); F. A. Berezin, Teor. Mat. Fiz. 6, 194 (1971) [Theor. Math. Phys. 6, 141 (1971)]. See also L. D. Faddeev, *Methods in Field Theory*, 1975 Les Houches Lectures, edited

by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976), p. 1.

[4]J. R. Klauder, in *Path Integrals*, Proceedings of the NATO Advanced Summer Institute, edited by G. J. Papadopoulos and J. T. Devreese (Plenum, New York, 1978), p. 5.

[5]Reference 2, cf. Eq. (65).

[6]J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968), Chap. 7.

[7]See also V. Bargmann, Commun. Pure Appl. Math. 14, 187 (1961); R. J. Glauber, Phys. Rev. 131, 2766 (1963).

[8]Reference 2, Eq. (5).

[9]See, e.g., J. Schwinger, *Quantum Kinematics and Dynamics* (Benjamin, New York, 1970), especially Secs. 4.6 and 4.7.

[10]R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4114 (1974); see, however, K. F. Freed, Faraday Disc. Chem. Soc. 55, 68 (1973).

[11]Thanks are expressed to H. Neuberger for a discussion of this point.

[12]See, e.g., Ref. 2, Eq. (83); J. R. Klauder, J. Math. Phys. 4, 1058 (1963), Eq. (38). Compare J. M. Radcliffe, J. Phys. A 4, 313 (1971) and F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A 6, 2211 (1972) for a related discussion (making allowance for a different phase convention).

[13]See, e.g., P. A. M. Dirac, Can. J. Math. 2, 129 (1950); 3, 1 (1951).

[14]See, e.g., C. F. Valenti and M. Lax, Phys. Rev. B 16, 4936 (1977).

[15]A. Jevicki and N. Papanicolaou, Ann. Phys. (N. Y.) (to be published).

[16]Reference 2, Eq. (18).

[17]J. R. Klauder, J. Math. Phys. 4, 1058 (1963), Sec. 3.

[18]See, e.g., E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge Univ. Press, Cambridge, England, 1953), Chap. III; A. S. Davydov, *Quantum Mechanics*, translated by D. ter Harr (Addison-Wesley, Reading, Mass, 1965), Chap. VI.