Rapidly rotating Bose-Einstein condensates in harmonic traps*

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*for general references, see [1, 2, 3, 4]

1 Physics of one vortex line in harmonic trap

Assume general three-dimensional trap potential

$$V_{\rm tr}(\boldsymbol{r}) = \tfrac{1}{2}M\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$$

Basic idea (Bogoliubov): for weak interparticle potentials, nearly all particles remain in condensate for $T \ll T_c$

- dilute: s-wave scattering length $a_s \ll$ interparticle spacing $n^{-1/3}$
- equivalently, require $na_s^3 \ll 1$
- assume self-consistent condensate wave function $\Psi(\boldsymbol{r})$
- gives nonuniform condensate density $n(\boldsymbol{r}) = |\Psi(\boldsymbol{r})|^2$
- for $T \ll T_c$, normalization requires $N = \int dV\, |\Psi(\boldsymbol{r})|^2$
- assume an energy functional

$$E[\Psi] = \int dV \left[\underbrace{\Psi^*\left(\mathcal{T} + V_{\rm tr}\right)\Psi}_{\text{harmonic oscillator}} + \underbrace{\tfrac{1}{2}g|\Psi|^4}_{2-body\ term}\right],$$

where $\mathcal{T} = -\hbar^2\nabla^2/2M$ is kinetic energy operator and $g = 4\pi a_s\hbar^2/M$ is interaction coupling parameter

- balance of kinetic energy $\langle \mathcal{T} \rangle$ and trap energy $\langle V_{\mathrm{tr}} \rangle$ gives mean oscillator length $d_0 = \sqrt{\hbar / M\omega_0}$ where $\omega_0 = (\omega_x \omega_y \omega_z)^{1/3}$ is geometric mean

- balance of kinetic energy $\langle \mathcal{T} \rangle$ and interaction energy $\langle gn \rangle$ gives healing length

$$\xi = \frac{\hbar}{\sqrt{2Mgn}} = \frac{1}{\sqrt{8\pi a_s n}}$$

- treat energy $E[\Psi]$ as a functional of Ψ and seek stationary solution

- with fixed normalization and μ the chemical potential, this gives Gross-Pitaevskii (GP) equation

$$\left(\mathcal{T} + V_{\mathrm{tr}} + \underbrace{g|\Psi|^2}_{Hartree} \right) \Psi = \mu \Psi$$

- can interpret nonlinear term as a Hartree potential $V_H(\boldsymbol{r}) = gn(\boldsymbol{r})$, giving interaction with nonuniform condensate density

- generalize to time-dependent GP equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(\mathcal{T} + V_{\mathrm{tr}} + V_H \right) \Psi$$

- this result implies that stationary solutions have time dependence $\exp(-i\mu t/\hbar)$

Introduce hydrodynamic variables

- write $\Psi(\boldsymbol{r},t) = |\Psi(\boldsymbol{r},t)| \exp[iS(\boldsymbol{r},t)]$ with phase S
- condensate density is $n(\boldsymbol{r},t) = |\Psi(\boldsymbol{r},t)|^2$
- current is
$$\boldsymbol{j} = \frac{\hbar}{2Mi}\left[\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^*\right] = |\Psi|^2\frac{\hbar\boldsymbol{\nabla}S}{M} = n\boldsymbol{v}$$
- identify last factor as velocity $\boldsymbol{v} = \hbar\boldsymbol{\nabla}S/M$
- note that $\boldsymbol{v}$ is irrotational so $\boldsymbol{\nabla}\wedge\boldsymbol{v} = 0$
- general property: circulation around contour $\mathcal{C}$ is
$$\oint_{\mathcal{C}} d\boldsymbol{l}\cdot\boldsymbol{v} = \frac{\hbar}{M}\oint_{\mathcal{C}} d\boldsymbol{l}\cdot\boldsymbol{\nabla}S = \frac{\hbar}{M}\,\Delta S|_{\mathcal{C}}$$
- change of phase $\Delta S|_{\mathcal{C}}$ must be integer times 2π since Ψ is single-valued
- hence circulation in BEC is *quantized* in units of $\kappa \equiv 2\pi\hbar/M = h/M$
- rewrite time-dependent GP equation in terms of $|\Psi|$ and S
  - imaginary part: $\partial n/\partial t + \boldsymbol{\nabla}\cdot(n\boldsymbol{v}) = 0$
  - real part: generalized Bernoulli equation

Introduction of harmonic trap yields much richer system than a uniform interacting Bose gas

- trap gives new *energy* scale $\hbar\omega_0$ and new *length* scale $d_0 = \sqrt{\hbar/M\omega_0}$
- assume repulsive interactions with $a_s > 0$
- repulsive interactions expand the condensate to larger mean radius $R_0 > d_0$
- as order of magnitude, ground-state energy E_g has the form [5]

$$\frac{E_g}{N} \sim \hbar\omega_0 \left(\underbrace{\frac{1}{\mathcal{R}^2}}_{kinetic} + \underbrace{\mathcal{R}^2}_{potential} + \underbrace{\frac{Na_s}{d_0}\frac{1}{\mathcal{R}^3}}_{interaction} \right),$$

  with $\mathcal{R} = R_0/d_0$ the dimensionless expansion ratio of radius

- new dimensionless parameter Na_s/d_0 arises from trap
- minimize E_g with respect to $\mathcal{R}$
- if $Na_s/d_0 \lesssim 1$, minimum E_g gives $\mathcal{R} \sim 1$ (ideal gas)

Properties of Thomas-Fermi (TF) limit

- if $Na_s/d_0 \gg 1$, kinetic energy is small and minimum E_g gives

$$\mathcal{R} = \frac{R_0}{d_0} \sim \left(\frac{Na_s}{d_0}\right)^{1/5} \gg 1 \quad \text{(“Thomas-Fermi” limit)}$$

- typically, $a_s \sim$ a few nm and $d_0 \sim$ a few μm
- thus $Na_s/d_0 \sim 10^3$ for $N \sim 10^6$
- ignore kinetic energy (radial gradient of density) and GP equation reduces to simple equation for density

$$gn(\boldsymbol{r}) = g|\Psi(\boldsymbol{r})|^2 = \mu - V_{\rm tr}(\boldsymbol{r})$$

  where right side is positive and zero elsewhere

- central density is $n(0) = \mu/g$
- TF density is $n(\boldsymbol{r}) = n(0)\left(1 - r^2/R_0^2\right)$ for spherical condensate in isotropic harmonic trap
- condensate radius given by $R_0^2 = 2\mu/M\omega_0^2$
- easily generalized to anisotropic trap: take $R_j^2 = 2\mu/M\omega_j^2$ for $j = x, y, z$

- normalization integral $\int dV\, n(r) = N$ for TF density gives $N(\mu_{TF})$
- easy to obtain $\mu_{TF}/\hbar\omega_0 = \frac{1}{2}\left(15Na_s/d_0\right)^{2/5} \gg 1$
- expansion ratio is $R_0/d_0 = \left(15Na_s/d_0\right)^{1/5} \gg 1$
- define healing length in terms of the central density

$$\xi^2 = \frac{1}{8\pi n(0) a_s}$$

- easily obtain the result $\xi R_0 = d_0^2$
- TF limit gives hierarchy of length scales $\xi \ll d_0 \ll R_0$
- ξ will be seen to characterize the vortex-core radius, so TF limit corresponds to vortices with small cores in a large condensate

(a) One vortex line in trapped BEC

First assume bulk condensate with uniform density n and a single straight vortex line along z axis

- Gross and Pitaevskii [6, 7]: take condensate wave function
$$\Psi(\boldsymbol{r}) = \sqrt{n}\, e^{i\phi} f\left(\frac{r_\perp}{\xi}\right)$$
where $r_\perp$ and ϕ are two-dimensional polar coordinates
- chemical potential is $\mu = gn$
- speed of sound is $s = \sqrt{\mu/M}$
- assume $f(0) = 0$ and $f(x) \to 1$ for $x \gg 1$
- velocity has circular streamlines with $\boldsymbol{v} = (\hbar/Mr_\perp)\,\hat{\boldsymbol{\phi}}$
- this is a quantized vortex line with $\oint d\boldsymbol{l} \cdot \boldsymbol{v} = h/M$
- $v \sim s$ when $r_\perp \sim \xi$, so vortex core forms by cavitation
- equivalently, centrifugal barrier gives vortex core of radius ξ
- energy per unit length of vortex is
$$E_v \approx \frac{\pi\hbar^2 n}{M} \ln\left(1.46\,\frac{R}{\xi}\right)$$

Static behavior of straight vortex line in a trap

Assume axisymmetric trap with

$$V_{\rm tr}(r_\perp, z) = \tfrac{1}{2}M\left(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2\right)$$

- If $\omega_z/\omega_\perp \gg 1$, strong axial confinement gives disk-shaped condensate
- If $\omega_z/\omega_\perp \ll 1$, strong radial confinement gives cigar-shaped condensate
- axisymmetric shape means angular momentum L_z is conserved for a single vortex on symmetry axis
- condensate wave function has the form

$$\Psi(\boldsymbol{r}_\perp, z) = e^{i\phi}|\Psi(r_\perp, z)|$$

- velocity is $\boldsymbol{v} = (\hbar/Mr_\perp)\hat{\boldsymbol{\phi}}$, like uniform condensate
- centrifugal energy again forces wave function to vanish for $r_\perp \lesssim \xi$
- hence density is now toroidal, with a hole along the symmetry axis

In TF limit, the separation of length scales $\xi \ll d_0 \ll R_0$ means that TF density is essentially unchanged

- to calculate energy, use the density of vortex-free TF condensate and cut off logarithmic divergences at core radius ξ
- if condensate is in rotational equilibrium at angular velocity $\boldsymbol{\Omega}$, the appropriate energy functional is [8] $E'[\Psi] = E[\Psi] - \boldsymbol{\Omega} \cdot \boldsymbol{L}[\Psi]$ where $\boldsymbol{L}$ is the angular momentum
- let E'_0 be energy of rotating vortex-free condensate
- let $E'_1(r_0, \Omega)$ be energy of a rotating condensate with straight vortex that is displaced laterally by distance r_0 from symmetry axis
- approximation of straight vortex works best for disk-shaped condensate ($\omega_z \gtrsim \omega_\perp$)
- Difference of these two energies is energy associated with formation of vortex $\Delta E'(r_0, \Omega) = E'_1(r_0, \Omega) - E'_0$
- $\Delta E'(r_0, \Omega)$ depends on position r_0 of vortex and on Ω

Plot $\Delta E'(r_0, \Omega)$ as function of $\zeta_0 = r_0/R_\perp$ for various fixed Ω [9]

curve (a) is $\Delta E'(r_0, \Omega)$ for $\Omega = 0$

- $\Delta E'(r_0, 0)$ decreases monotonically with increasing ζ_0
- curvature is negative at $\zeta_0 = 0$
- for no dissipation, fixed energy means constant ζ_0, so that only allowed motion is uniform precession at a fixed distance from origin
- angular velocity is given by variational Lagrangian method [10, 11, 3]

$$\dot{\phi}_0 = \frac{\partial E(r_0)/\partial r_0}{\partial L_z(r_0)/\partial r_0} = \frac{\Omega_m}{1 - r_0^2/R_\perp^2},$$

  where $\Omega_m = \frac{3}{2}\left(\hbar/MR_\perp^2\right)\ln\left(R_\perp/\xi\right)$ is critical angular velocity for onset of metastability for central vortex (discussed below)

- $E(r_0)$ and $L_z(r_0)$ are energy and angular momentum of off-center vortex in nonrotating TF condensate

- precession arises from nonuniform trap potential (*not image vortex*) and nonuniform condensate density
- for vortex near the center, $\dot{\phi}_0 \approx \Omega_m$
- for larger r_0, precession increases because of reduced TF density near the edge (*not from image vortex*)
- compare with experimental studies at JILA [12]
  - theory predicts $\dot{\phi}/2\pi \approx 1.58 \pm 0.16$ Hz, and
  - experiment finds $\dot{\phi}/2\pi \approx 1.8 \pm 0.1$ Hz
- in presence of weak dissipation, vortex slowly moves outward along curve (a), following spiral orbit in xy plane

As Ω increases, curvature near $r_0 = 0$ decreases

- curve (b) is when curvature near $r_0 = 0$ vanishes
- it corresponds to angular velocity

$$\Omega_m = \frac{3}{2}\frac{\hbar}{MR_\perp^2}\ln\left(\frac{R_\perp}{\xi}\right) = \frac{3}{5}\Omega_c$$

- for $\Omega \gtrsim \Omega_m$, energy $\Delta E'(r_0, \Omega)$ has local *minimum* near $r_0 = 0$
- dissipation would now drive vortex back toward symmetry axis
- Ω_m is angular velocity for onset of *metastability*
- vortex at center is *locally* stable for $\Omega > \Omega_m$, but not *globally* stable, since $\Delta E'(0, \Omega_m)$ is positive

As Ω increases beyond Ω_m, local minimum of $\Delta E'(r_0, \Omega)$ near center decreases

- curve (c) is for Ω_c when $\Delta E'(0, \Omega_c)$ vanishes
- central vortex is degenerate in energy with vortex-free state at Ω_c

$$\Omega_c = \frac{5}{2}\frac{\hbar}{MR_\perp^2} \ln\left(\frac{R_\perp}{\xi}\right) = \frac{5}{3}\Omega_m$$

- for $\Omega > \Omega_c$, central vortex is both locally and globally stable
- as Ω increases beyond Ω_c, energy barrier near outer edge becomes thinner
- curve (d) illustrates behavior for $\Omega = \frac{3}{2}\Omega_c$

(b) Feynman's relation for vortex density in rotating superfluids

- solid-body rotation has $\boldsymbol{v}_{\rm sb} = \boldsymbol{\Omega} \wedge \boldsymbol{r}$
- $\boldsymbol{v}_{\rm sb}$ has constant vorticity $\boldsymbol{\nabla} \wedge \boldsymbol{v}_{\rm sb} = 2\boldsymbol{\Omega}$
- each quantized vortex at $\boldsymbol{r}_j$ has localized vorticity

$$\boldsymbol{\nabla} \wedge \boldsymbol{v} = \frac{2\pi\hbar}{M}\,\delta^{(2)}(\boldsymbol{r}_\perp - \boldsymbol{r}_j)\,\hat{\boldsymbol{z}}$$

- assume $\mathcal{N}_v$ vortices uniformly distributed in area $\mathcal{A}$ bounded by contour $\mathcal{C}$
- circulation around $\mathcal{C}$ is $\mathcal{N}_v \times 2\pi\hbar/M$
- but circulation in $\mathcal{A}$ is also $2\Omega\mathcal{A}$
- hence vortex density is $n_v = \mathcal{N}_v/\mathcal{A} = M\Omega/\pi\hbar$
- area per vortex $1/n_v$ is $\pi\hbar/M\Omega \equiv \pi l^2$ which defines radius $l = \sqrt{\hbar/M\Omega}$ of circular cell
- intervortex spacing $\sim 2l$ decreases like $1/\sqrt{\Omega}$
- analogous to quantized flux lines (charged vortices) in type-II superconductors

(c) Experimental creation and detection of vortices in a dilute trapped BEC

- first vortex made at JILA (1999) [13]
- used nearly spherical ^{87}Rb condensate containing two different hyperfine components
- spin up condensate by coupling the two components with a stirring perturbation
- turn off coupling, leaving one component with trapped quantized vortex surrounding nonrotating core of other component
- use selective tuning to make nondestructive image of either component
- study precession of this vortex with filled core around trap center (also with empty core [12])
- find good fit to theory
- see no outward radial motion for ~ 1 s, so dissipation is small on this time scale

École Normale Supérieure (ENS) group in Paris studied vortex creation in elongated rotating cigar-shaped condensate with one component [14, 15]

- used off-center toggled rotating laser beam to deform the transverse trap potential and stir the condensate at an applied frequency $\Omega/2\pi \lesssim 200$ Hz
- find vortex appears at a critical frequency $\Omega_c \approx 0.7\omega_\perp$ (detected by expanding the condensate, which now has a disk shape, with vortex core as expanded hole)
- this value of Ω_c is significantly ($\sim 70\%$) higher than that predicted by TF thermodynamic critical angular velocity
- vortex nucleation is dynamical process associated with surface instability (quadrupole oscillation)

- ENS group observed small vortex arrays of up to 11 vortices (arranged in two concentric rings)
- like patterns predicted and seen in superfluid ^{4}He
- MIT group has prepared considerably larger rotating condensates in less elongated trap
- they have observed triangular vortex lattices with up to 130 vortices [16]
- like Abrikosov lattice of quantized flux lines (which are charged vortices) in type-II superconductors
- JILA group has now made large rotating condensates with several hundred vortices and angular velocity $\Omega/\omega_\perp \approx 0.995$ [17]
- these rapidly rotating systems open many exciting new possibilities (discussed below)

2 Vortex arrays in mean-field (GP) regime (these are *coherent* states)

Qualitative features

As Ω increases, the vortex density $n_v = M\Omega/\pi\hbar$ increases linearly following the Feynman relation

- in addition, centrifugal forces expand the condensate radially, so that the area also increases
- hence the number of vortices $\mathcal{N}_v = n_v \pi R_\perp^2 = M\Omega R_\perp^2/\hbar$ increases faster than linearly with Ω
- conservation of particles implies that the condensate also shrinks axially
- TF approximation assumes that interaction energy $\langle g|\Psi|^4\rangle$ and trap energy $\langle V_{\text{tr}}|\Psi|^2\rangle$ are large relative to kinetic energy for density variations $(\hbar^2/M)\langle(\boldsymbol{\nabla}|\Psi|)^2\rangle$
- expansion of condensate means that central density eventually becomes small and TF picture fails

(a) Mean-field Thomas-Fermi regime

Quantitative description of rotating TF condensate

Kinetic energy of condensate involves

$$\frac{\hbar^2}{2M}\int dV\, |\boldsymbol{\nabla}\Psi|^2 = \underbrace{\int dV\, \frac{1}{2}Mv^2|\Psi|^2}_{superflow\ energy} + \underbrace{\frac{\hbar^2}{2M}\int dV\, \left(\boldsymbol{\nabla}|\Psi|\right)^2}_{density\ variation}$$

where $\Psi = \exp(iS)|\Psi|$ and $\boldsymbol{v} = \hbar\boldsymbol{\nabla}S/M$ is flow velocity

- generalized TF approximation: retain the energy of superflow but ignore the energy from density variation
- this approximation will fail eventually when vortex lattice becomes dense and cores start to overlap
- assume axisymmetric trap $V_{\rm tr} = \frac{1}{2}M\left(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2\right)$
- in rotating frame, generalized TF energy functional is

$$E'[\Psi] = \int dV \quad \left[\left(\tfrac{1}{2}Mv^2 + V_{\rm tr} - M\boldsymbol{\Omega}\cdot\boldsymbol{r}\wedge\boldsymbol{v}\right)|\Psi|^2 + \tfrac{1}{2}g|\Psi|^4\right]$$

For $\boldsymbol{\Omega}$ along z, can rewrite $E'[\Psi]$ as

$$E'[\Psi] = \int dV \left[\frac{1}{2}M\left(\boldsymbol{v} - \boldsymbol{\Omega} \wedge \boldsymbol{r}\right)^2 |\Psi|^2 + \frac{1}{2}M\omega_z^2 z^2 |\Psi|^2 + \frac{1}{2}\left(\omega_\perp^2 - \Omega^2\right) r_\perp^2 |\Psi|^2 + \frac{1}{2}g|\Psi|^4\right]$$

- here, $\boldsymbol{v}$ is flow velocity generated by all the vortices and $\boldsymbol{v}_{\rm sb} \equiv \boldsymbol{\Omega} \wedge \boldsymbol{r}$ is solid-body rotation
- in the rotating frame, the dominant effect of the dense vortex array is that spatially averaged flow velocity $\langle \boldsymbol{v} \rangle$ is close to $\boldsymbol{v}_{\rm sb}$
- hence can ignore first term in $E'[\Psi]$, giving

$$E'[\Psi] \approx \int dV \left[\frac{1}{2}M\omega_z^2 z^2 |\Psi|^2 + \frac{1}{2}\left(\omega_\perp^2 - \Omega^2\right) r_\perp^2 |\Psi|^2 + \frac{1}{2}g|\Psi|^4\right]$$

- E' now looks exactly like TF energy for nonrotating condensate but with a *reduced* radial trap frequency $\omega_\perp^2 \to \omega_\perp^2 - \Omega^2$

Now have TF wave function that depends explicitly on Ω through the altered radial trap frequency $\omega_\perp^2 \to \omega_\perp^2 - \Omega^2$

$$|\Psi(r_\perp, z)|^2 = n(0)\left(1 - \frac{r_\perp^2}{R_\perp^2} - \frac{z^2}{R_z^2}\right)$$

where $R_\perp^2 = 2\mu/[M(\omega_\perp^2 - \Omega^2)]$ and $R_z^2 = 2\mu/M\omega_z^2$

- must have $\Omega < \omega_\perp$ to retain radial confinement
- normalization $\int dV|\Psi|^2 = N$ shows that

  $$\frac{\mu(\Omega)}{\mu(0)} = \left(1 - \frac{\Omega^2}{\omega_\perp^2}\right)^{2/5}$$

  in three dimensions
- central density given by $n(0) = \mu(\Omega)/g$
- $n(0)$ decreases with increasing Ω because of reduced radial confinement

- TF formulas for condensate radii show that

$$\frac{R_z(\Omega)}{R_z(0)} = \left(1 - \frac{\Omega^2}{\omega_\perp^2}\right)^{1/5}, \quad \frac{R_\perp(\Omega)}{R_\perp(0)} = \left(1 - \frac{\Omega^2}{\omega_\perp^2}\right)^{-3/10}$$

  confirming axial shrinkage and radial expansion

- aspect ratio changes

$$\frac{R_z(\Omega)}{R_\perp(\Omega)} = \frac{R_z(0)}{R_\perp(0)}\left(1 - \frac{\Omega^2}{\omega_\perp^2}\right)^{1/2}$$

- this last effect provides an important diagnostic tool to determine actual angular velocity Ω [18]
- these JILA experiments [18] obtain rapidly rotating condensates by rotating thermal cloud above T_c and then cooling to $T \ll T_c$
- method works by deforming the normal cloud from disk-shaped to cigar-shaped, then removing atoms near the long ends (they have small angular momentum)
- measured aspect ratio indicated that $\Omega/\omega_\perp$ became as large as 0.94

How uniform is the vortex array?

The analysis of the TF density profile $|\Psi_{TF}|^2 = n_{TF}$ in the rotating condensate assumed that the flow velocity $\boldsymbol{v}$ was precisely the solid-body value $\boldsymbol{v}_{\rm sb} = \boldsymbol{\Omega} \wedge \boldsymbol{r}$

- this led to the cancellation of the contribution
$$\int dV \,(\boldsymbol{v} - \boldsymbol{\Omega} \wedge \boldsymbol{r})^2 \, n_{TF}$$
in the TF energy functional
- a more careful study [19] shows that there is a small nonuniformity in the vortex lattice
- specifically, each regular vortex lattice position vector $\boldsymbol{r}_j$ experiences a small displacement field $\boldsymbol{u}(\boldsymbol{r})$, so that $\boldsymbol{r}_j \to \boldsymbol{r}_j + \boldsymbol{u}(\boldsymbol{r}_j)$
- as a result, the two-dimensional vortex density changes to
$$n_v(\boldsymbol{r}) \approx \overline{n_v}\,(1 - \boldsymbol{\nabla} \cdot \boldsymbol{u})$$
where $\overline{n_v} = M\Omega/\pi\hbar$ is the uniform Feynman value

- near the jth vortex core, the flow velocity is a singular part

$$\boldsymbol{v}_{\text{sing}} = \frac{\hbar}{M}\frac{\hat{\boldsymbol{z}} \wedge (\boldsymbol{r} - \boldsymbol{r}_j)}{|\boldsymbol{r} - \boldsymbol{r}_j|^2}$$

plus a smooth background $\overline{\boldsymbol{v}}(\boldsymbol{r})$

- the smooth background velocity can be evaluated as an integral over the slightly nonuniform vortex density

$$\begin{aligned}\overline{\boldsymbol{v}}(\boldsymbol{r}) &\approx \frac{\hbar}{M}\int d^2r'\,\overline{n_v}\,[1 - \boldsymbol{\nabla}'\cdot\boldsymbol{u}\,(\boldsymbol{r}')]\,\frac{\hat{\boldsymbol{z}} \wedge (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^2} \\ &\approx \boldsymbol{\Omega} \wedge [\boldsymbol{r} - 2\boldsymbol{u}\,(\boldsymbol{r})]\end{aligned}$$

where the second term follows with an integration by parts using $\nabla^2 \ln|\boldsymbol{r} - \boldsymbol{r}'| = 2\pi\delta^{(2)}(\boldsymbol{r} - \boldsymbol{r}')$

- the first term is the usual solid-body rotation $\boldsymbol{\Omega} \wedge \boldsymbol{r}$, and the second term shows how the distortion in the vortex lattice affects the mean induced velocity

- the new term in the energy is nonzero contribution from the local integral inside the jth unit cell

$$\sum_j \int_j dV_j \, \frac{M}{2} \left(\boldsymbol{v}_{\text{sing}} + \overline{\boldsymbol{v}} - \boldsymbol{\Omega} \wedge \boldsymbol{r}_j\right)^2 \, n_{TF}(\boldsymbol{r}_j)$$

  and then summed over the vortex lattice

- since the particle density and the vortex density vary slowly over each unit cell, replace $\sum_j$ with an integral weighted with the nonuniform vortex density $n_v(\boldsymbol{r})$

- the dominant solid-body contribution $\boldsymbol{\Omega} \wedge \boldsymbol{r}$ cancels, and the remaining parts are the energy of the jth vortex inside the local circular cell (from $\boldsymbol{v}_{\text{sing}}$) and the contribution from the distortion of the lattice

- the radius of the local unit cell is $l(\boldsymbol{r}) = 1/\sqrt{\pi n_v(\boldsymbol{r})}$, which includes the lattice distortion

- the additional kinetic energy becomes approximately

$$\int dV \, n_{TF} \left[\frac{\pi \hbar^2}{2M} \, \overline{n_v} \, (1 - \boldsymbol{\nabla} \cdot \boldsymbol{u}) \ln\left(\frac{1}{\pi \overline{n_v} \xi^2}\right) + 2M\Omega^2 u^2\right]$$

  with no other dependence on $\boldsymbol{u}$ to leading logarithmic order

- vary this energy with respect to $\boldsymbol{u}$ and obtain the Euler-Lagrange equation, which can be solved to give

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{r}) &\approx -\frac{1}{8\pi\overline{n_v}}\ln\left(\frac{\bar{l}^2}{\xi^2}\right)\boldsymbol{\nabla}\ln n_{TF}(r)\\ &\approx \frac{\bar{l}^2}{4R_\perp^2}\ln\left(\frac{\bar{l}^2}{\xi^2}\right)\frac{\boldsymbol{r}}{1-r^2/R_\perp^2}\end{aligned}$$

  where $\bar{l}^2 = 1/\pi\overline{n_v}$ can be taken as the mean circular cell radius inside the slowly varying logarithm

- the deformation of the regular vortex lattice is purely radial (as expected from symmetry)

- $R_\perp^2/\bar{l}^2$ is the number of vortices $\mathcal{N}_v$ in the rotating condensate, so that the nonuniform distortion is small, of order $1/\mathcal{N}_v$ (at most a few %), even though the TF number density n_{TF} changes dramatically near edge

- recent JILA experiments [20] confirm these predicted small distortions for relatively dense vortex lattices

- correspondingly, the vortex density becomes

$$n_v(r) \approx \overline{n_v} - \frac{1}{2\pi R_\perp^2}\ln\left(\frac{\bar{l}^2}{\xi^2}\right)\frac{1}{\left(1-r^2/R_\perp^2\right)^2}$$

  (the correction is again of order $1/\mathcal{N}_v$)

Tkachenko oscillations of the vortex lattice

In 1966, Tkachenko [21] studied the equilibrium arrangement of a rotating vortex array as model for superfluid ^{4}He

- assumed two-dimensional incompressible fluid with straight vortices
- showed that a triangular lattice has lowest energy in rotating frame
- studied small perturbations about the equilibrium positions and found unusual collective motion in which the vortices undergo a nearly transverse wave of lattice distortions (analogous to two-dimensional transverse "phonons" in the vortex lattice, but with no change in fluid density)
- for long wavelengths (small k), Tkachenko found a linear dispersion relation $\omega_k \approx c_T k$
- speed of Tkachenko wave is $c_T = \sqrt{\frac{1}{4}\hbar\Omega/M} = \frac{1}{2}\bar{l}\Omega$, where $\bar{l} = \sqrt{\hbar/M\Omega}$ is radius of circular vortex cell

More generally, the vortices can also undergo bending motions, leading to a collective version of Kelvin helical wave on a single vortex (not discussed here)

- analysis of small perturbations in a vortex lattice yields the long-wavelength dispersion relation [22, 23]

$$\omega^2 \approx (2\Omega)^2 \frac{k_z^2 + \frac{1}{16} k_\perp^4 \bar{l}^2}{k_z^2 + k_\perp^2}$$

  where k_z and $k_\perp$ are the components of $\boldsymbol{k}$ parallel and perpendicular to the rotation axis

- for $k_z \to 0$, this expression reproduces the Tkachenko result $\omega \approx c_T k_\perp$

- for $k_\perp \to 0$, reproduces classical inertial waves with $\omega = \pm 2\Omega$

- these modes have not been observed in superfluid ^{4}He because visualizing vortices is very difficult

In a rotating gas, the compressibility becomes important, as shown by Sonin [24, 25] and Baym [26]

- let the speed of sound in the compressible gas be c_s
- for a wave propagating in the xy plane, the coupling between the vortices and the compressible fluid leads to generalized dispersion relation

$$\omega^2 = c_T^2 \frac{c_s^2 k^4}{4\Omega^2 + c_s^2 k^2}$$

- if $k \gg \Omega/c_s$, recover Tkachenko's result $\omega = c_T k$ (incompressible limit)
- but if $k \ll \Omega/c_s$, mode becomes *soft* with $\omega \propto k^2$
- Sonin [25] obtains dynamical equations for waves in a nonuniform condensate, along with appropriate boundary conditions at the outer surface
- Baym [26] uses theory for uniform condensate plus approximate boundary conditions from Anglin and Crescimanno [27]
- rough agreement with JILA experiments [28] on low-lying Tkachenko modes in rapidly rotating BEC (up to $\Omega/\omega_\perp \approx 0.975$)

What happens to vortex core radius as $\Omega \to \omega_\perp$?

- recall simple estimate

$$\xi^2 = \frac{1}{8\pi n(0) a_s}$$

- this expression implies that vortex core size ξ diverges for $\Omega \to \omega_\perp$ because $n(0) \propto \mu(\Omega) \propto \left(1 - \Omega^2/\omega_\perp^2\right)^{2/5}$
- improved description generalizes TF model to include the circulating flow velocity around each core with a mean Wigner-Seitz circular cell of radius $l = \sqrt{\hbar/M\Omega}$
- includes spatial variation of density near core and treats ξ as a variational parameter [29, 30]
- as Ω increases and l decreases, predict that ξ increases until $\xi^2/l^2 \sim 0.5$ and this ratio then remains fixed as Ω continues to increase
- in this limit, the vortex cores occupy a constant finite fraction (~ 0.5) of unit cell
- Recent JILA experiments [17] have reached rotation rates $\Omega/\omega_\perp \gtrsim 0.99$, and more detailed studies confirm this growth and saturation of the core size [20]

(b) Mean-field quantum Hall regime

Lowest-Landau-Level (quantum Hall) behavior

When the vortex cores overlap, kinetic energy associated with density variation around each vortex core becomes important

- hence the TF approximation breaks down (it ignores this kinetic energy from density variations)
- it is preferable to return to full GP energy functional $E'[\Psi]$ in the rotating frame.
- in this limit of rapid rotations ($\Omega \lesssim \omega_\perp$), Ho [31] suggested an important rewriting of the same quantity that incorporates kinetic energy *exactly*
- in this limit of rapid rotation, the condensate expands and becomes effectively two dimensional
- for simplicity, treat a two-dimensional condensate that is *uniform* in the z direction over a length Z
- condensate wave function $\Psi(\boldsymbol{r}_\perp, z)$ can be written as $\sqrt{N/Z}\,\psi(\boldsymbol{r}_\perp)$, where $\psi(\boldsymbol{r}_\perp)$ is a two-dimensional wave function with unit normalization $\int d^2r\,|\psi|^2 = 1$

General two-dimensional energy functional in rotating frame becomes

$$E'[\psi] = \int d^2r\, \psi^* \left(\frac{p^2}{2M} + \frac{1}{2} M \omega_\perp^2 r_\perp^2 - \Omega L_z + \frac{1}{2} g_{2D} |\psi|^2 \right) \psi,$$

where $\boldsymbol{p} = -i\hbar\boldsymbol{\nabla}$, $L_z = \hat{\boldsymbol{z}} \cdot \boldsymbol{r} \times \boldsymbol{p}$, and $g_{2D} = Ng/Z$

This energy functional can be rewritten *exactly* as

$$E'[\psi] = \int d^2r\, \psi^* \left[\frac{(\boldsymbol{p} - M\boldsymbol{\omega}_\perp \times \boldsymbol{r}_\perp)^2}{2M} \right.$$
$$\left. + (\omega_\perp - \Omega)\, L_z + \frac{1}{2} g_{2D} |\psi|^2 \right] \psi,$$

where $\boldsymbol{\omega}_\perp \equiv \hat{\boldsymbol{z}} \omega_\perp$

- assume that $\Omega/\omega_\perp \to 1$ and that interaction energy $\int d^2r\, \frac{1}{2} g_{2D} |\psi|^4$ is small
- hence focus on the first line

- in this limit, energy becomes

$$E'_L[\psi] = \int d^2r\, \psi^* \frac{(\boldsymbol{p} - M\boldsymbol{\omega}_\perp \times \boldsymbol{r}_\perp)^2}{2M} \psi$$

- define *equivalent* uniform magnetic field $\boldsymbol{B} = -2M\boldsymbol{\omega}_\perp/|e|$
- define *equivalent* vector potential $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B} \times \boldsymbol{r}$
- here, we use *symmetric gauge* to describe magnetic field, with $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$
- approximate $E'_L[\psi]$ is *precisely* the Hamiltonian of a single particle with charge $-|e|$ moving in the xy plane in this magnetic field $\boldsymbol{B}$

$$\mathcal{H}_L = \int d^2r\, \psi^* \frac{(\boldsymbol{p} - |e|\boldsymbol{A})^2}{2M} \psi$$

- this one-body Hamiltonian was solved by Landau in 1930, but in *different gauge* (now known as "Landau gauge")
- for solution in symmetric gauge, see Ref. [32]

Here, the exact eigenfunctions can be written $\psi_{nm}(\boldsymbol{r}_\perp)$, where $n \geq 0$ and $m \geq 0$ are non-negative integers and n specifies the "Landau level"

- for these Landau eigenfunctions, the eigenvalues of $\mathcal{H}_L$ are $\epsilon_{nm} = \hbar\omega_\perp (2n + 1)$
- evidently, the eigenvalues are independent of m, so that the states in a given Landau level are massively degenerate
- these eigenfunctions are also eigenstates of L_z with eigenvalues $\hbar\,(m - n)$
- apart from the interaction energy, the Landau-level eigenfunction ψ_{nm} is an eigenstate of full one-particle Hamiltonian

  $$\frac{(\boldsymbol{p} - |e|\boldsymbol{A})^2}{2M} + (\omega_\perp - \Omega)\,L_z$$

  with eigenvalue

  $$\hbar\,[(\omega_\perp + \Omega)\,n + (\omega_\perp - \Omega)\,m + \omega_\perp]$$

- small positive value of $\omega_\perp - \Omega \ll \omega_\perp + \Omega$ lifts the degeneracy associated with the index m

Interaction effects tend to mix the various single-particle eigenfunctions ψ_{nm}

- if $\omega_\perp - \Omega$ is sufficiently small and if interaction energy is small, then there is energy gap $2\hbar\omega_\perp$ between the lowest Landau level and the excited Landau levels, and energy is independent of m
- this requires $g_{2D}n \lesssim \hbar\omega_\perp$, where n is the mean two-dimensional particle density (note that $g_{2D}n \sim \mu$, where μ is the chemical potential)
- the assumption of small interaction energy may be valid because centrifugal forces dramatically expand the condensate as $\Omega \to \omega_\perp$
- hence assume that the system is solely in the lowest Landau level ("LLL") and construct the approximate solution of the GP equation from this restricted set of eigenfunctions ψ_{0m}

LLL eigenfunctions have a very simple form

$$\psi_{0m}(\boldsymbol{r}_\perp) \propto r_\perp^m e^{im\phi}\, e^{-r_\perp^2/2d_\perp^2}$$

- here, $d_\perp = \sqrt{\hbar/M\omega_\perp}$ is analogous to the "magnetic length" in the Landau problem
- in terms of a complex variable $\zeta \equiv x + iy$, these LLL eigenfunctions have an extremely simple form

  $$\psi_{0m} \propto \zeta^m\, e^{-r_\perp^2/2d_\perp^2}$$

  with $m \geq 0$ (note that $\zeta = r_\perp\, e^{i\phi}$ when expressed in two-dimensional polar coordinates)
- assume that the GP wave function is a finite linear combination of these LLL eigenfunctions

  $$\psi_{GP}(\boldsymbol{r}_\perp) = \sum_m c_m \psi_{0m}(\boldsymbol{r}_\perp) = f(\zeta)\, e^{-r_\perp^2/2d_\perp^2}$$

  where $f(\zeta)$ is an *analytic function* of the complex variable ζ
- specifically, $f(\zeta)$ is a finite polynomial and thus can be factorized as $f(\zeta) = \prod_j (\zeta - \zeta_j)$ apart from overall constant

In this way, we are led to the very simple approximate GP solution

$$\psi_{LLL}(\boldsymbol{r}_\perp) = C \prod_j (\zeta - \zeta_j)\, e^{-r_\perp^2/2d_\perp^2}$$

where C is a normalization constant

- the product $\prod_j (\zeta - \zeta_j)$ is a complex polynomial that vanishes at each of the points $\{\zeta_j\}$, so that these are the positions of the nodes of ψ
- in addition, phase of wave function increases by 2π whenever ζ moves around any of these zeros $\{\zeta_j\}$
- we conclude that the LLL trial solution has singly quantized vortices located at positions of zeros $\{\zeta_j\}$
- spatial variation of number density $n(\boldsymbol{r}_\perp) = |\psi_{LLL}(\boldsymbol{r}_\perp)|^2$ is determined by spacing of the vortices, so that core size is comparable with the intervortex spacing $\bar{l} = \sqrt{\hbar/M\Omega}$ which is simply $d_\perp$ in the limit $\Omega \approx \omega_\perp$
- this approximate solution thus generalizes previous TF wave function in the limit $\Omega \to \omega_\perp$

Take this LLL trial function seriously and study its properties

- since LLL wave functions play a crucial role in the quantum Hall effect (two-dimensional electrons in a strong magnetic field), this LLL regime has been called "mean-field quantum Hall" limit [33]
- note that we are still in the regime governed by GP equation, so there is still a BEC
- corresponding many-body ground state is simply a Hartree product with each particle in *same* one-body solution $\psi_{LLL}(\boldsymbol{r}_\perp)$, namely

$$\Psi_{GP}(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N) \propto \prod_{j=1}^{N} \psi_{LLL}(\boldsymbol{r}_j)$$

- this is *coherent (superfluid)* state, since a single GP state ψ_{LLL} has *macroscopic* occupation

- study logarithm of the particle density for this LLL state $\ln n_{LLL}(\boldsymbol{r}_\perp) = \ln |\psi_{LLL}(\boldsymbol{r}_\perp)|^2$
- use ψ_{LLL} to find

$$\ln n_{LLL}(\boldsymbol{r}_\perp) = -\frac{r_\perp^2}{d_\perp^2} + 2\sum_j \ln |\boldsymbol{r}_\perp - \boldsymbol{r}_j|$$

- apply two-dimensional Laplacian
- use $\nabla^2 \ln |\boldsymbol{r} - \boldsymbol{r}_j| = 2\pi\delta^{(2)}(\boldsymbol{r} - \boldsymbol{r}_j)$
- find

$$\nabla^2 \ln n_{LLL}(\boldsymbol{r}_\perp) = -\frac{4}{d_\perp^2} + 4\pi \sum_j \delta^{(2)}(\boldsymbol{r}_\perp - \boldsymbol{r}_j)$$

- here, sum over delta functions is precisely the *vortex* density $n_v(\boldsymbol{r}_\perp)$
- this result relates *particle* density $n_{LLL}(\boldsymbol{r}_\perp)$ in LLL approximation to *vortex* density $n_v(\boldsymbol{r}_\perp)$ [31, 33]

$$\frac{1}{4}\nabla^2 \ln n_{LLL}(\boldsymbol{r}_\perp) = -\frac{1}{d_\perp^2} + \pi n_v(\boldsymbol{r}_\perp)$$

- if vortex lattice is exactly uniform (so n_v is constant), then density profile is strictly Gaussian, with $n_{LLL}(\boldsymbol{r}_\perp) \propto \exp(-r_\perp^2/\sigma^2)$ and $\sigma^{-2} = d_\perp^{-2} - \pi n_v$
- in this case, $\sigma^2 \gg d_\perp^2$
- more precisely, $\sigma^{-2} \propto \omega_\perp - \Omega$
- Watanabe *et al.* [33] argue that the density profile should independently have an inverted parabolic (TF) shape $n_{LLL}(\boldsymbol{r}_\perp) \propto 1 - r_\perp^2/R_\perp^2$
- then find *nonuniform* vortex density with

$$n_v(r_\perp) \approx \frac{1}{\pi d_\perp^2} - \frac{1}{\pi R_\perp^2}\frac{1}{\left(1 - r_\perp^2/R_\perp^2\right)^2}$$

  similar to result at lower Ω [19]

- independently, numerical work by Cooper *et al.* [34] shows that allowing the vortices in the LLL to deviate from the triangular array near the outer edge lowers the energy

3 Behavior for $\Omega \gtrsim \omega_\perp$

What happens beyond the "mean-field quantum Hall" regime is still subject to vigorous debate

(a) Beyond GP regime (*correlated* states)

- define the ratio $\nu \equiv N/\mathcal{N}_v$ of the number of atoms per vortex
- because of similarities to a two-dimensional electron gas in a strong magnetic field, ν is called the "filling fraction" [35, 36]
- current experiments [17] have $N \sim 10^5$ and $\mathcal{N}_v \sim$ several hundred, so $\nu \sim$ a few hundred
- numerical studies [36] for small number of vortices ($\mathcal{N}_v \lesssim 8$) and variable N indicate that the coherent GP state is favored for $\nu \gtrsim 6$

- for smaller ν there is a sequence of *highly correlated* states similar to some known from the quantum Hall effect, in particular a bosonic version of the Laughlin state [36] (here $z_j = x_j + iy_j$ refers to jth particle)

$$\Psi_{\text{Lau}}(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N) \propto \prod_{j<k}^{N} (z_j - z_k)^2 \exp\left(-\sum_{j=1}^{N} \frac{|z_j|^2}{2d_\perp^2}\right)$$

- these correlated many-body states are *qualitatively different* from coherent GP form
  - $\Psi_{GP}(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N) \propto \prod_j \psi(\boldsymbol{r}_j)$ is the Hartree product of N factors of *same* one-body function $\psi(\boldsymbol{r})$
  - the product $\prod_{jk}(z_j - z_k)^2$ in $\Psi_{\text{Lau}}(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N)$ involves $N(N-1)/2$ factors for all possible *pairs* of particles and vanishes whenever two particles are close together
  - this is the source of the correlations
  - for large N, correlated form Ψ_{Lau} is much more difficult to use

(b) Addition of quartic potential

One way to avoid singularity when $\Omega \to \omega_\perp$ is to add a quartic confining potential [37, 38, 39]

- now have a total potential with quadratic and quartic terms
$$V_{\rm tr} = \frac{1}{2} M \omega_\perp^2 \left(r^2 + \lambda \frac{r^4}{d_\perp^2} \right)$$
where the dimensionless constant λ fixes the quartic admixture
- allows access to regime $\Omega/\omega_\perp \geq 1$
- studied experimentally at ENS, Paris [40], where a blue-detuned axial laser provided the weak quartic confinement ($\lambda \sim 10^{-3}$ and $\omega_\perp/2\pi \approx 64.8$ Hz)
- find regular vortex lattice for $\Omega \lesssim \omega_\perp$
- find disordered vortex lattice for $\omega_\perp \lesssim \Omega$
- near $\Omega \approx 1.05\,\omega_\perp$, the system seems to break up
- TF theory predicts a reduced density at center, which is observed

What is happening?

- ENS condensate is nearly spherical for $\Omega \sim \omega_\perp$, so three-dimensional effects are important
- they suggest repeating the experiment with strong axial confinement to see if three-dimensional effects dominate and cause instability
- GP analysis in two dimensions finds nothing like the observed break up [38, 39, 41]
- is there some sort of transition from a GP state to a highly correlated state in the regime $\Omega \gtrsim \omega_\perp$?
- this issue remains very uncertain

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