Lecture 2 3. Mean curvature of a surface

At any point on a regular surface, the plane of maximum curvature (with radius of curvature R_1) is perpendicular to the plane of minimum curvature (with radius of curvature R_2). These minimum and maximum radii of curvature are termed the principal radii of curvature.

The mean curvature of the surface (often denoted □ or 2H) is given by the sum of the inverse principal radii of curvature, i.e.

$$\kappa = 2H = \frac{1}{R_1} + \frac{1}{R_2}$$

The maximum and minimum of the normal curvature κ_1 and κ_2 at a given point on a surface are called the principal curvatures. The principal curvatures measure the maximum and minimum bending of a regular surface at each point**.**

For the surface described above, $\kappa = 2H = \frac{1}{R_1} + \frac{1}{R_2}$ For a sphere of radius R: $\kappa = 2H = \frac{1}{R_1} + \frac{1}{R_2}$.

For an arbirtrary area element □A of a curved surface, the change in area caused by a normal expansion of the surface □d is:

$$\Delta A + \delta\Delta A = (R_1 + \delta d)d\Theta_1 (R_2 + \delta d)d\Theta_2$$

$$\Delta A + \delta\Delta A = R_1 R_2 d\Theta_1 d\Theta_2 (1 + \delta d(\frac{1}{R_1} + \frac{1}{R_2}))$$

$$\delta\Delta A = \delta d(\frac{1}{R_1} + \frac{1}{R_2})\Delta A$$

Easy derivation of Young Laplace equation: Consider a surface element at equilibrium between two phases with principal radii R1 and R2. Let there be an isothermal perturbation about this equilibrium state. Suppose that the two bulk fluids are incompressible, so the displacement occurs at constant volume for the two bulk phases.

$$\delta F = 0 = \gamma\delta\Delta A - P_\alpha \delta V_\alpha - P_\beta \delta V_\beta = \delta d \gamma(\frac{1}{R_1} + \frac{1}{R_2})\Delta A - P_\alpha \delta d \Delta A + P_\beta \delta d \Delta A$$

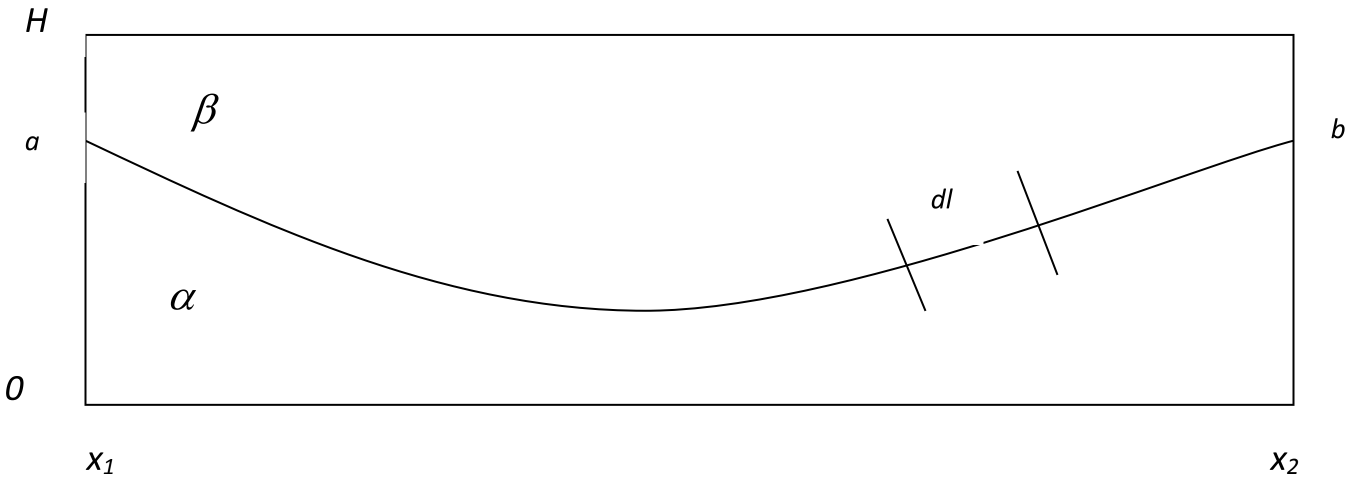
$$0 == \delta d \Delta A \left\{ \gamma(\frac{1}{R_1} + \frac{1}{R_2}) - (P_\alpha - P_\beta) \right\}$$

$$\gamma(\frac{1}{R_1} + \frac{1}{R_2}) = P_\alpha - P_\beta$$

4. Derivation of the Equations of Hydrostatics by Quasi-thermodynamics

(Here F is the Helmholtz free energy)

SYSTEM:



Consider a curved interface between two phases at equilibrium enclosed in a solid box. Energy can come and go from the system by heat only. We will consider isothermal disturbances:

$$dU = TdS$$

$$dF = dU - TdS - SdT = 0$$

Assume:

0<z<H; z=h(x) locates the surface

h(x)<z<H is in the □ phase

0<z<h(x) is in the □ phase

Assume 0<y<L

Let the interface intersect the solid sides at fixed locations:

$h(x_1)=a$; $h(x_2)=b$

Assume: no motion; system at equilibrium

Let gravity (body force) disturb the system in time □t:

$$h(x,t) = h_{eq}(x) + \delta h_1(x,t)$$

This disturbance creates a change in the Helmholtz free energy:

$$F = F_{eq} + \delta F_1$$

The disturbance in free energy is zero when $\delta h_1(x,t)$ =0. By looking at F($\delta h_1(x,t)$ =0) we can learn about the minimum free energy state (equilibrium) of the system.

To derive F for the system, consider the interface as being divided into small elements *dl*

Ignore the curvature in the dl elements; use thermo of planar interfaces there, then integrate:

$$F = \int_{x1}^{x2} F_{element} dl$$

$$F = F^s + F^\alpha + F^\beta + F^{\alpha-wall} + F^{\beta-wall}$$

Since the three phase contact lines are fixed, the wall contributions do not change

$$f^s = \gamma + \sum \mu_i \Gamma_i$$

Chose interface location so that the sum can be neglected in this 2-component (⍰□⍰ system)

$$F^s = L\int_{x1}^{x2} \gamma dl$$

The disturbance of theinterface sets up a flow field:

$\delta \dot{h}$ sets up δu^α ; δu^β

If fluids ⍰□□⍰ are incompressible, this will constrain the system:

$$\nabla \bullet \delta u^\beta = 0; \quad \nabla \bullet \delta u^\alpha = 0$$

This constraint would restrict the perturbations in F⍰ and F⍰.

To do: ADJOIN THE CONSTRAINTS TO F. CALCULATE $\delta F = 0$ for an arbitrary disturbance.

1. Contribution from interface: $F^s = L\int_{x1}^{x2} \gamma dl = L\gamma \int_{x1}^{x2} \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2} dx$

For small disturbance can of h about equilibrium can expand:

$$\delta F^{s} = L\gamma \int_{x1}^{x2} \frac{\frac{dh_{eq}}{dx}}{\sqrt{1+\left(\frac{dh_{eq}}{dx}\right)^{2}}} (\frac{d\delta h}{dx}) dx$$

2. Contribution from bulk fluids:

$$\delta F^{\alpha} = \int_{V\alpha} \rho \mathbf{g}^{\alpha} \bullet \delta \mathbf{u}^{\alpha} \Delta t dV$$

$$\delta F^{\beta} = \int_{V\beta} \rho^{\beta} \mathbf{g} \bullet \delta \mathbf{u}^{\beta} \Delta t dV$$

3. Adjoin constraints; P⍰⍰t and P⍰⍰t are convenient forms for Lagrange multipliers

$$C_{\beta} = \int_{V\beta} (\nabla \bullet \delta \mathbf{u}^{\beta}) P^{\beta} \Delta t dV;$$

$$C_{\alpha} = \int_{V\alpha} (\nabla \bullet \delta \mathbf{u}^{\alpha}) P^{\alpha} \Delta t dV$$

Adding up these terms:

$$\delta F = L\gamma \int_{x1}^{x2} \frac{\frac{dh}{dx}}{\sqrt{1+\left(\frac{dh}{dx}\right)^{2}}} (\frac{d\delta h}{dx}) dx + \int_{V\alpha} \rho \mathbf{g} \bullet \delta \mathbf{u}^{\alpha} \Delta t dV + \int_{V\beta} \rho \mathbf{g} \bullet \delta \mathbf{u}^{\beta} \Delta t dV$$

$$+ \int_{V\beta} P^{\beta} \Delta t \nabla \bullet \delta \mathbf{u}^{\beta} dV + \int_{V\alpha} P^{\alpha} \Delta t \nabla \bullet \delta \mathbf{u}^{\alpha} dV = 0$$

Manipulating the surface contribution gives:

$$TERM1 = L\gamma \int_{x1}^{x2} \frac{d}{dx} \left(\frac{\frac{dh_{eq}}{dx}}{\sqrt{1+\left(\frac{dh_{eq}}{dx}\right)^{2}}} \right) \delta h dx - \gamma \delta h \frac{\frac{dh_{eq}}{dx}}{\sqrt{1+\left(\frac{dh_{eq}}{dx}\right)^{2}}} \Bigg|_{x1}^{x2} = L\gamma \int_{x1}^{x2} \kappa \delta h dx$$

Manipulating the constraints gives:

$$C^{\alpha} = \int_{V\alpha} \left[\nabla \bullet (P^{\alpha} \Delta t \delta \mathbf{u}^{\alpha}) - \delta \mathbf{u}^{\alpha} \Delta t \bullet \nabla P^{\alpha} \right] dV$$

$$C^{\alpha} = \int_{V\alpha} -\delta \mathbf{u}^{\alpha} \Delta t \bullet \nabla P^{\alpha} dV + \int_{S\alpha} n \bullet (P^{\alpha} \Delta t \delta \mathbf{u}^{\alpha}) dA$$

$$C^{\alpha} = \int_{V\alpha} -\delta \mathbf{u}^{\alpha} \Delta t \bullet \nabla P^{\alpha} dV + L \int_{x1}^{x1} n \bullet P^{\alpha} \Delta t \delta \mathbf{u}^{\alpha} dx$$

Substituting these into a single expression and manipulating gives:

$$L \int_{x1}^{x2} \{\gamma \kappa \delta h dx + \} dx + \int_{l1}^{l2} P^{\alpha} (n \bullet \Delta t \delta \mathbf{u}^{\alpha}) + P^{\beta} (n \bullet \Delta t \delta \mathbf{u}^{\beta}) dl$$

$$\int_{V\alpha} -\delta \mathbf{u}^{\alpha} \Delta t \bullet (\nabla P^{\alpha} + \rho^{\alpha} \mathbf{g}) dV + \int_{V^{\beta}} -\delta \mathbf{u}^{\beta} \Delta t \bullet (\nabla P^{\beta} + \rho^{\beta} \mathbf{g}) dV = 0$$

Using the kinematic condition for a small displacement of a material surface:

$$\frac{\delta h}{\sqrt{1 + h_{xeq}{}^{2}}} = (n \bullet \Delta t \delta \mathbf{u}^{\beta})$$

continuity of velocity at surface gives, so true also for alpha phase, but with opposite sign for outward pointing normal.

$$\frac{\delta h}{\sqrt{1 + h_{xeq}{}^{2}}} dl = (n \bullet \Delta t \delta \mathbf{u}^{\beta}) dl = \delta h dx$$

So the variation in Helmholtz free energy becomes:

$$L \int_{x1}^{x2} \delta h \{\gamma \kappa - P^{\alpha} + P^{\beta}\} dx +$$

$$\int_{V\alpha} -\delta \mathbf{u}^{\alpha} \Delta t \bullet \{\nabla P^{\alpha} + \rho^{\alpha} \mathbf{g}\} dV + \int_{V^{\beta}} -\delta \mathbf{u}^{\beta} \Delta t \bullet \{\nabla P^{\beta} + \rho^{\beta} \mathbf{g}\} dV = 0$$

This variation should be zero for an arbitrary disturbance so expressions in curly brackets should be equal to zero.

The Young Laplace equation:

$$\gamma\kappa = P^\alpha - P^\beta$$

The equations of hydrostatics in the bulk fluids:

$$\nabla P^\alpha + \rho^\alpha \mathbf{g} = 0$$
$$\nabla P^\beta + \rho^\beta \mathbf{g} = 0$$