Granular Materials: The Plan

Two regimes:
- Statics
- Dynamics

Both regimes have unique properties

How are these regimes related?
- Jamming

Lecture 1) Jamming and Forces

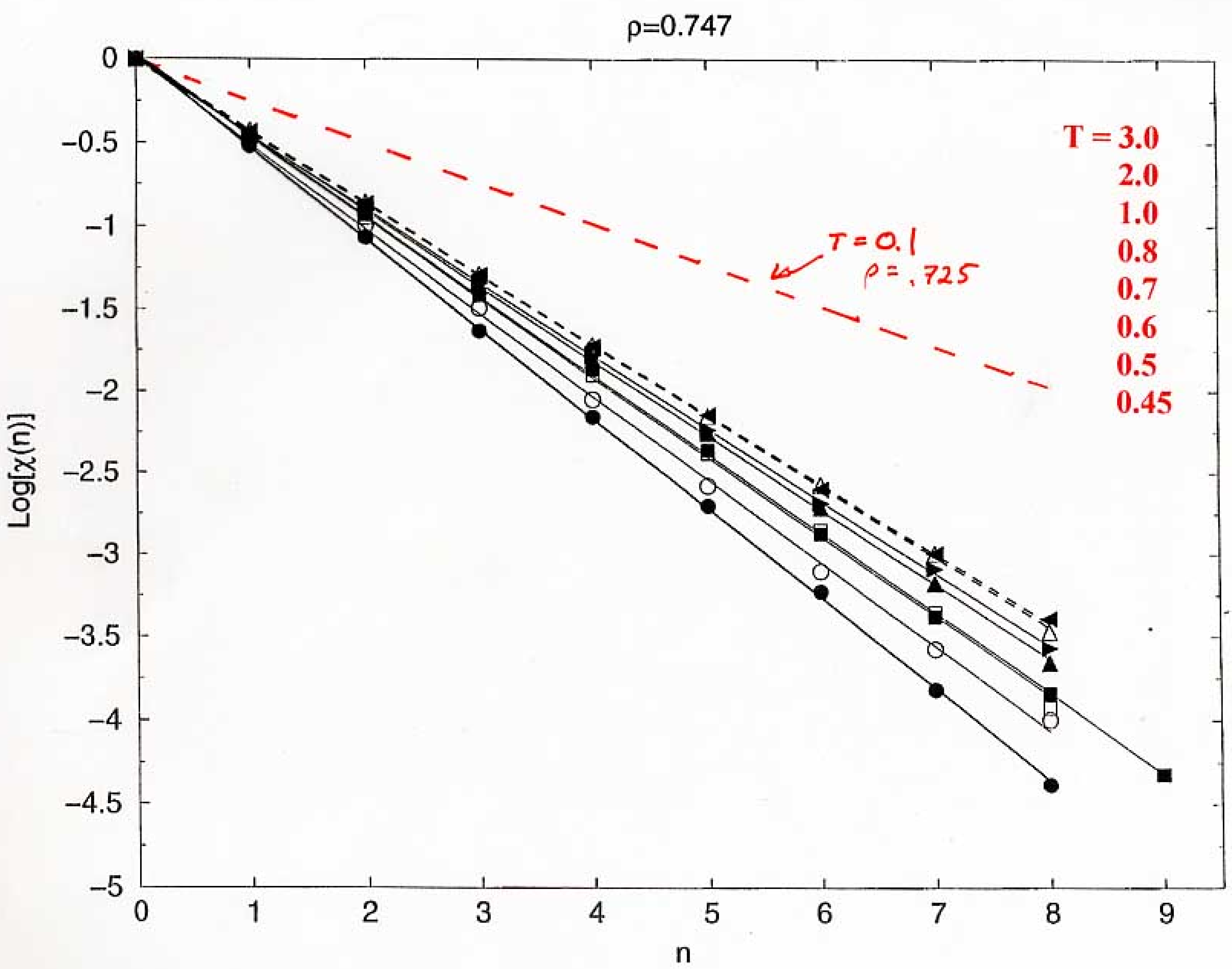
Lecture 2) Compaction and Sound

Lecture 3) Public Lecture - General remarks about granular materials

Lecture 4) Flow and Convection

Lecture 5) Flow (continued); Introduction of a Second Fluid

Equilibrium Force-Chain Correlation Function



Jamming:

from Glasses to Granular Matter

Corey S. O'Hern — University of Chicago; UCLA
Stephen A. Langer — NIST
Andrea J. Liu — UCLA

ITP - Santa Barbara 1997
$ - NSF - $

Things get stuck far from equilibrium - Do it to themselves (Self-organized).

Is jamming
(lowering stress or increasing ρ)
related to the glass transition
(lowering T)?

Relation to close-packing.

Relate Granular Material to Glasses

Glasses: Heat

solid → liquid.

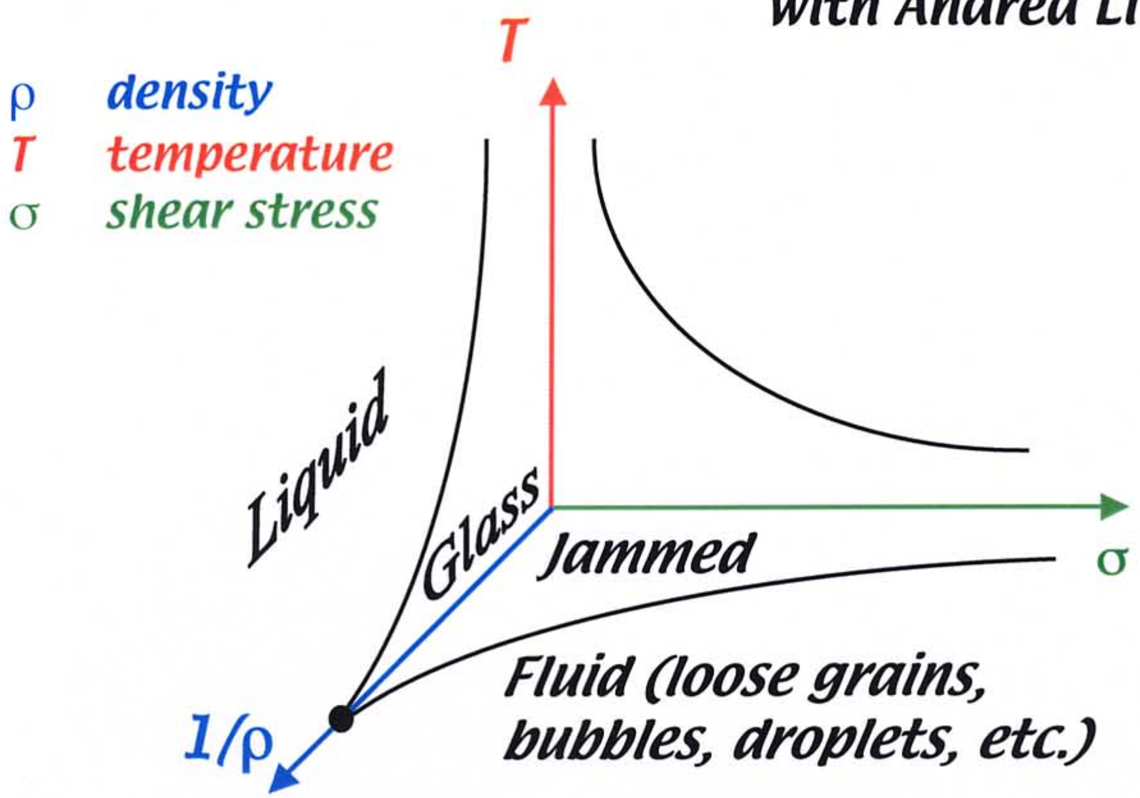
Granular Matter: Shake, shear

solid → liquid.

Both perched precariously at a transition
⇒ Jamming

"Jamming Phase Diagram" links both phenomena

with Andrea Liu



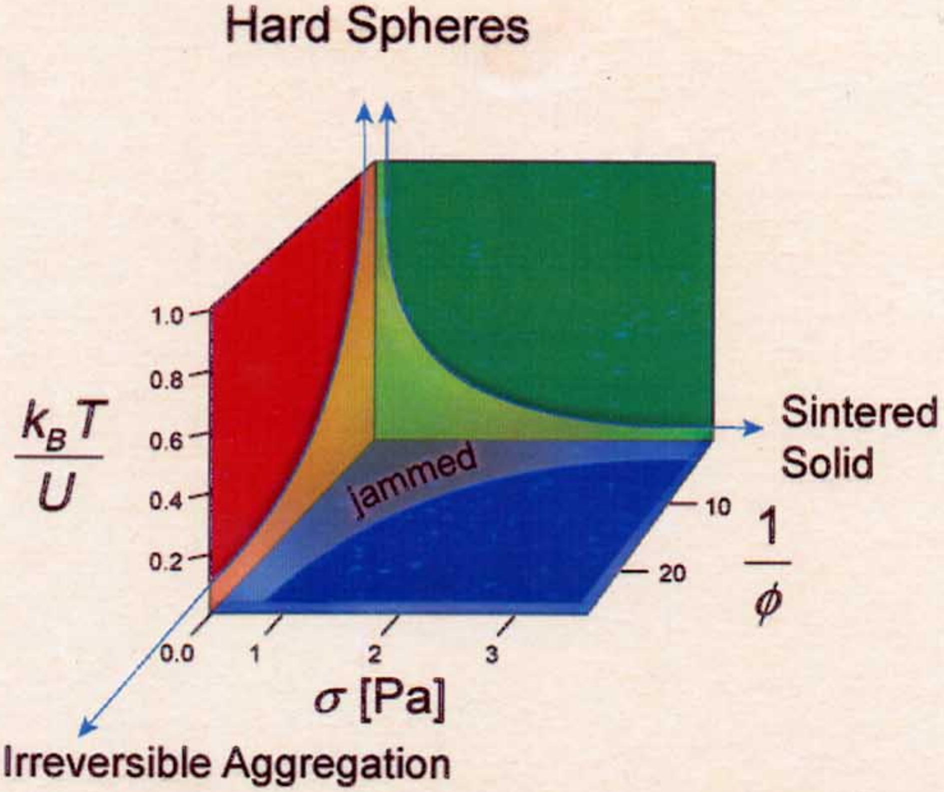


Fig. 4: Composite jamming phase diagram for attractive colloidal particles. The values indicated on the axes are typical; the actual values depend to some extent on the details of the system.

FROM:

V. TRAPPE, V. PRASAD, L. CIPELLETTI, P.N. SEGRE, AND D.A. WEITZ.

"JAMMING PHASE DIAGRAM FOR ATTRACTIVE PARTICLES"

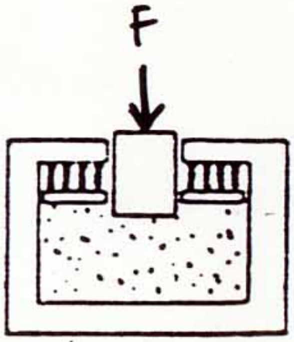
NATURE, 411, 772 (2001).

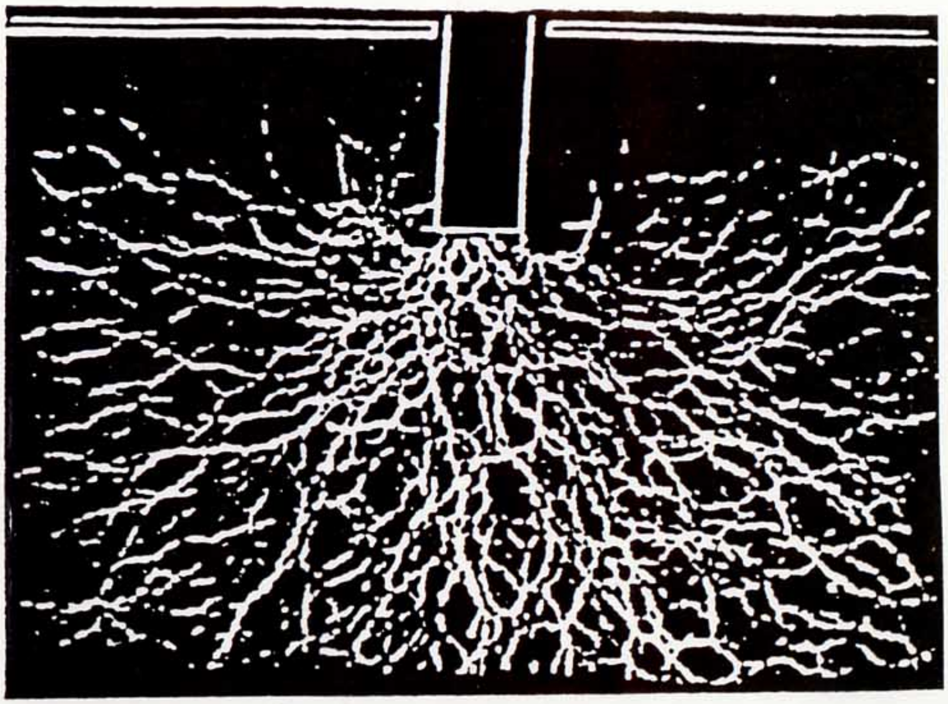
Force Chains

View using stress-induced birefringence:

Dantu 1967

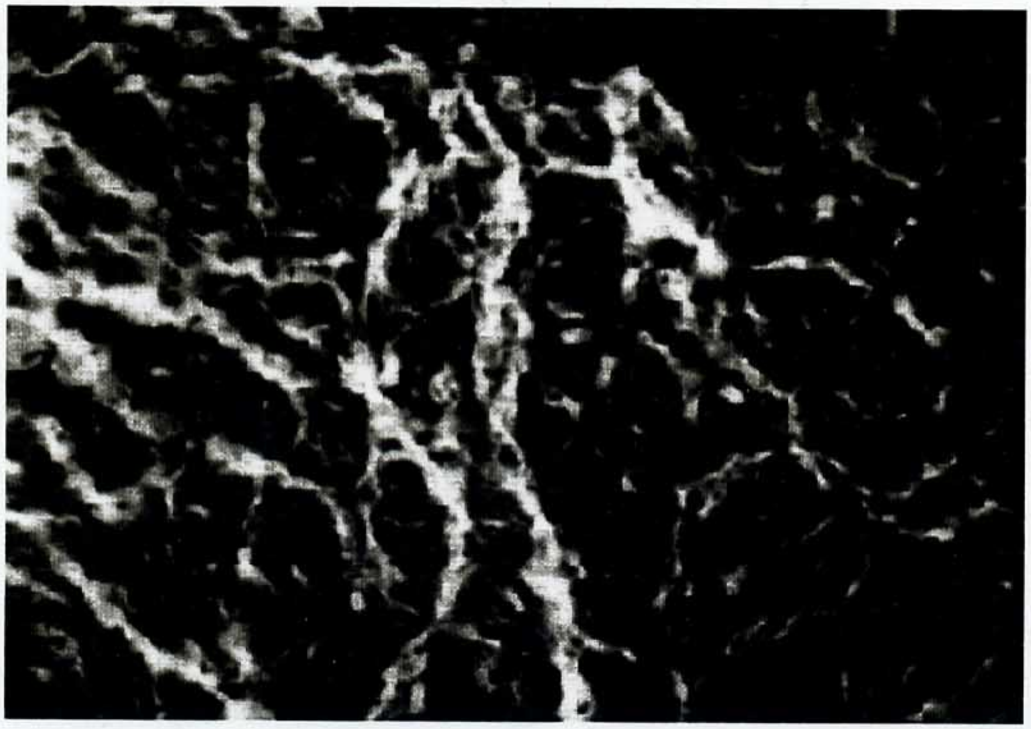
2-dimensional pile





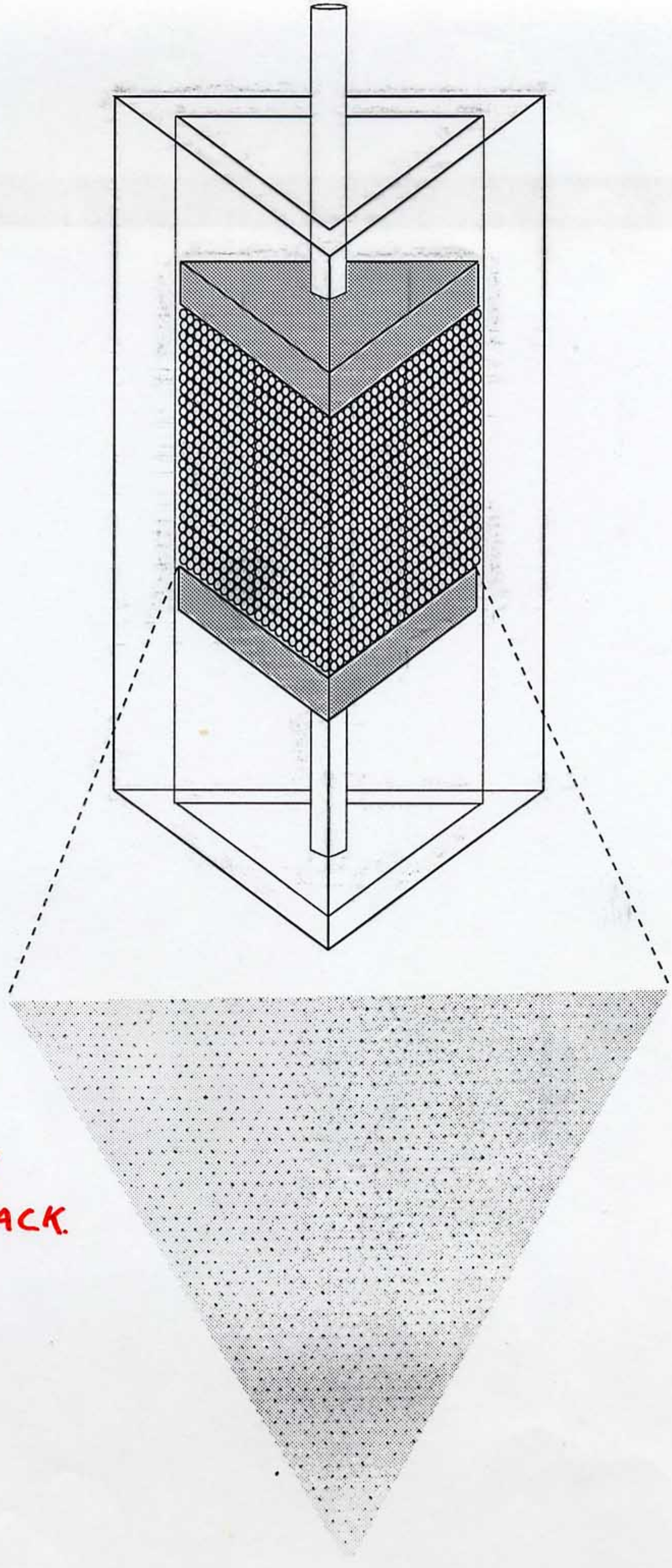
Force Chains in Bead Pack

3-D Visualization



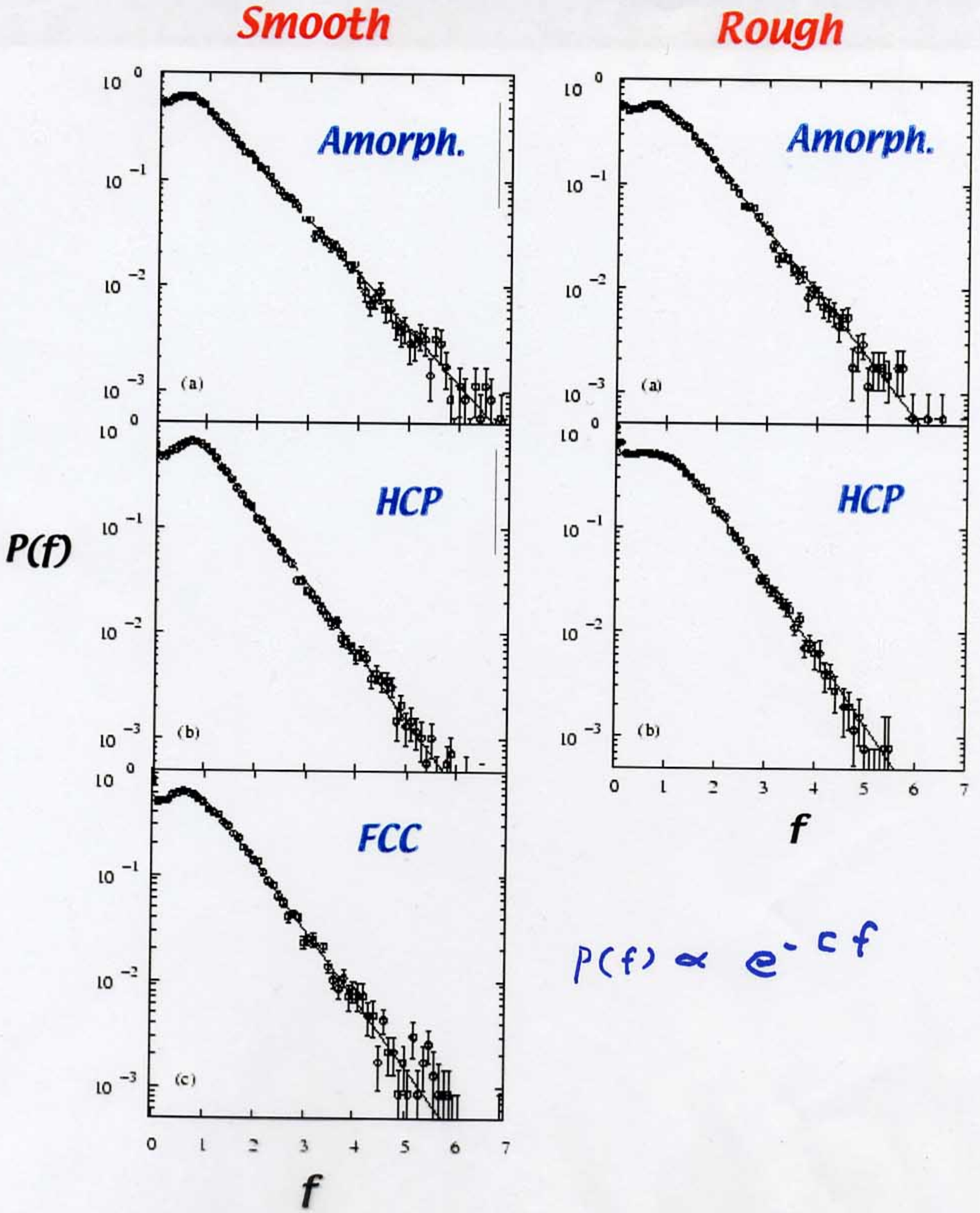
Liu, Nagel, Schecter, Coppersmith, Majumdar, Narayan, Witten

MAKE A
CRYSTAL



MEASURE
FORCES
AT BOUNDARY
OF BEAD PACK.

Force Distributions in Bead Packs



$P(f) \propto e^{-cf}$

Histogram: number of contacts with force f = F/<F>

Blair, Mueggenburg, Marshall, Jaeger and Nagel

Questions

How to understand shape of $P(f)$ (at small f)?

How is arching manifested in $P(f)$?

What is correlation length for force chains?

$\Rightarrow$ Are these results the same or different from those in liquids and glasses?

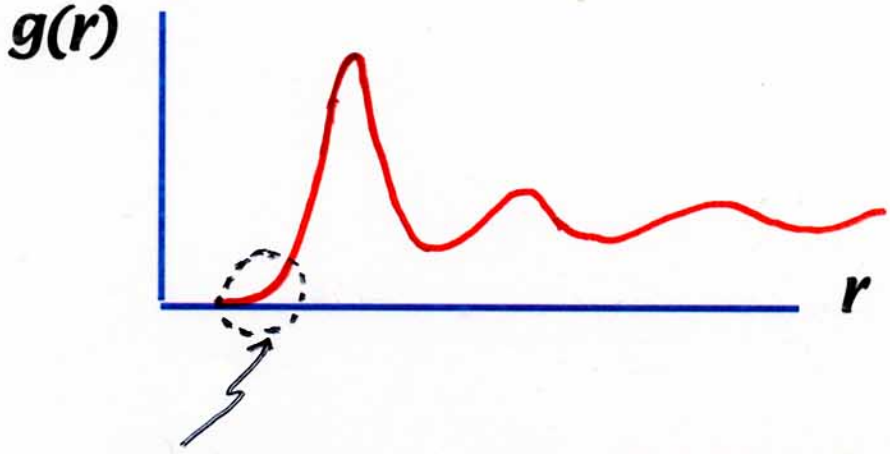
P(f) for Liquids

Force from inter-atomic potential, $V(r)$:

$$f = -dV(r)/dr$$

P(f) from Radial-Distribution Function, $G(r)$:

$$P(f)\, df = G(r)\, dr = (\pi r^2)\, g(r)\, dr$$



$$g(r)_{\lim r \to 0} = a \exp[-V(r)/k_B T]$$

Exact asymptotically!

For e.g., $V(r) = r^{-12}$

$$\Rightarrow \log P(f) = A - (15/13)\log[1/f] - (C/T)\, f^{12/13}$$

For hard spheres: $P(f) = a/f \exp[-cf]$

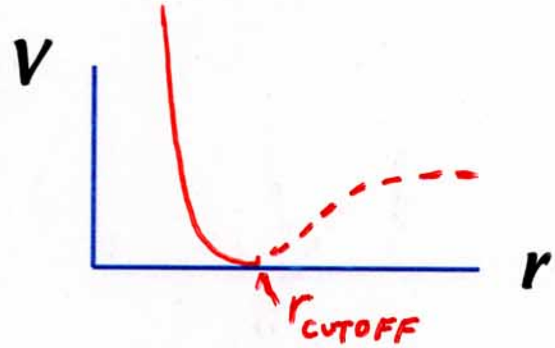
Truncated Lennard-Jones

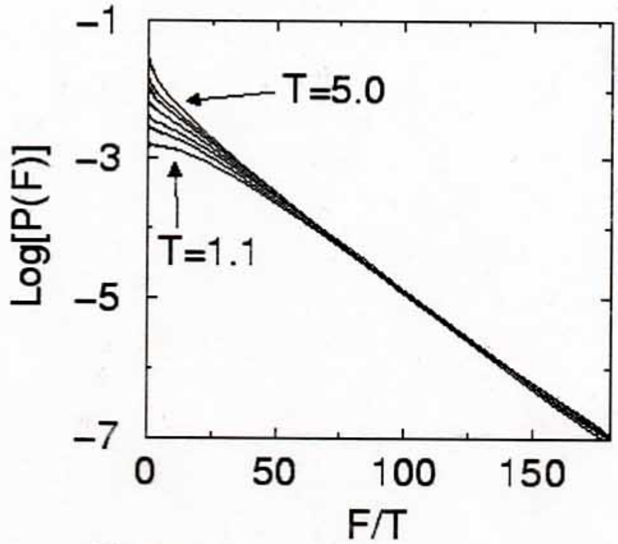
$$V(r) = 4\varepsilon[(\sigma/r)^{12} - (\sigma/r)^{6}]$$

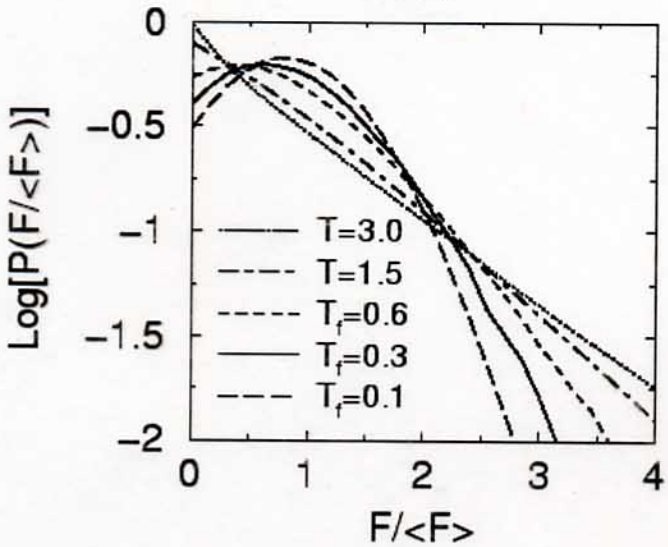
$T_g \approx 1.1$

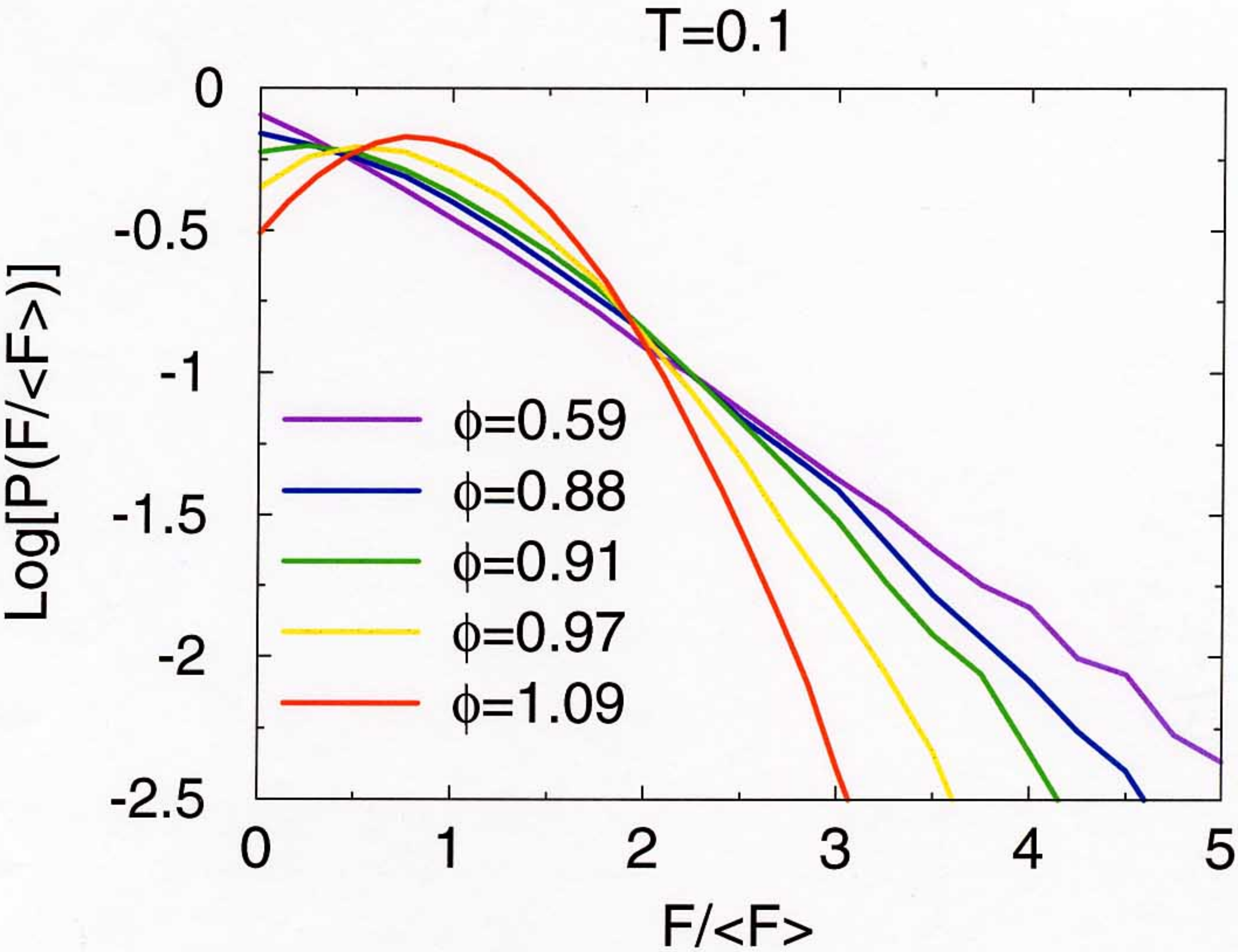
$\sigma_2 = 1.4\,\sigma_1$

50%/50%

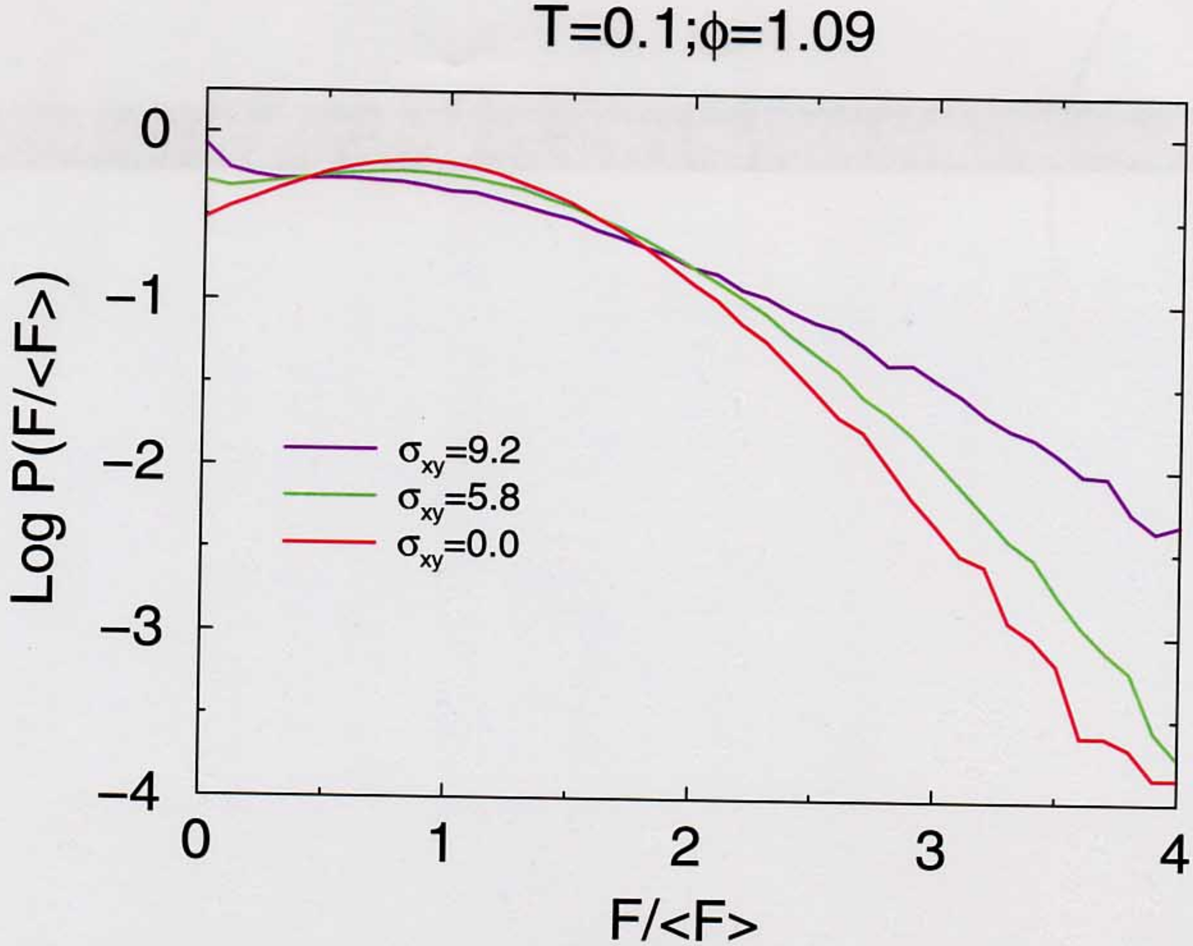








T=0.1
Log[P(F/<F>)]
F/<F>
ϕ=0.59
ϕ=0.88
ϕ=0.91
ϕ=0.97
ϕ=1.09
0
-0.5
-1
-1.5
-2
-2.5
0
1
2
3
4
5


T=0.1;ϕ=1.09
Log P(F/<F>)
F/<F>
σxy=9.2
σxy=5.8
σxy=0.0
0
−1
−2
−3
−4
0
1
2
3
4

Relevance of Peak in P(f)

Jammed state, glasses, crystals - have peak

Peak disappears with
-increasing temperature and shear
-decreasing density

Peak → Signature of jammed state.
Yield stress developed.

Peak in P(f) has slowest decaying forces.
Cause of slow relaxation at T_g?

Peak in P(f) from peak in g(r).
⇒ Criterion for height and width of peak in g(r).

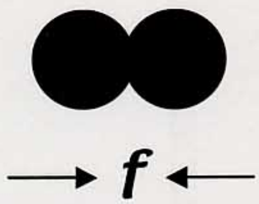
Physical interpretation

Why signature in P(f) and not in g(r)?
Forces must balance: acceleration or other forces.

Couples to development of force network.
Force network contains many-body correlations.
P(f) and g(r) contain only 2-body correlations.

Static Granular Material and Force Distributions

P(f) is the probability that normal force between two neighbors is f.



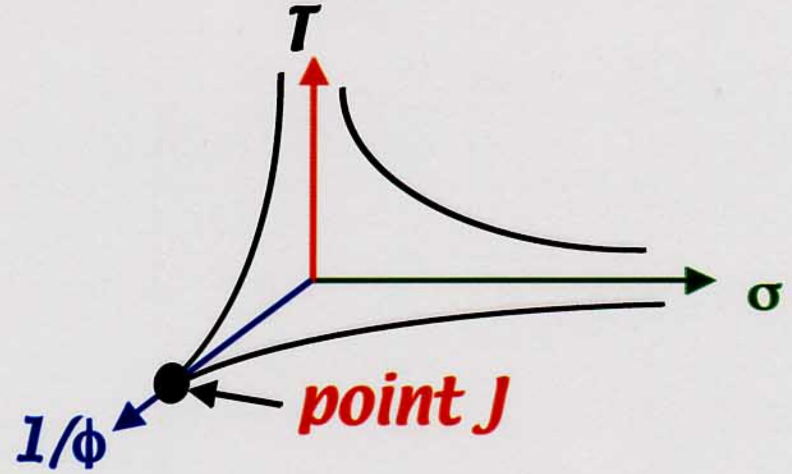
P(f) has "universal" properties.
At jamming develops a peak
(signature of a yield stress)

Implications?

Can one measure P(f) in the field and see how close to rupture a fault is?

Simplest example of jamming?

$T = 0;\ \sigma = 0;\ \phi = \phi_c$



What is nature of J?

In equilibrium for $\phi \le \phi_c$, $P = 0$.

Candidate for ideal glass.

Does point J "control" jamming surface?

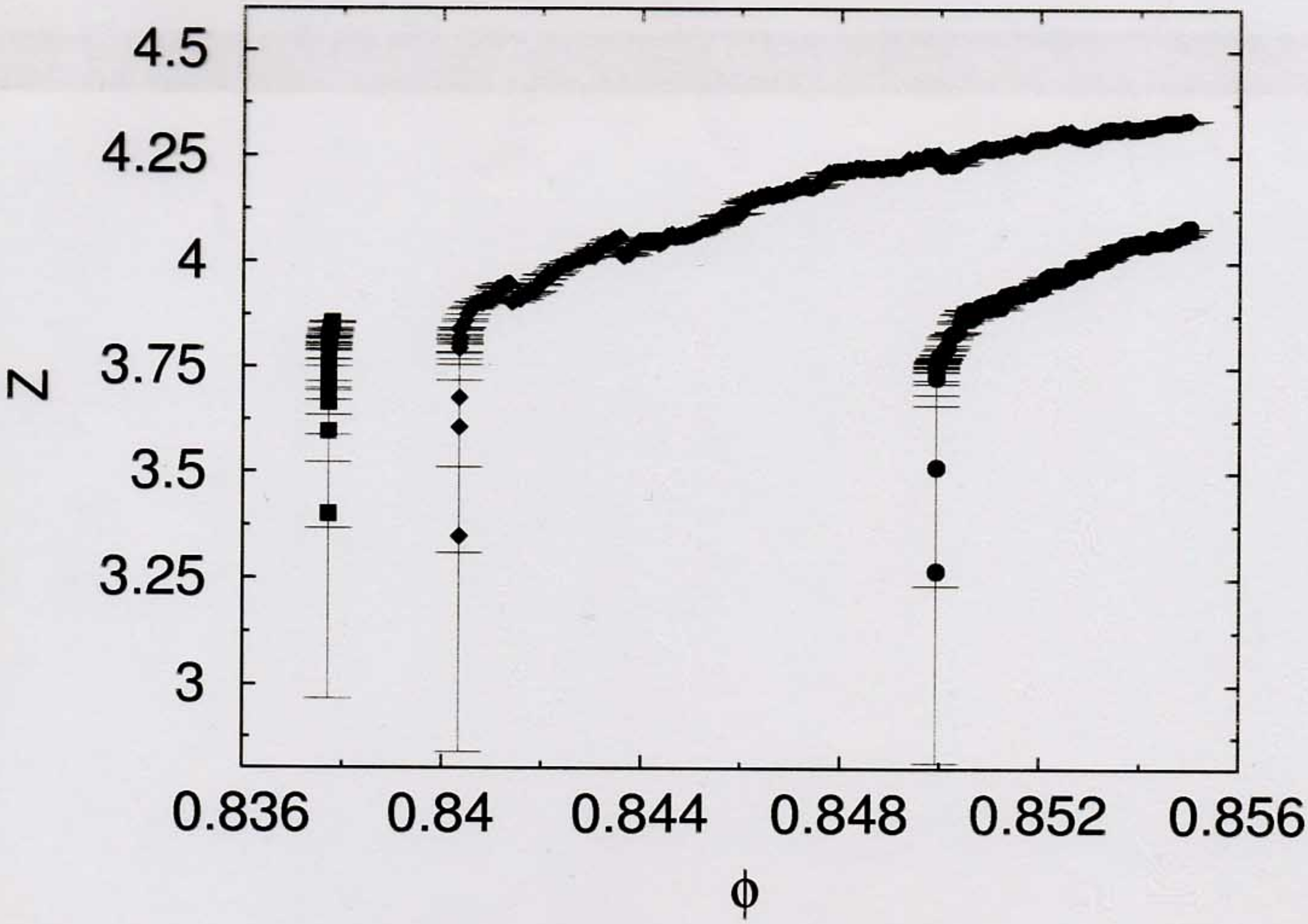
Random Close-Packing at J

$\phi_c = \phi_{RCP}$?

Bernal, Finney, Onoda and Liniger. . .
Stillinger and Weber (inherent structures).

Is ϕ_c unique? (Depends on quench rate?)
What does "Random" mean?
Is there loose packing (without friction)?

Average Coordination Number



3 configurations

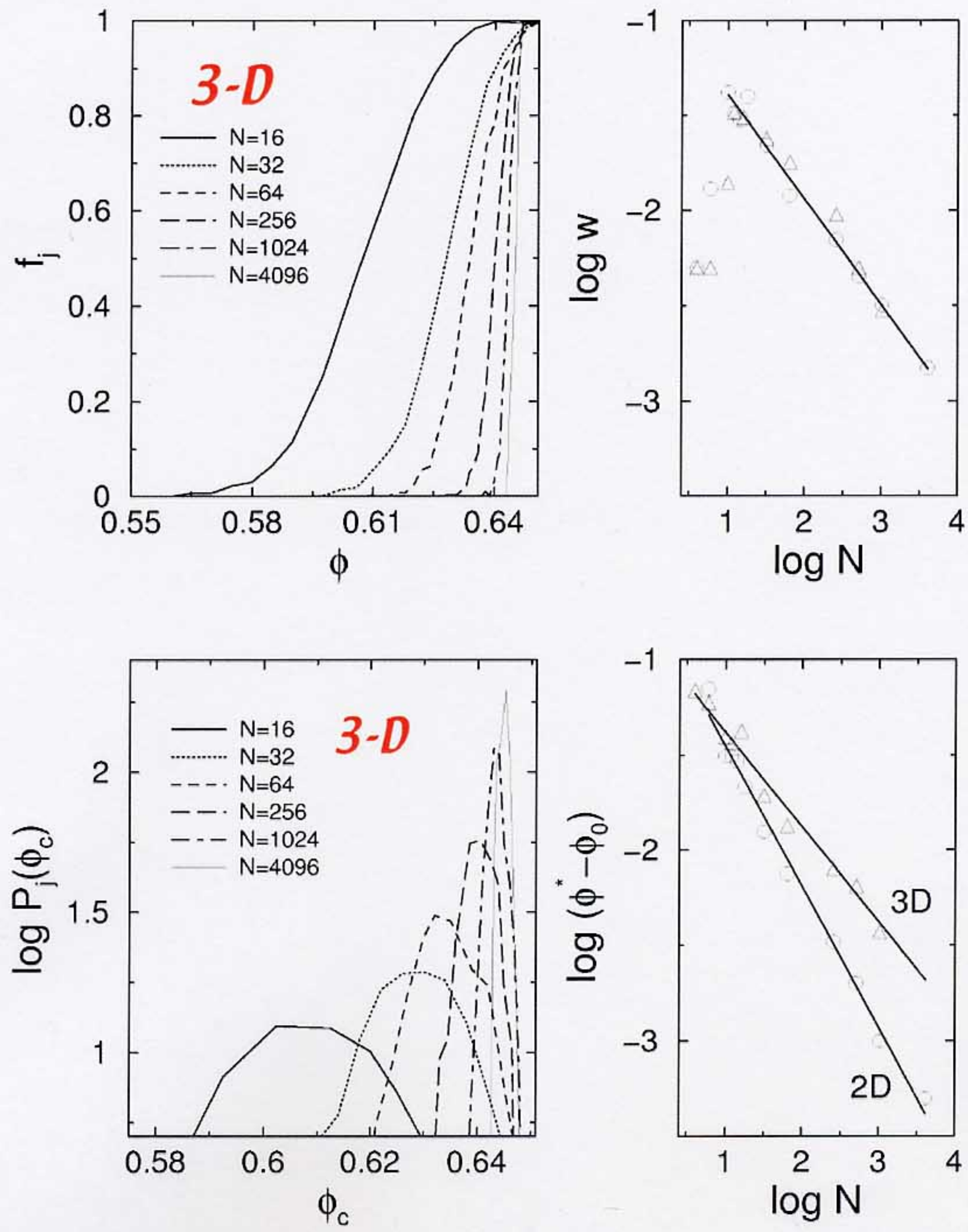
N = 1024

Soft-spheres (repulsive springs)

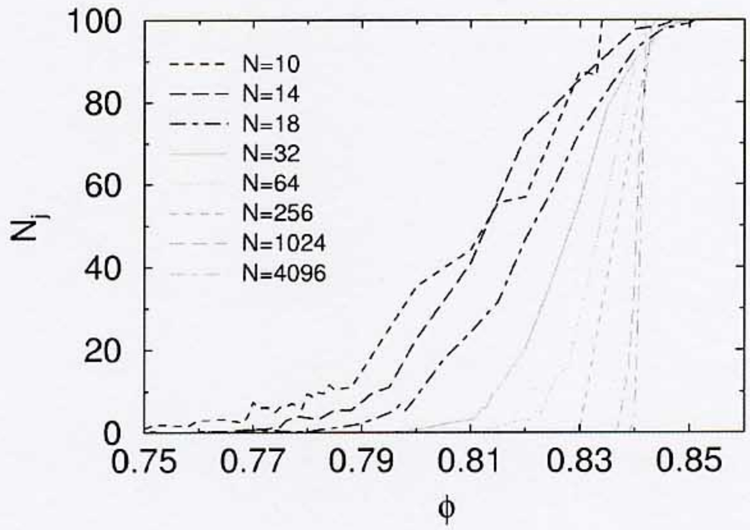
Frictionless

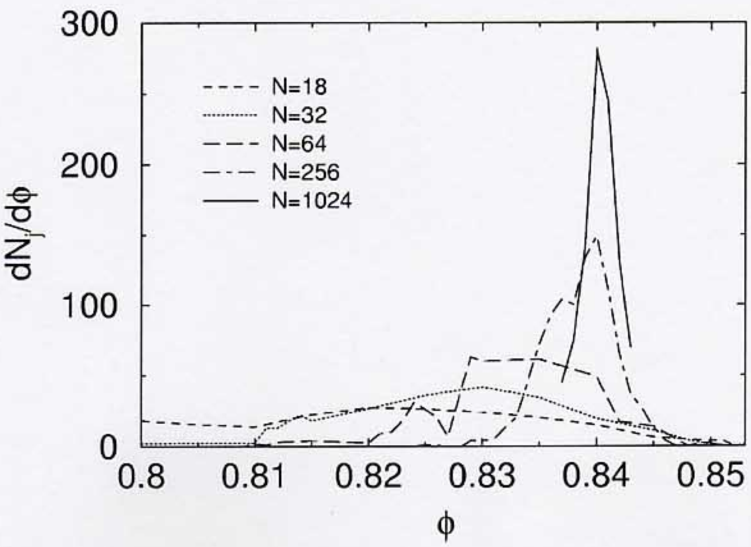
Isotropic

Onset of jamming:
Finite-size effects
Each configuration has own onset

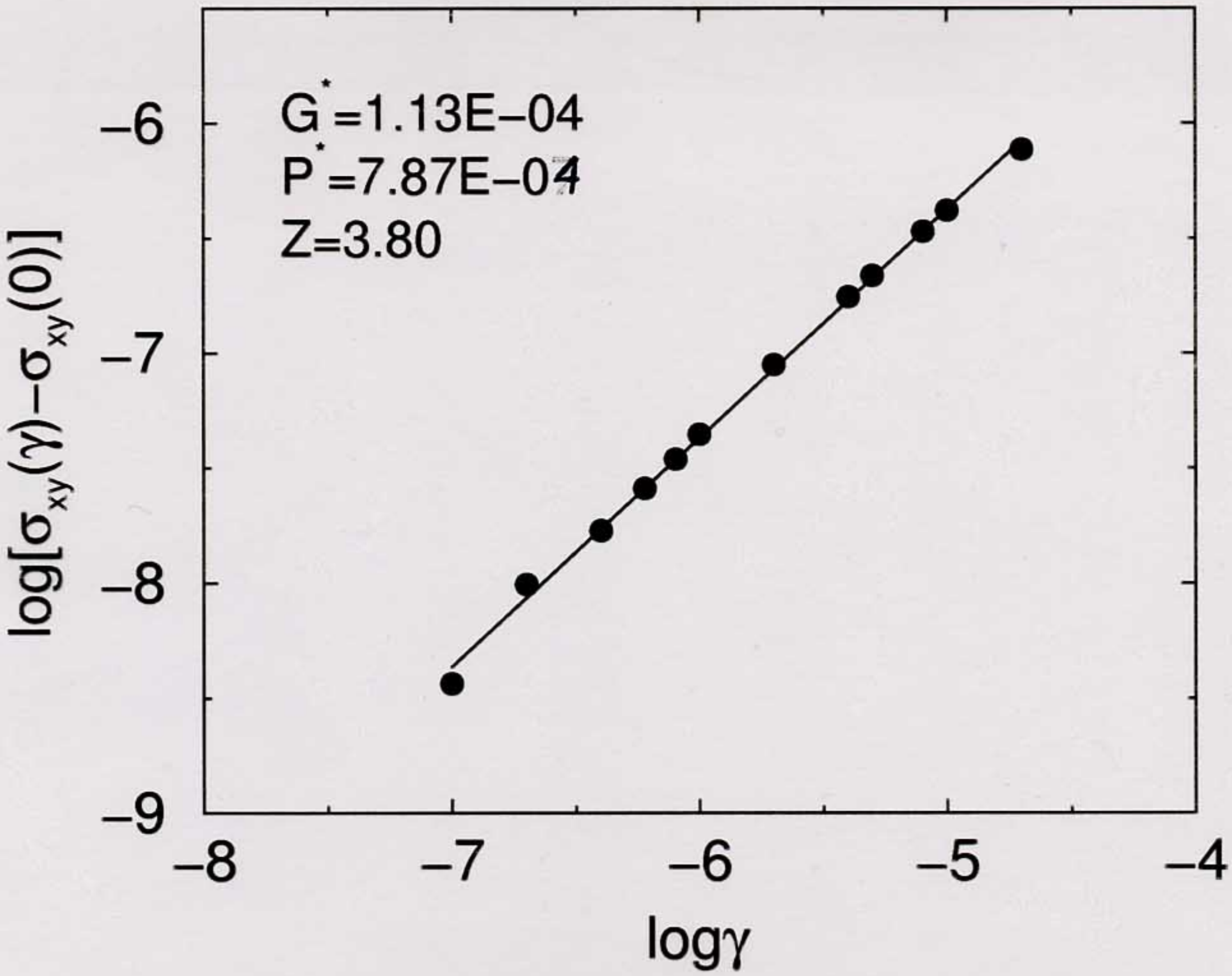


Onset of jamming:
Finite-size effects
2-D simulations





Stress vs. Strain

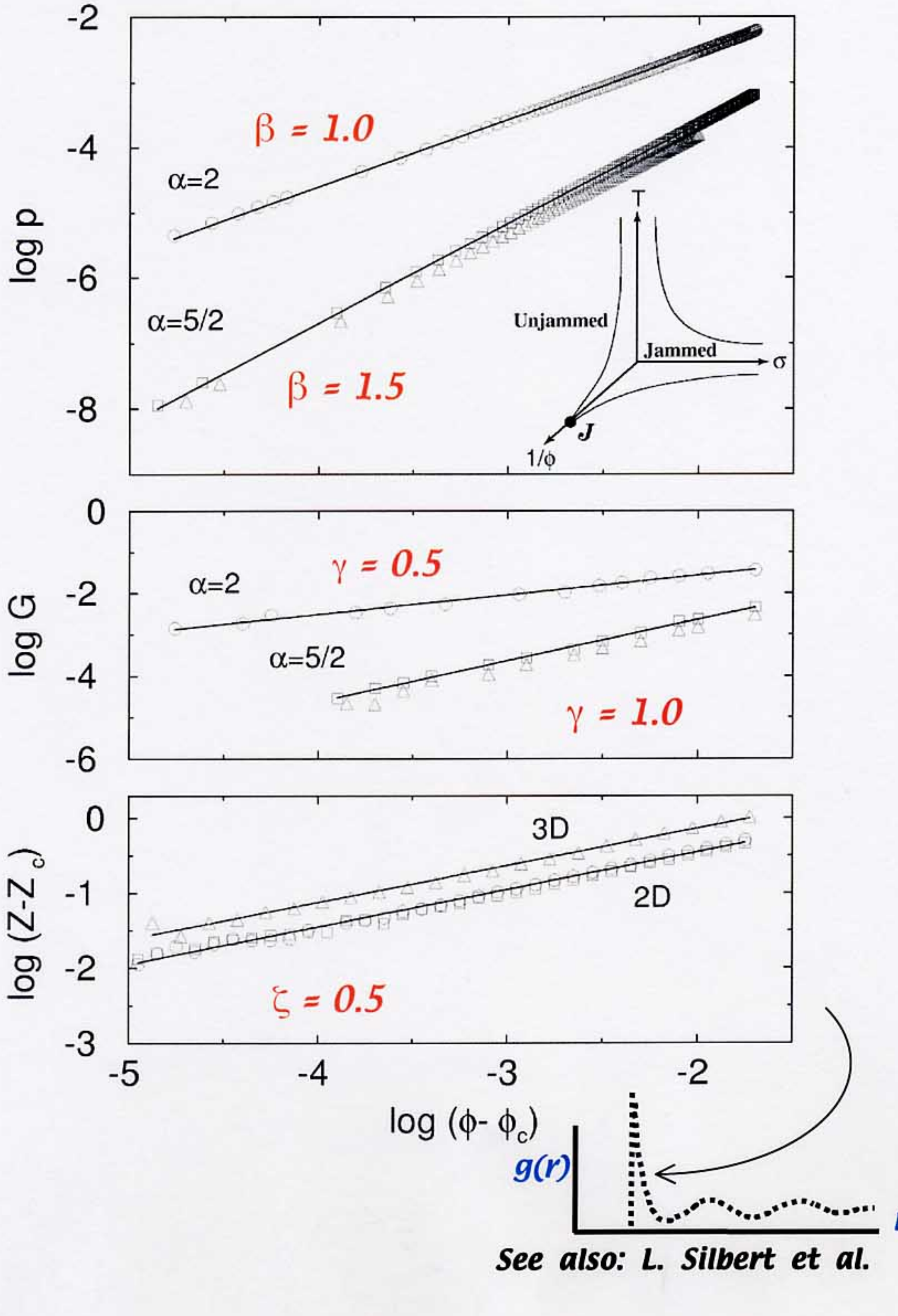


Shear modulus finite at z < 4

N = 1024
Soft-spheres (repulsive springs)
Frictionless
Isotropic

Scaling near J

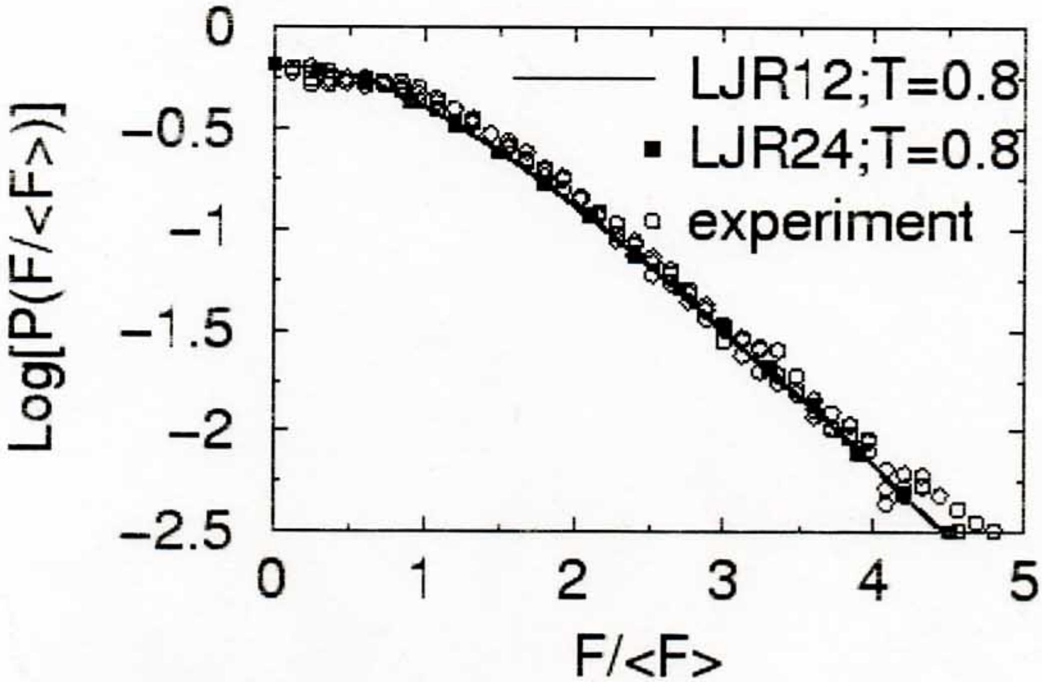
$V(r) = V_o(r_o - r)^{\alpha}$; $\alpha = 2, 5/2$; $D = 2, 3$



See also: L. Silbert et al.

Comparison of Simulations of Liquids with Experiments on Granular Material

LJR12: $V(r) \propto [(\sigma/r)^{12} - (\sigma/r)^{6}]$
LJR24: $V(r) \propto [(\sigma/r)^{24} - (\sigma/r)^{6}]$



Something is Wrong!

Granular material, foam should interact with Hertzian contacts:

$$V(r) \propto (r_0 - r)^{5/2}$$

or Hooke (springs):

$$V(r) \propto (r_0 - r)^2.$$

This would predict:

$$P(f) \propto (1/f)^{1/3} \exp[-f^{5/3}]$$

or

$$P(f) \propto \exp[-f^2].$$

NOT an exponential!

Why do they behave like hard spheres?

Lack of Self Averaging at J

For soft, finite-range potentials:

$$V(r) = V_o(r_o - r)^{\alpha}.$$

At J, independent of V(r)

$$P(F/\langle F\rangle) \propto e^{-F/\langle F\rangle}.$$

(Equilibrium: $P(F/\langle F\rangle) \propto e^{-V(r)/kT}$

$\Rightarrow$ Gaussian (not exponential) for $\alpha = 2$.)

Why?

At J, <F> varies enormously between configurations:

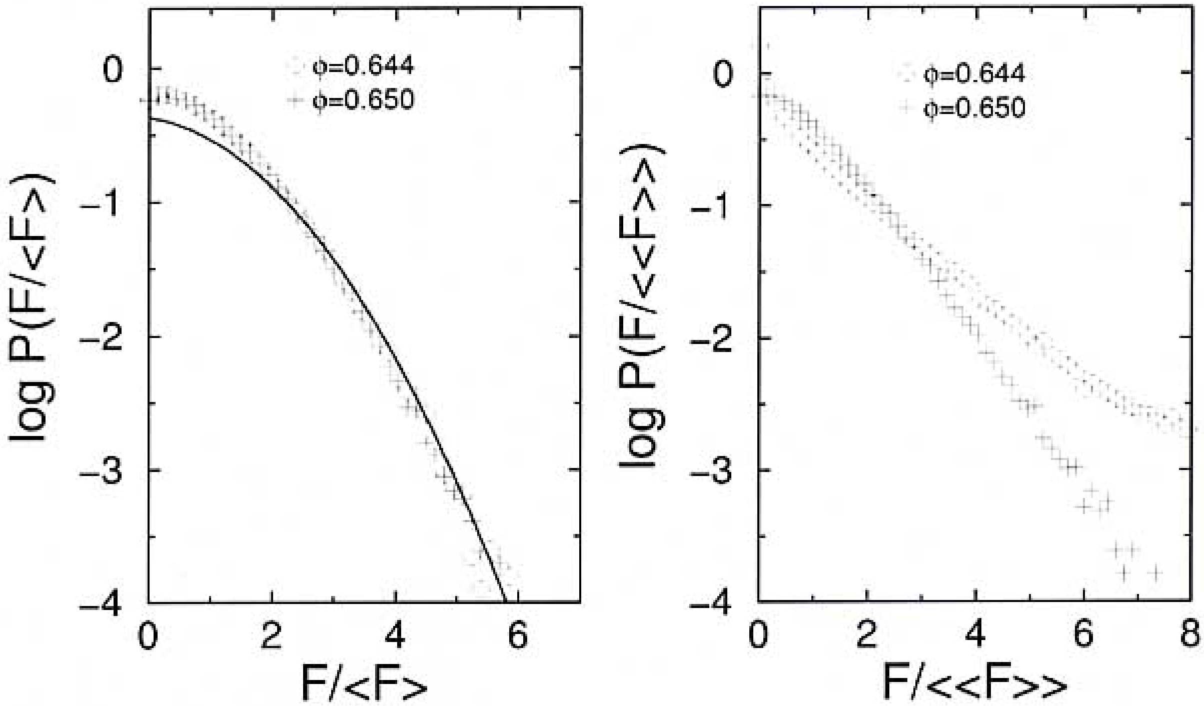
some configurations unjammed, and all sub-regions are unjammed (i.e., P = 0 everywhere);

some configurations jammed, and sub-regions are mostly jammed.

Compare averaging at J

<F> average for single configuration
<<F>> average over all configurations

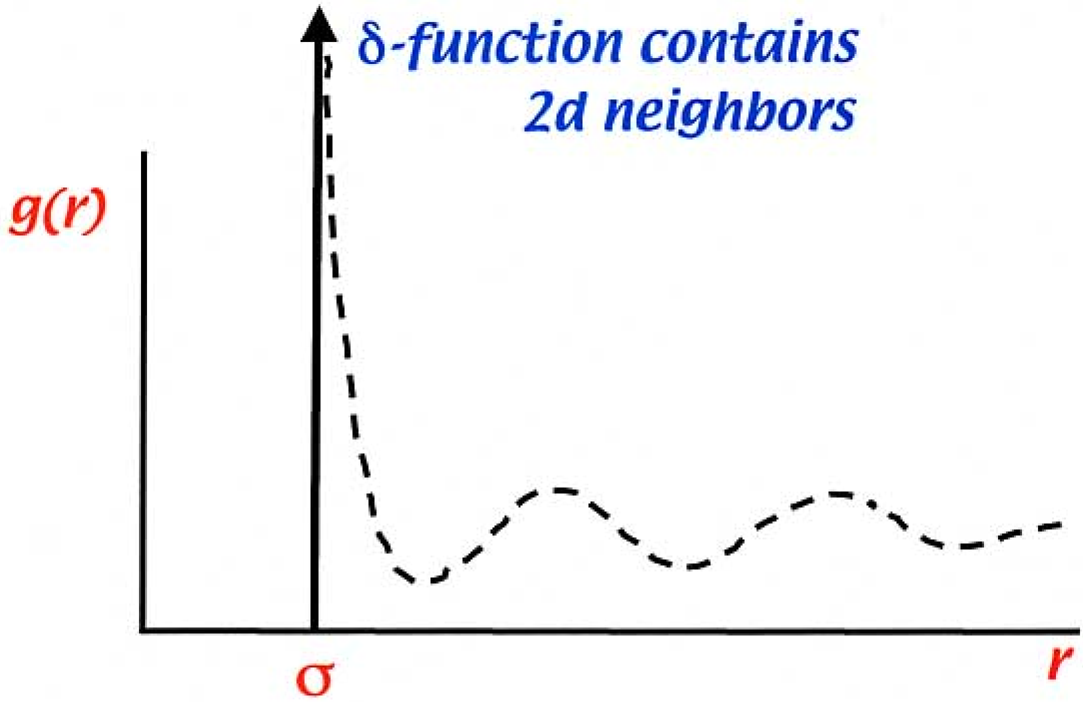
$$P(F/\langle F\rangle) \neq P(F/\langle\langle F\rangle\rangle)$$



$\alpha = 2; \; D = 3$

Divergence in $g(r)$ at $r = \sigma$

At J ($T = 0$, $\sigma = 0$, $\phi = \phi_c$)



$g(r)$ finite at $T > 0$ and $\phi \neq \phi_c$.

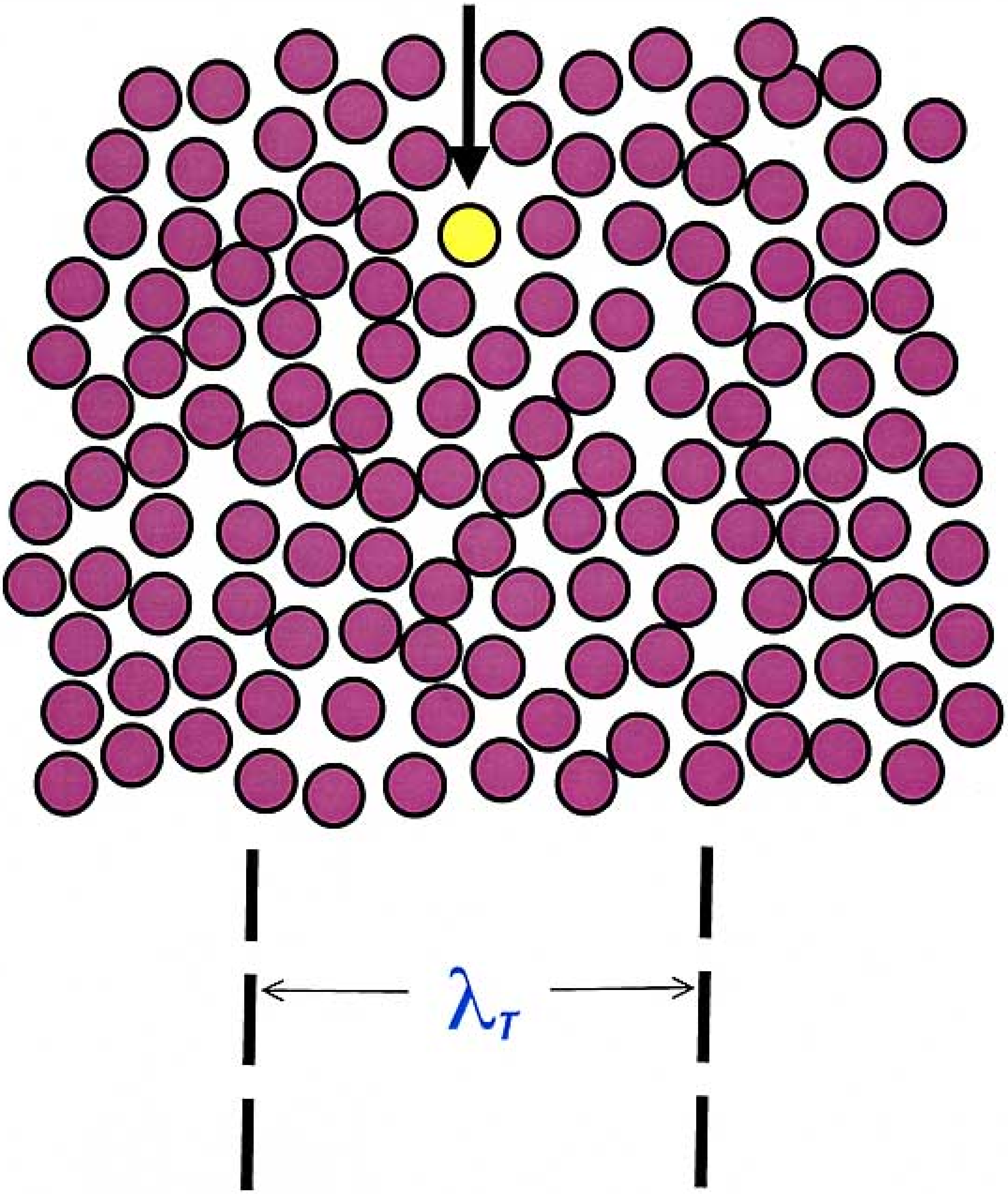
⇒ $S(k)$: no divergence (oscillations).

⇒ Peak in $P(f)$ and phase diagram.

Represents length scale going to zero.

Correlation length diverging?

Transverse correlation length



λ_T *measures rearrangement in transverse direction.*

Diverges at J ($\phi = \phi_c$)?

Special Properties of Point J

Jamming:
Develops bulk and shear modulus at same ϕ_c.

ϕ_c well-defined (unique) as $N \rightarrow \infty$

Critical Behavior

Scaling of P, G, $(Z - Z_c)$ versus $(\phi - \phi_c)$

$P(f)$ has robust exponential tail
(independent of $V(r)$)
No self-averaging

Divergence in $g(r)$ at $r = \sigma$

Diverging (transverse) length scale?

Unusual for Critical Behavior

Scaling exponents depend on $V(r)$ but not on dimension.

No fluctuations below ϕ_c (in P or G)

Jump in Z from $Z = 0$ to Z_c at $\phi = \phi_c$

No signature in $S(k)$

So where are we?

Jamming "Phase Diagram"

Useful - Common concepts in jamming and glass transitions.

Force Distributions

Exact liquid model.

Peak in $P(f) \Rightarrow$ glass transition.

Random Close-Packing

Meaning of random.

Finite-size effects $\Rightarrow$ long length scale.

Point J

Special kind of critical point.

* *

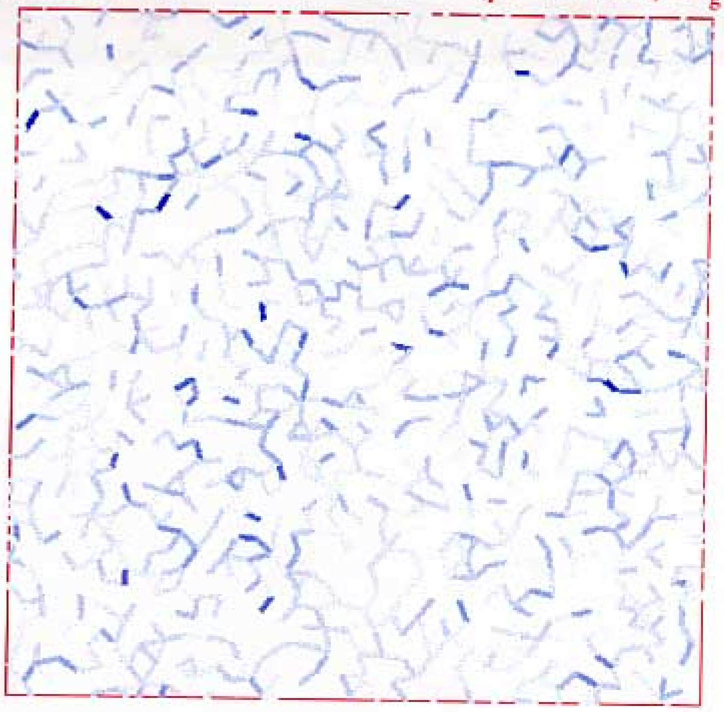
Does J "control" jamming phase diagram? Hints it does: divergent $g(r)$, $P(f)$ peak, oscillating $S(k)$.

Role of force networks?

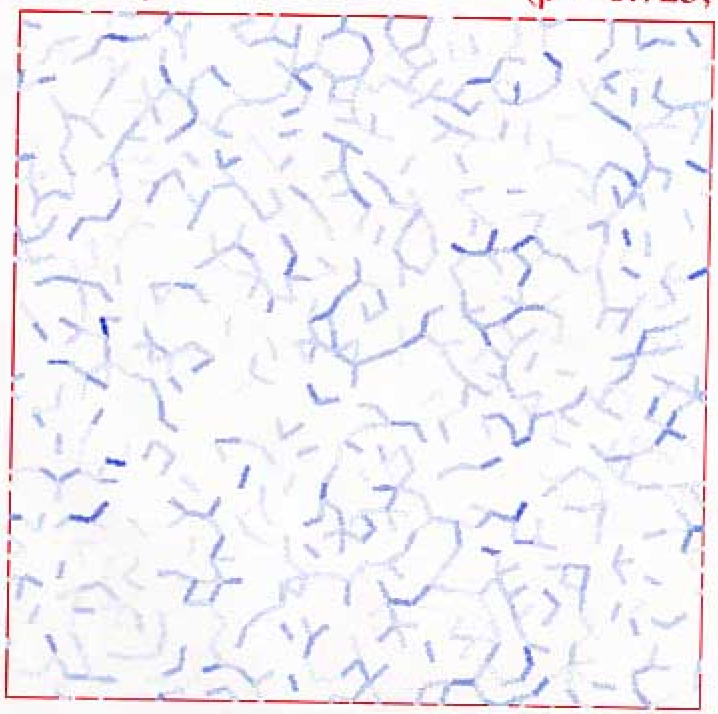
Kinetic heterogeneities in liquids

Order parameter for glass transition?

Liquid (Unjammed) **T = 0.8** ($\rho = 0.725$, $T_g = 0.33$)



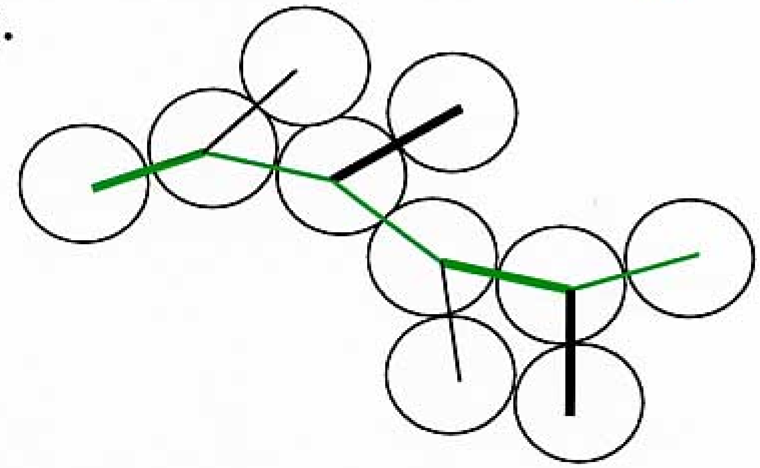
Glass (Jammed) **T = 0.1** ($\rho = 0.725$, $T_g = 0.33$)



Force Chains

Algorithm to define force chain:

Choose particle, k.
Find neighbor, l, with largest force, $\boldsymbol{F}_{k,l}$.
Find neighbor, m, to l with largest $\boldsymbol{F}_{l,m} \cdot \boldsymbol{F}_{k,l}$.
Keep going.



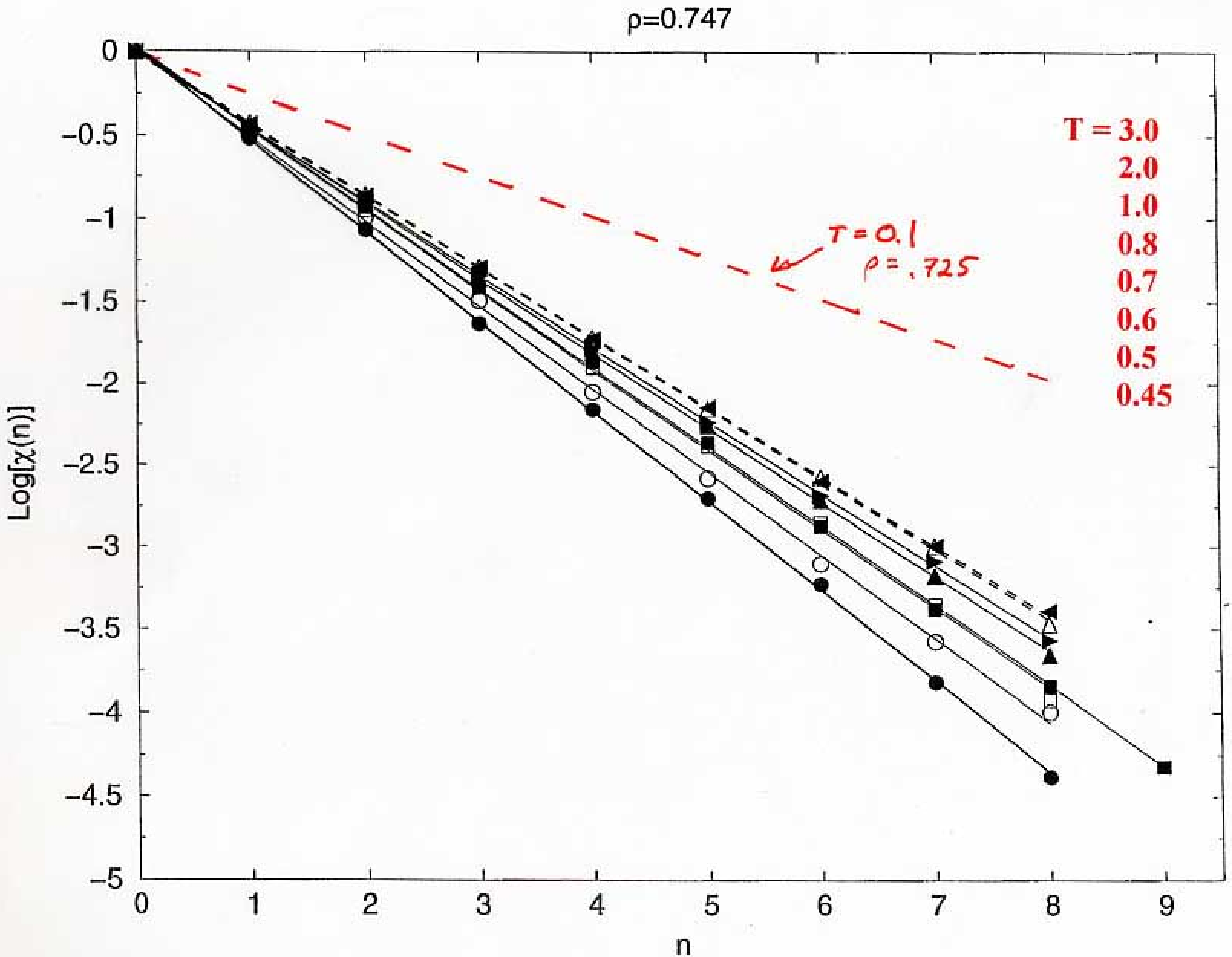
Algorithm for strength of chain:

$$\tilde{\chi}_i \equiv \left[F_{i,i+1}\right]^2 \cos(\theta_{i,i-1} - \theta_{i,i+1})$$

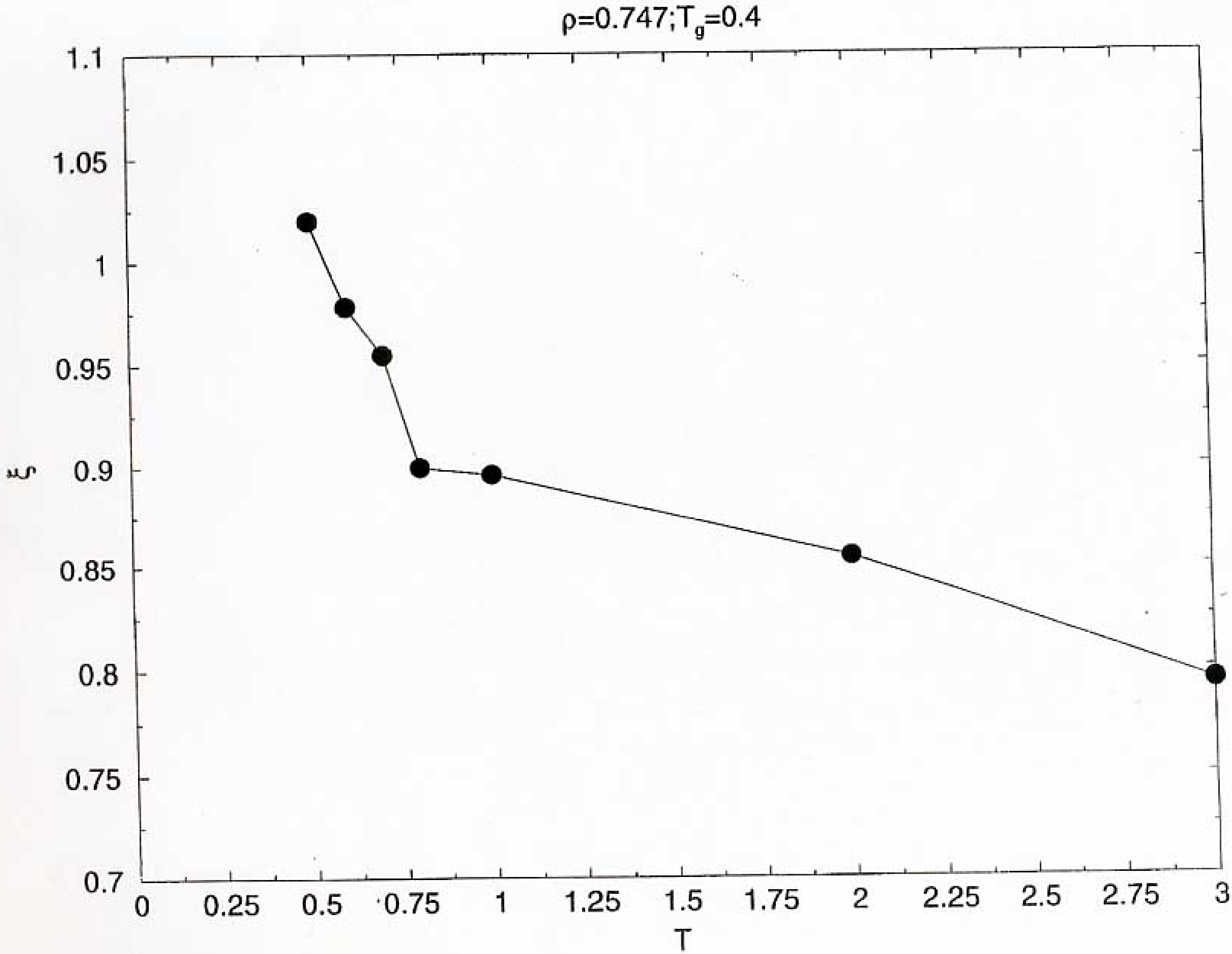
$$\chi(n) \equiv \left\langle \prod_{i=0}^{n} \left[\frac{\tilde{\chi}_i}{\left\langle \tilde{\chi}_0 \right\rangle} \right] \right\rangle$$

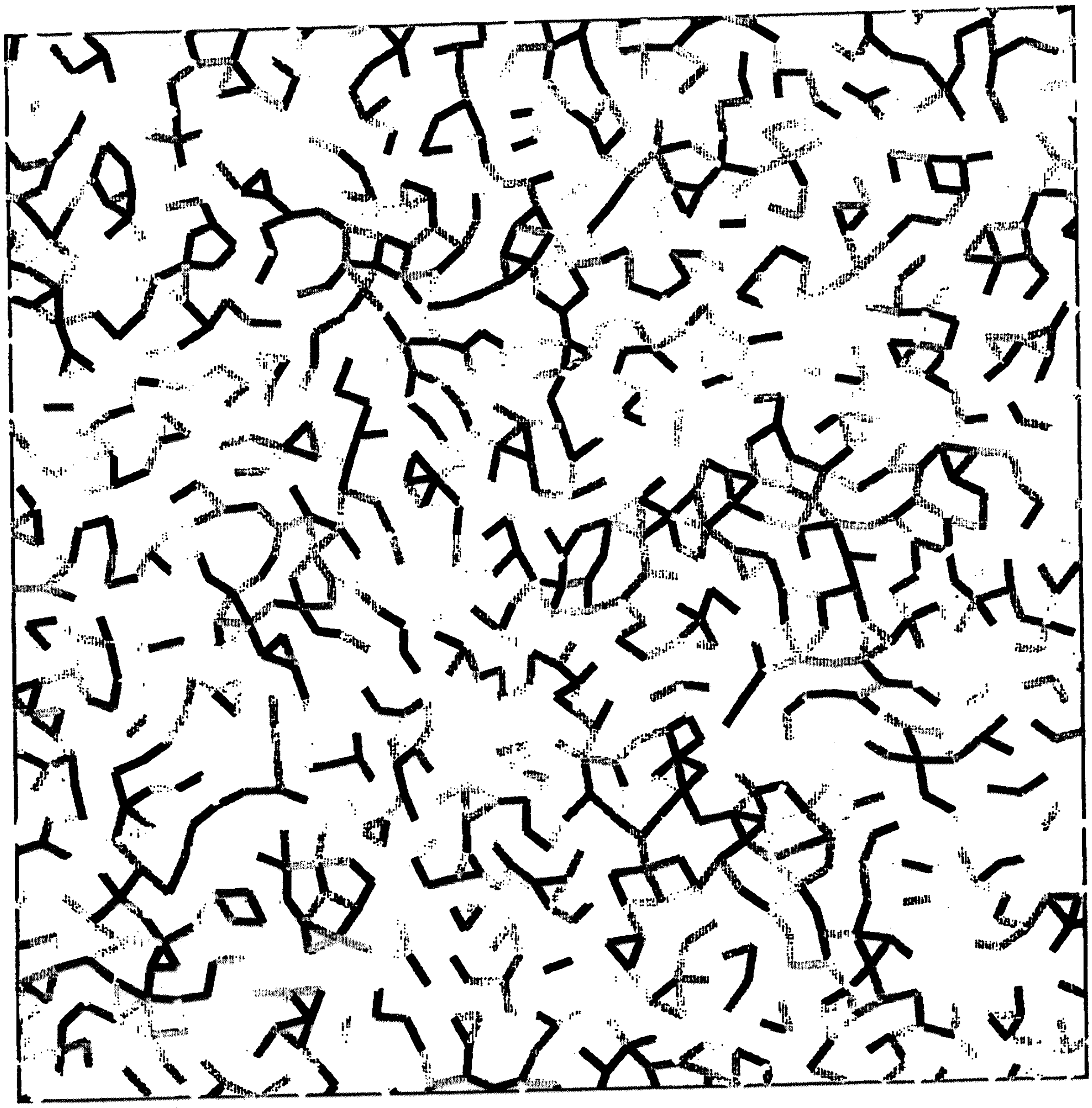
<...> average all particles and configurations.

Equilibrium Force-Chain Correlation Function

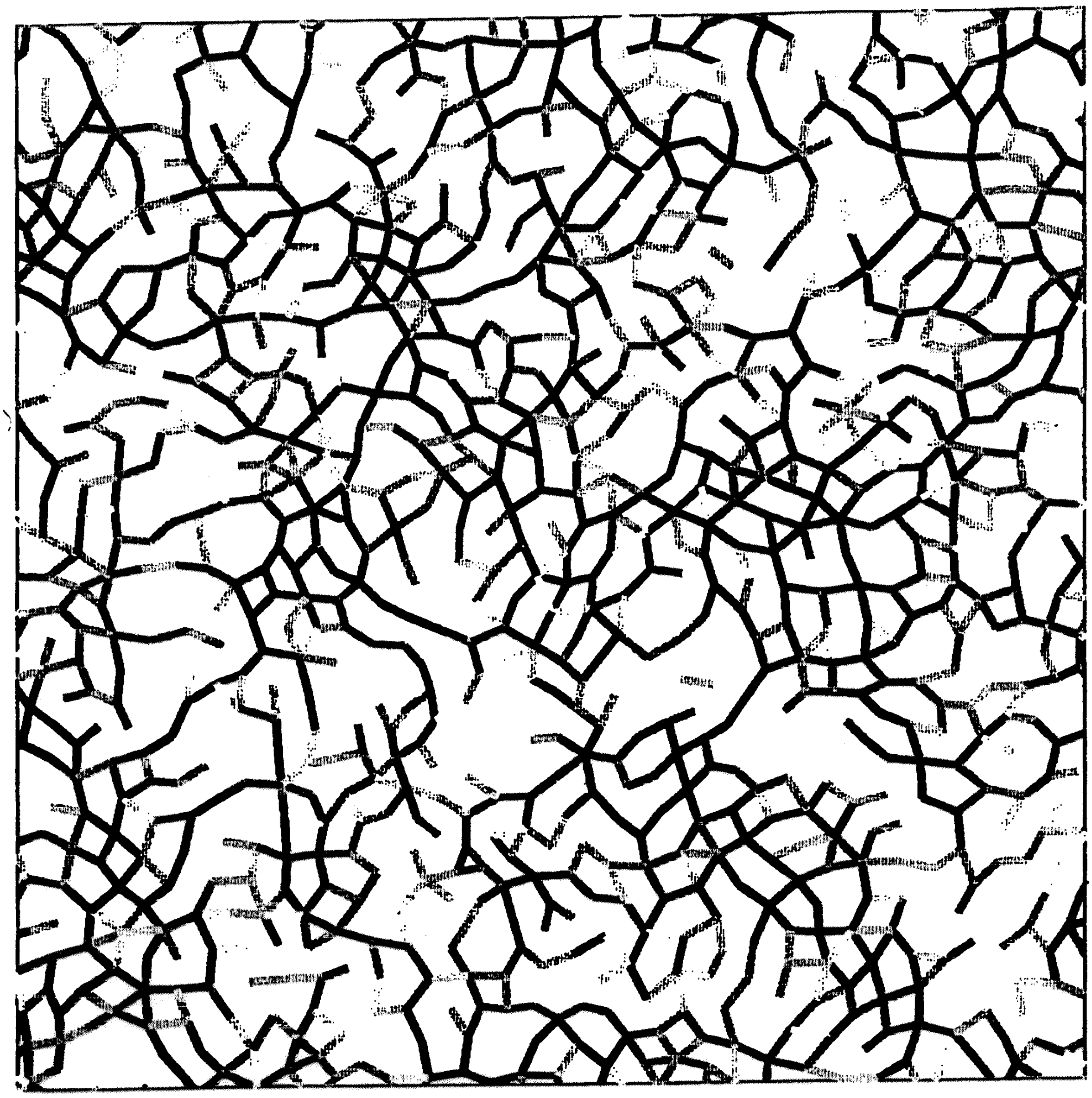


Equilibrium Force-Chain Correlation Length

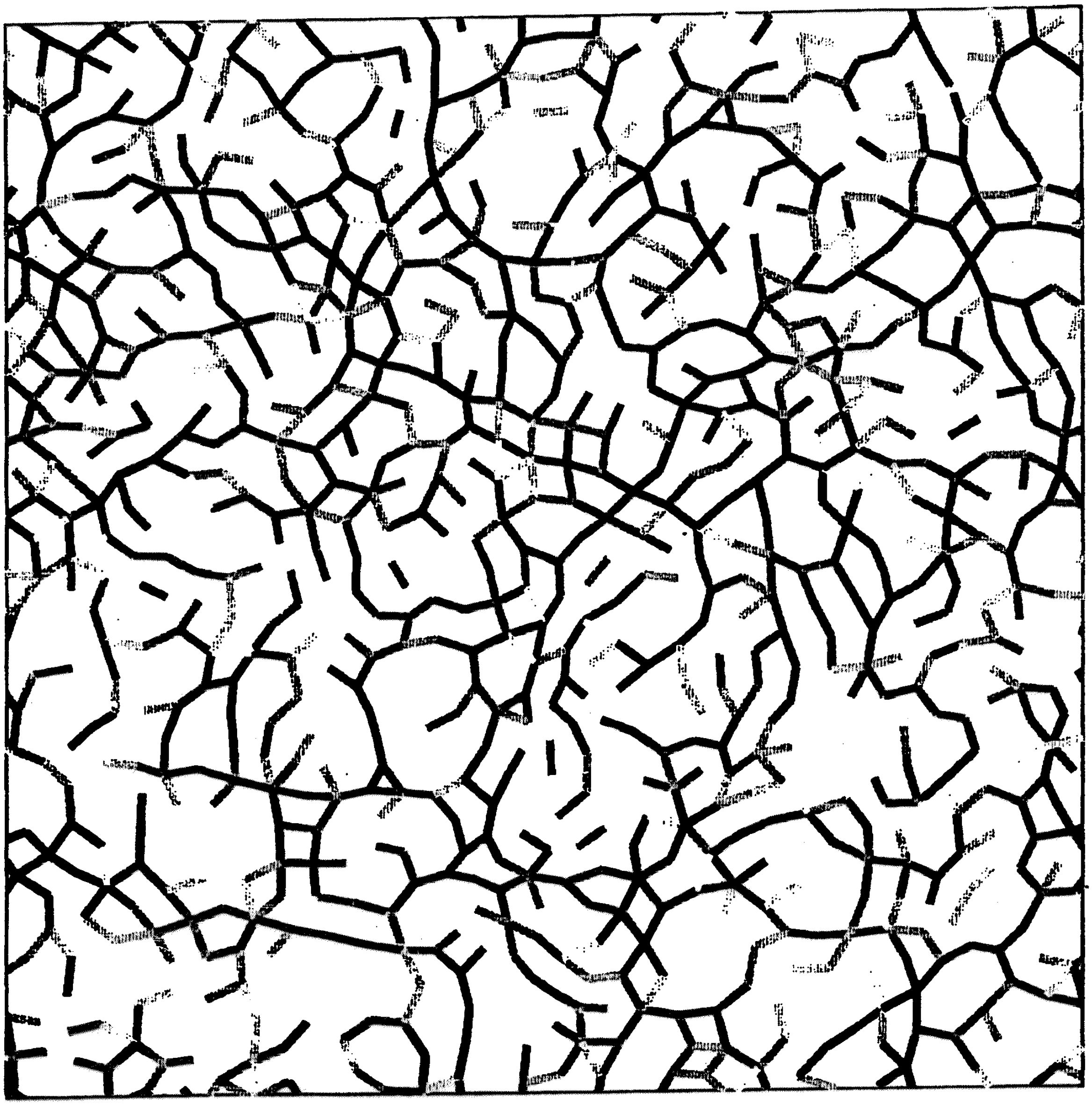




rho 0.725 T0.1 t1 f13.ps

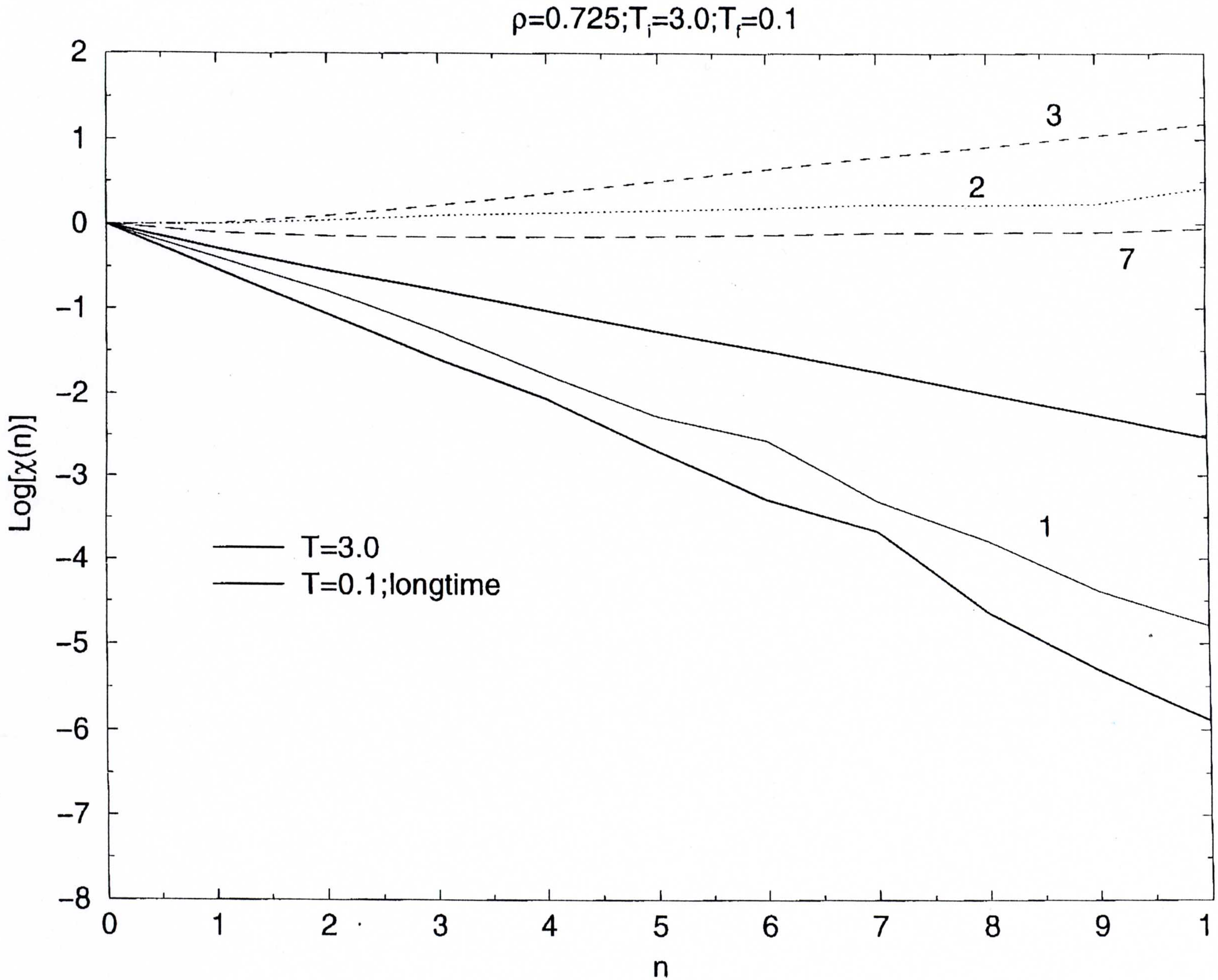


rho 0.725 T0.1 t 3 f13.ps

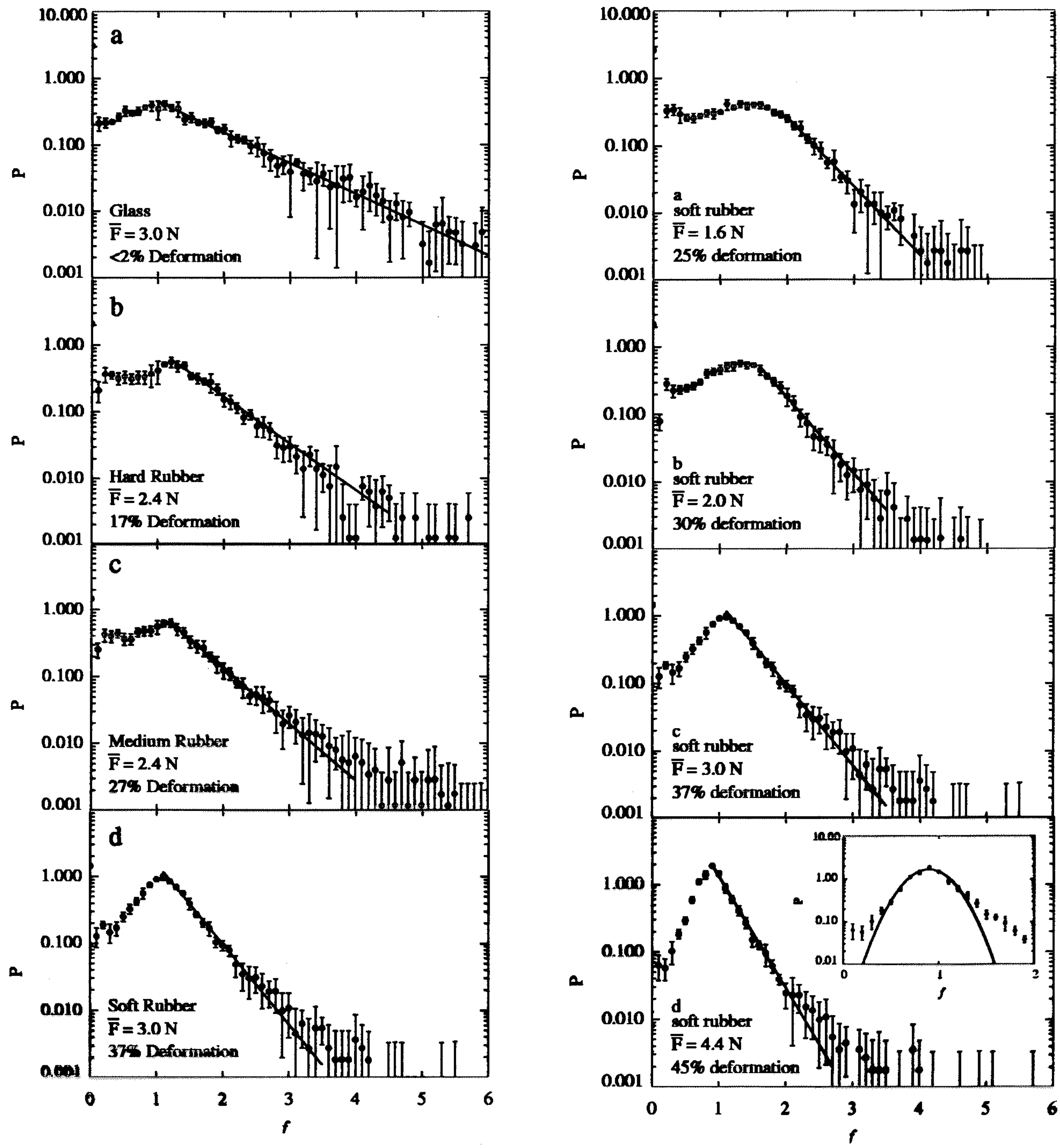


rho 0.725 T0.1 t7 f13.ps

Non-Equilibrium Force-Chain Correlation Function



Compressible Particles



J. Michael Erikson, Nathan Mueggenburg,
Heinrich Jaeger, and Sidney Nagel

Large Force Fluctuations in a Flowing Granular Medium

Emily Longhi and Nalini Easwar
Department of Physics, Smith College, Northampton, Massachusetts 01063

Narayanan Menon

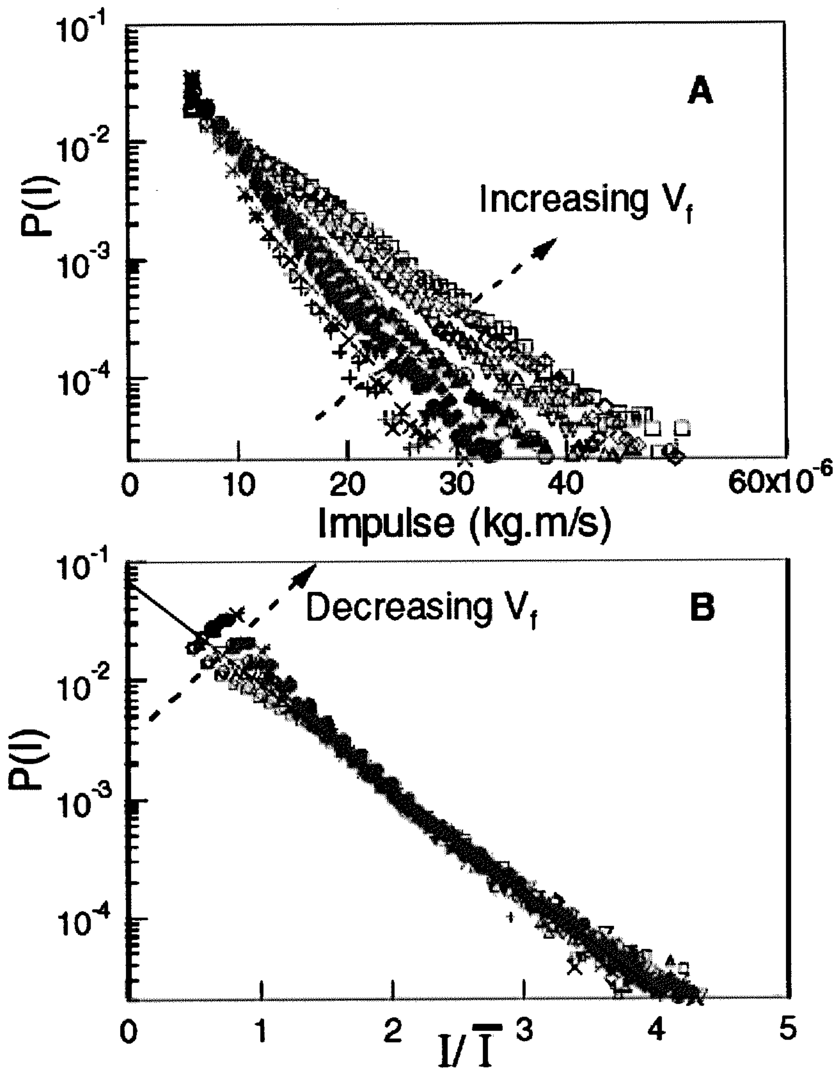


FIG. 3. (A) Impulse histograms on a log-linear scale for various V_f (all in cm/s): 9.4(X), 11.7(+), 14.7(•), 17.7(◆), 23.0(▼), 28.4(▲), 33.1(∘), 37.6(▽), 44.5(△), 50.1(◇), and 60.0(□). The flow velocities correspond to opening sizes ranging from $a = 3d$ to $16d$. The data at 9.4 and 23 cm/s correspond to the transducer located higher up in the flow. (B) Impulse histograms of (A) scaled to the average impulse $\bar{I}$ for each flow velocity.

Memories in sand: Experimental tests of construction history on stress distributions under sandpiles

Loic Vanel,[1] Daniel Howell,[2] D. Clark,[2] R. P. Behringer,[2] and Eric Clément[1]

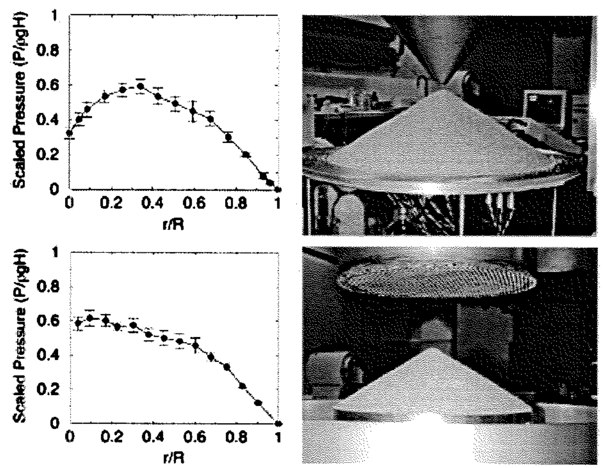


FIG. 2. Dimensionless normal stress profiles, $P/\rho gH$, vs dimensionless radial distance r/R, beneath conical piles of granular materials of height H and radius R. The construction techniques are illustrated by the accompanying photographs (see text).

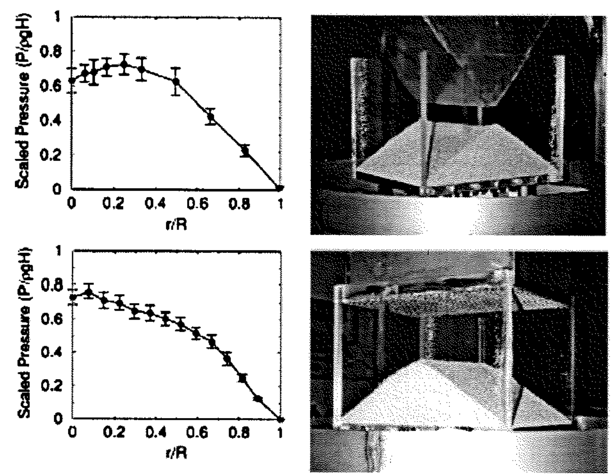


FIG. 3. Dimensionless normal stress profiles, $P/\rho gH$, vs transverse distance r/R, beneath wedge-shaped piles of granular material of height H and width $2R$. The piles are made by different construction techniques illustrated by the accompanying photographs (see text).

Green's Function Probe of a Static Granular Piling

Guillaume Reydellet and Eric Clément

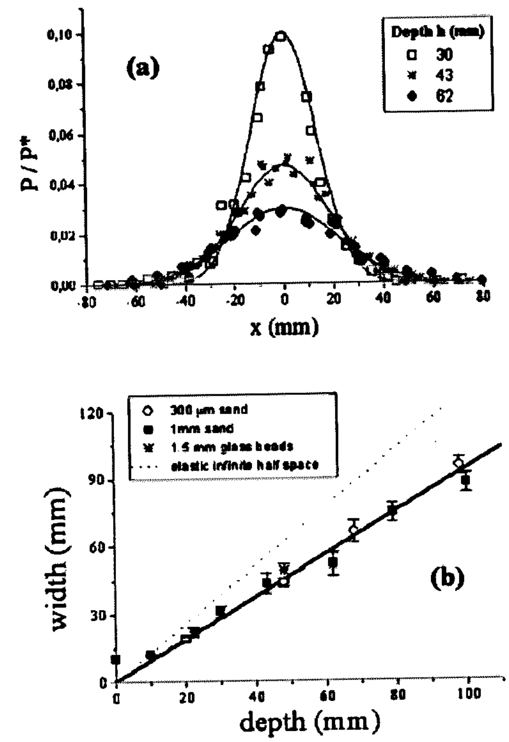


FIG. 2. Horizontal stress distribution in response to a localized solicitation (Green's function). (a) Green's function $P(x) = \sigma_{zz}(x)/P^*$ measured at three different depths for $d = 1$ mm sand. See text for definition of the rescaling factor P^*. (b) Half amplitude width W of the response function as a function of depth h for three different granular materials (see legend). The straight line is the best linear fit: $W = 0.94h$.

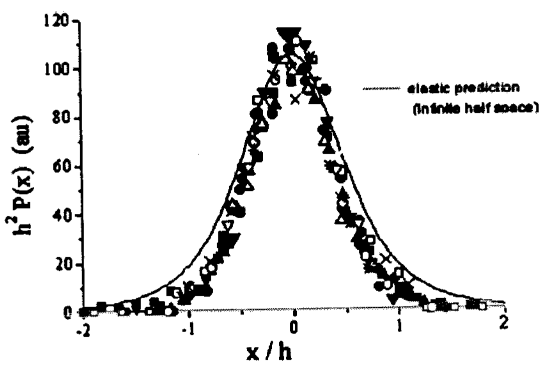
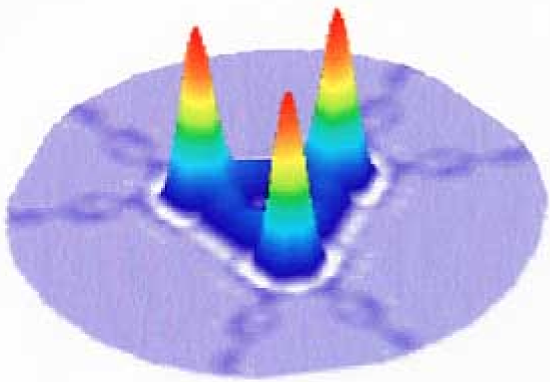
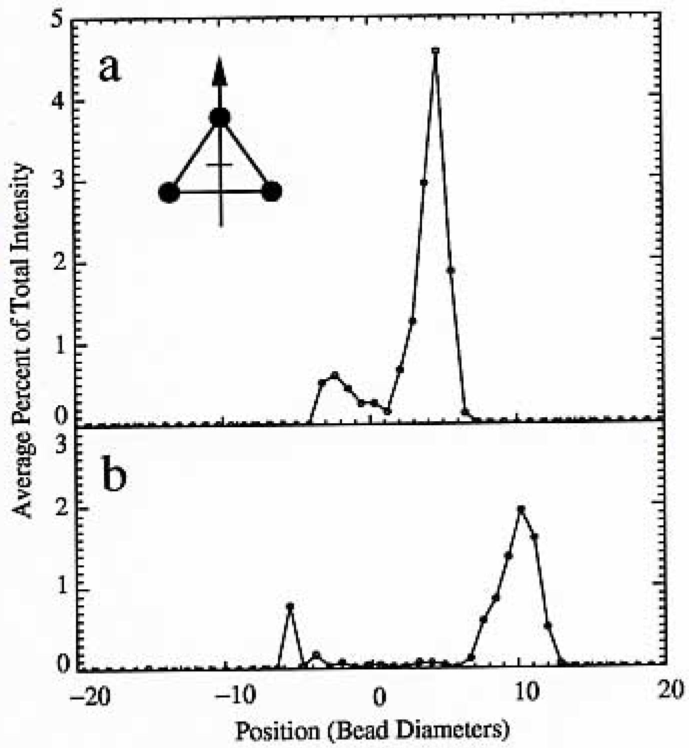
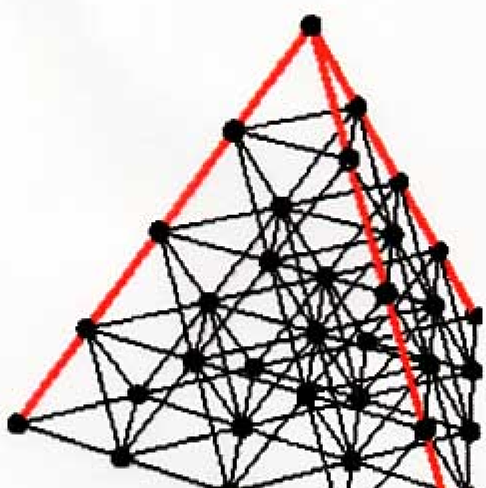


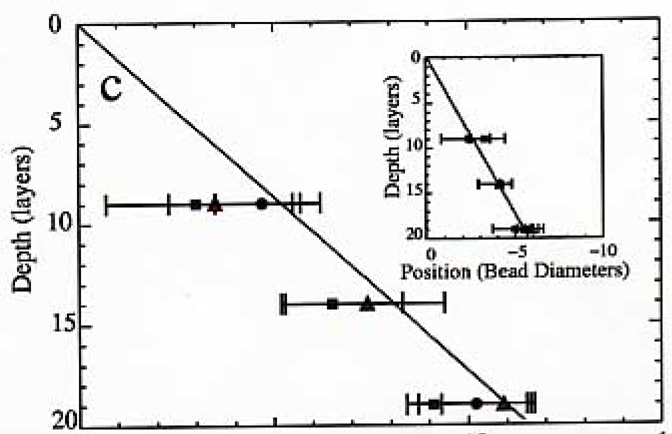
FIG. 3. Rescaled Green's function $h^2P(x)$ as a function of the rescaled horizontal axis x/h, for different depths and different types of granular materials. Aquarium sand: $d = 300$ μm [$h = 30$ mm (■), 62 mm (●), 79 mm (▲), 100 mm (▼)]; Fontainebleau sand: $d = 1$ mm [$h = 19$ mm (□), 48 mm (○), 68 mm (△), 97 mm (▽)]; glass beads: $d = 1$ mm [$h = 28$ mm (*), 48 mm (×)]. The straight line is the theoretical response of an elastic infinite half-space.

FCC Crystal



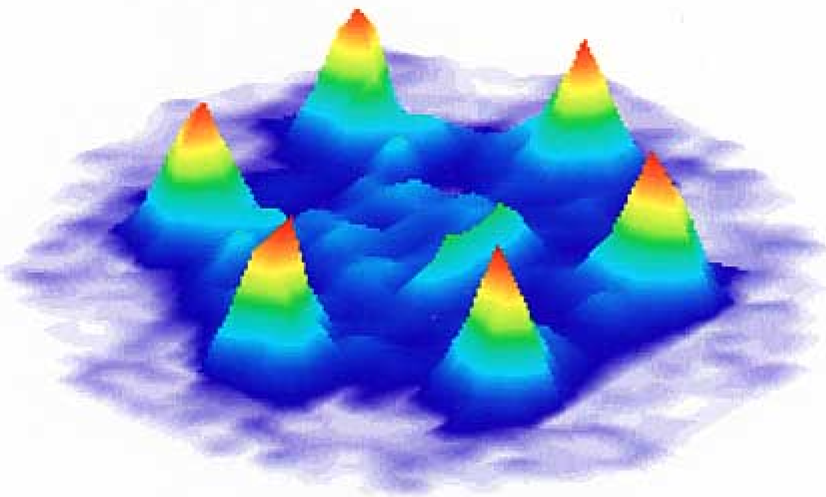






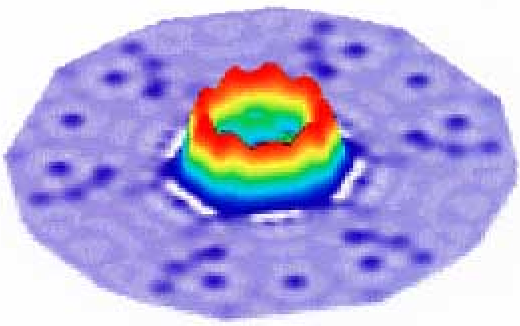
Nathan Mueggenburg, Heinrich Jaeger, and Sidney Nagel

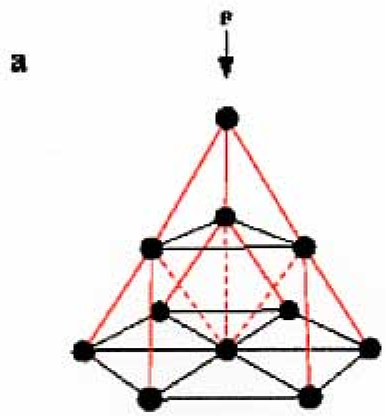
FCC with Stacking Fault

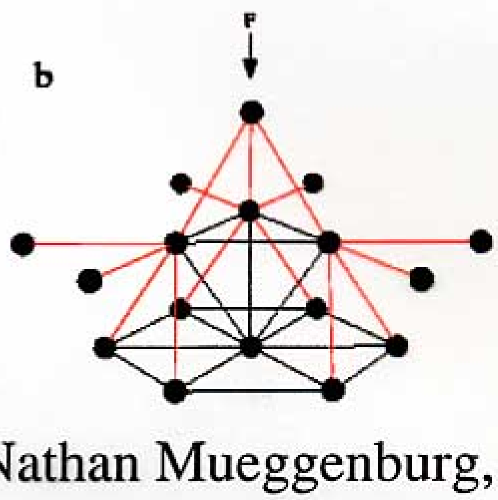


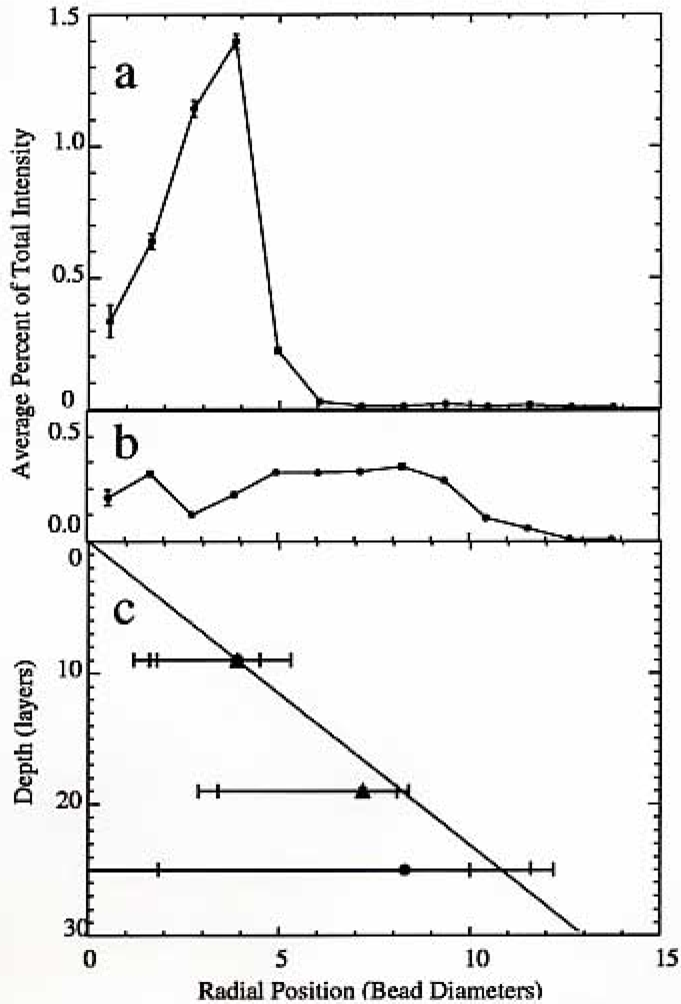
Melissa Spannuth, Nathan Mueggenburg
Heinrich Jaeger, and Sidney Nagel

HCP Crystal









Nathan Mueggenburg, Heinrich Jaeger and Sidney Nagel

Compaction

Is there a well-defined density?

Hard spheres:

Loose packing: $\rho_{LP} \approx 0.55$
Close packing: $\rho_{CP} \approx 0.64$
(Crystal: $\rho_{Xtal} \approx 0.74$)
Friction is important.

Is there an equilibrium?
What is the temperature?
If not, what is the ensemble?

How is steady state approached?

Role of Density Fluctuations

Fluctuations: A new probe of dynamics

Can driven fluctuations about steady state define "T" as do fluctuations in thermal systems?

Is there a fluctuation-dissipation theorem?

Is there a zeroeth law?

Use existence of reversible compaction line to define ensemble.

Physics of $\hbar = 0$ and $T = 0$

Compare thermal energy, $k_B T$, to rearrangement energy, mgd.

For $d = 1$ mm:

$$k_B T / mgd < 10^{-12}$$

$\Rightarrow T = 0!$

Why need a temperature?

Explore phase space

- allows averaging and escape from metastable states

Approach equilibrium

Fluctuations

Is there a fluctuation-dissipation theorem? (What is "T"?)

Provides velocity scale

No equilibrium - only steady state.

Is there a zeroeth law?

How do we describe material?

Can we average? (What ensemble?)

What properties depend on preparation?

Granular Temperature

Flow induces motion:

Random component as well as component in direction of flow.

e.g., Menon and Durian

Random part is like a "Granular Temperature"

But: Flow stops $\Rightarrow$ $T_{gran} \approx 0$.

How to describe a static powder?

Thermodynamics of Sand

Model of Edwards et al.

Replace E of ordinary thermodynamics with V.

Hamiltonian replaced by a function W(V).

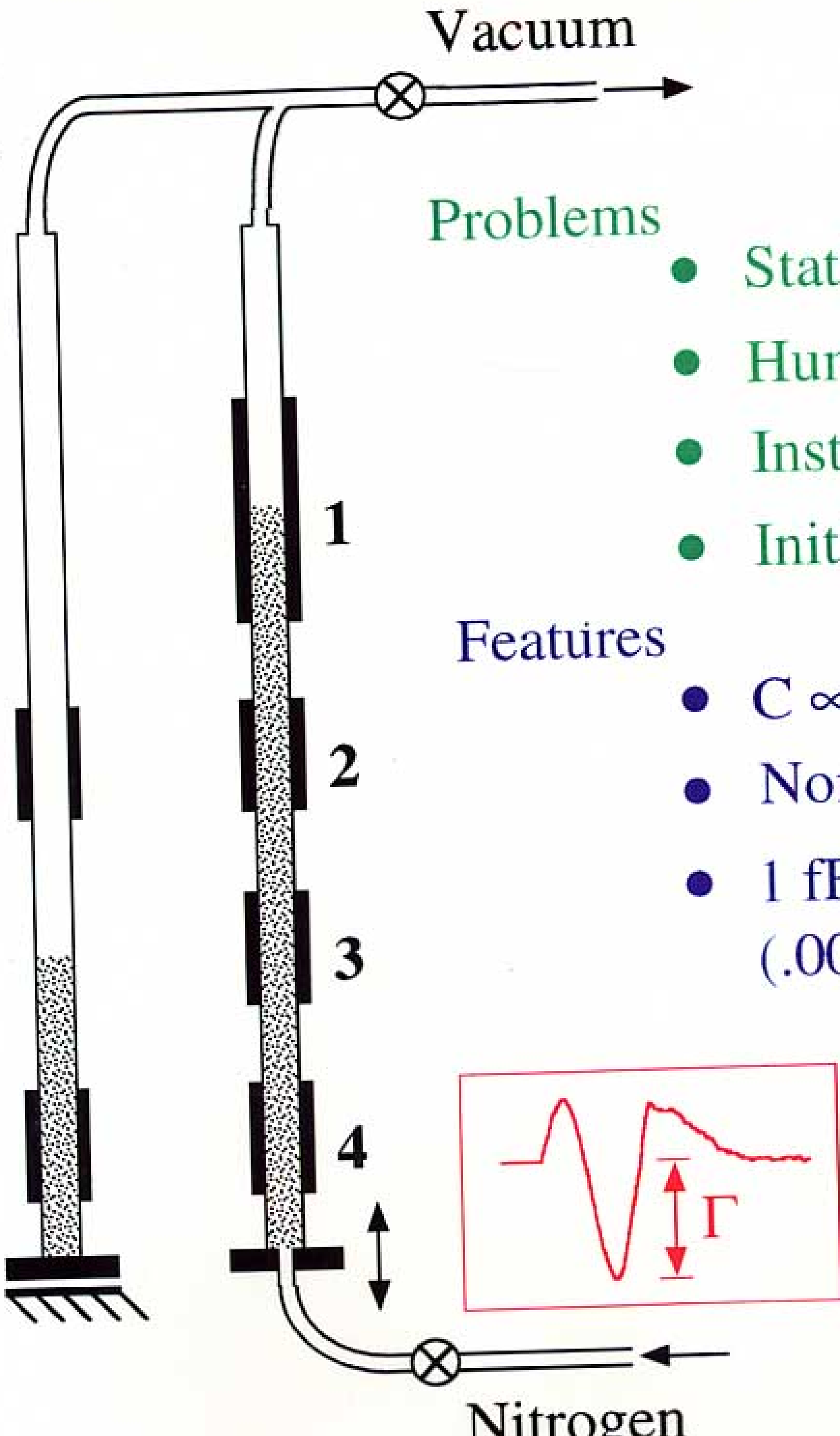
Entropy $\propto$ Log [number of configurations].

$\Rightarrow$ T replaced by "compactivity" χ.

Note: $\chi \neq T_{eff}$

How to measure χ?

What ensemble?
(Must include preparation of sample.)



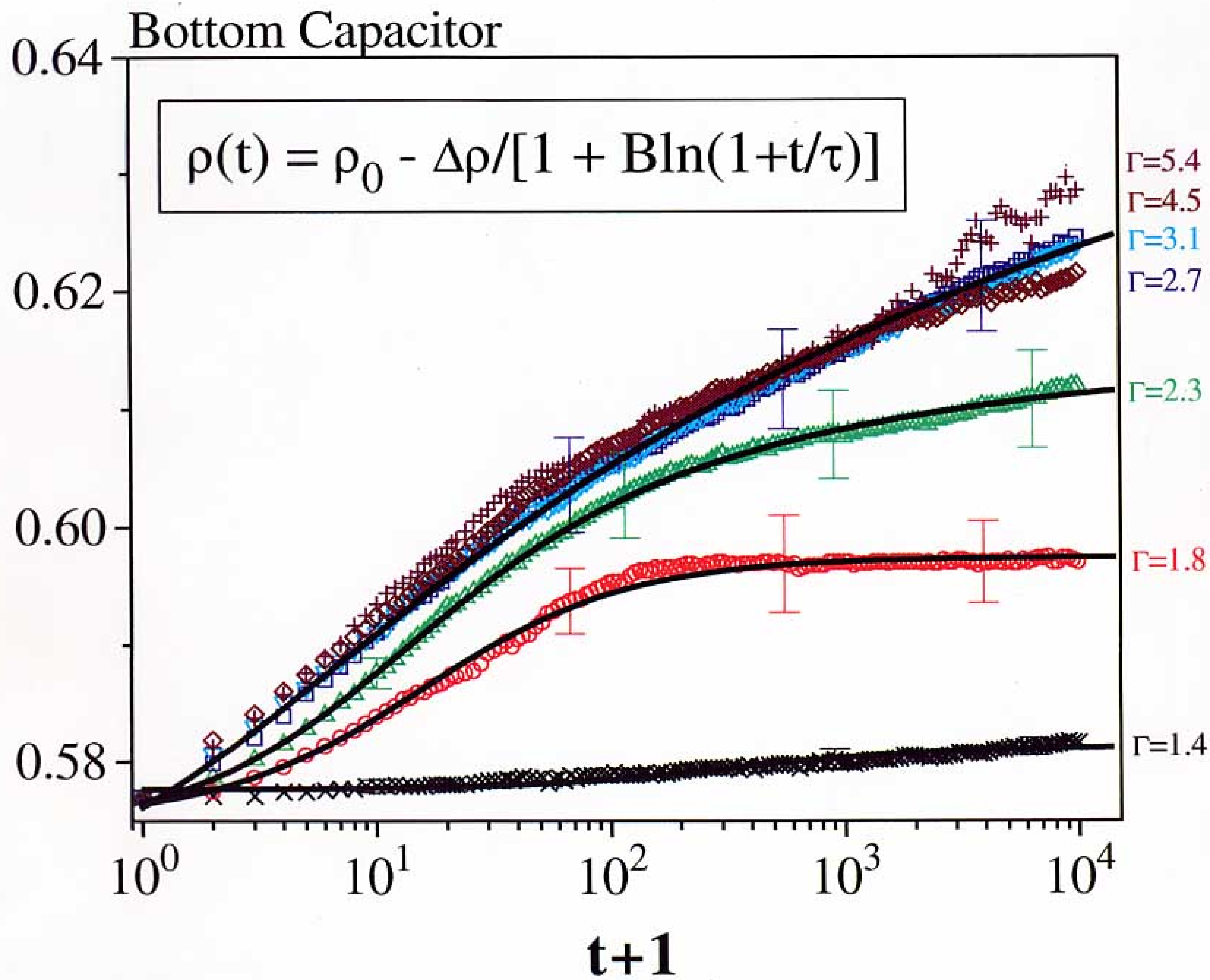
Problems

- Static Electricity
- Humidity
- Instrumental Drift
- Initial Conditions

Features

- $C \propto \rho$
- Non-invasive
- 1 fF resolution (.0006 in ρ)

$\Gamma = a/g$



Bottom Capacitor
ρ(t) = ρ0 - Δρ/[1 + Bln(1+t/τ)]
0.64
0.62
0.60
0.58
Γ=5.4
Γ=4.5
Γ=3.1
Γ=2.7
Γ=2.3
Γ=1.8
Γ=1.4
10^0
10^1
10^2
10^3
10^4
t+1

Comparison to Theory

Sum of 2 Exponentials

individual and cluster relaxation - two time constants
(Barker & Mehta, PRE **47**, 184 1993)

$$\rho(t) = \rho_f - \Delta\rho_{ind} e^{-t/\tau_{ind}} - \Delta\rho_{col} e^{-t/\tau_{col}}$$

Stretched Exponential

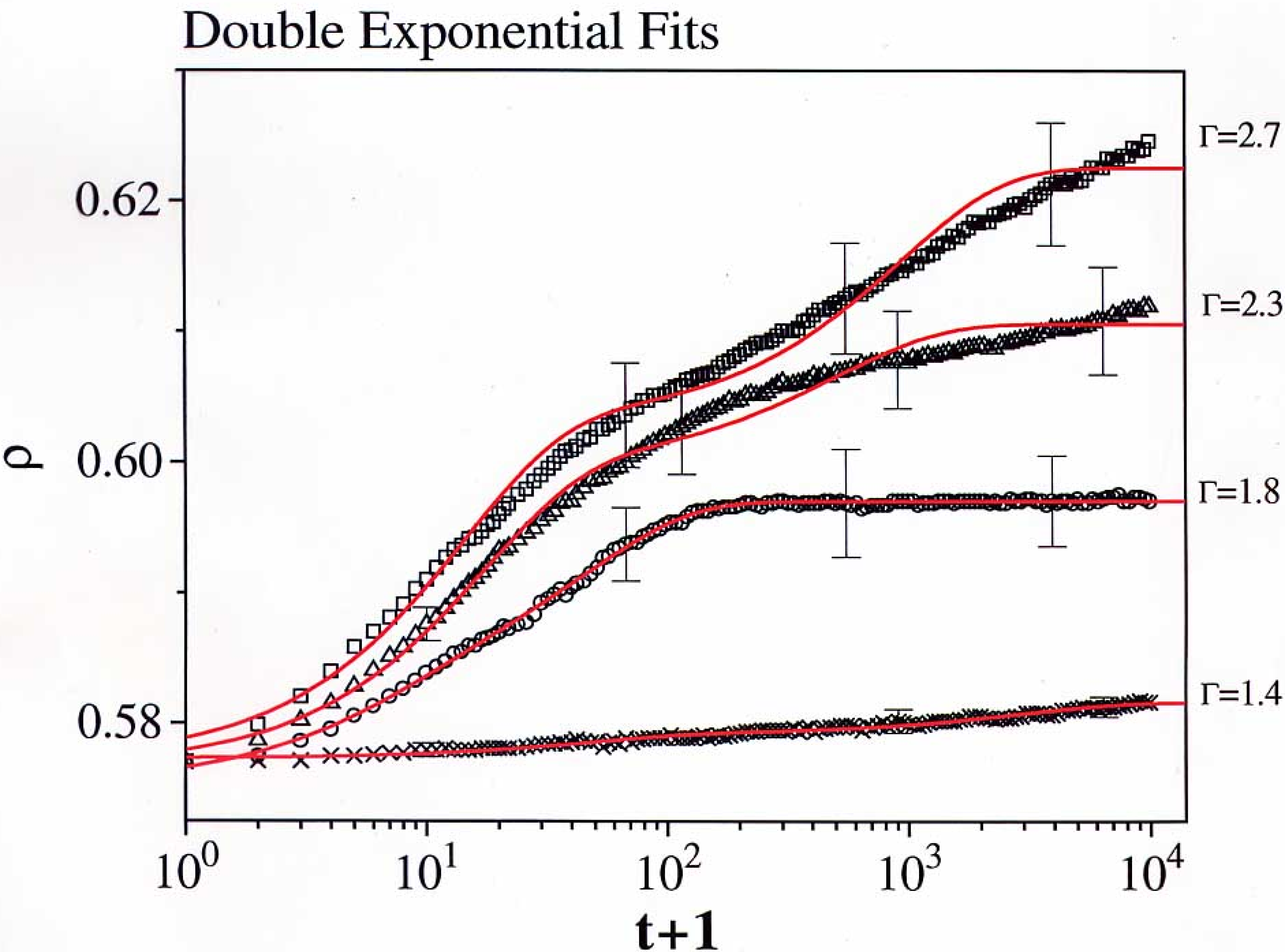
continuous range of time constants

$$\rho(t) = \rho_f - \Delta\rho_{\infty} e^{-\left(\frac{t+t_0}{\tau}\right)^{\beta}}$$

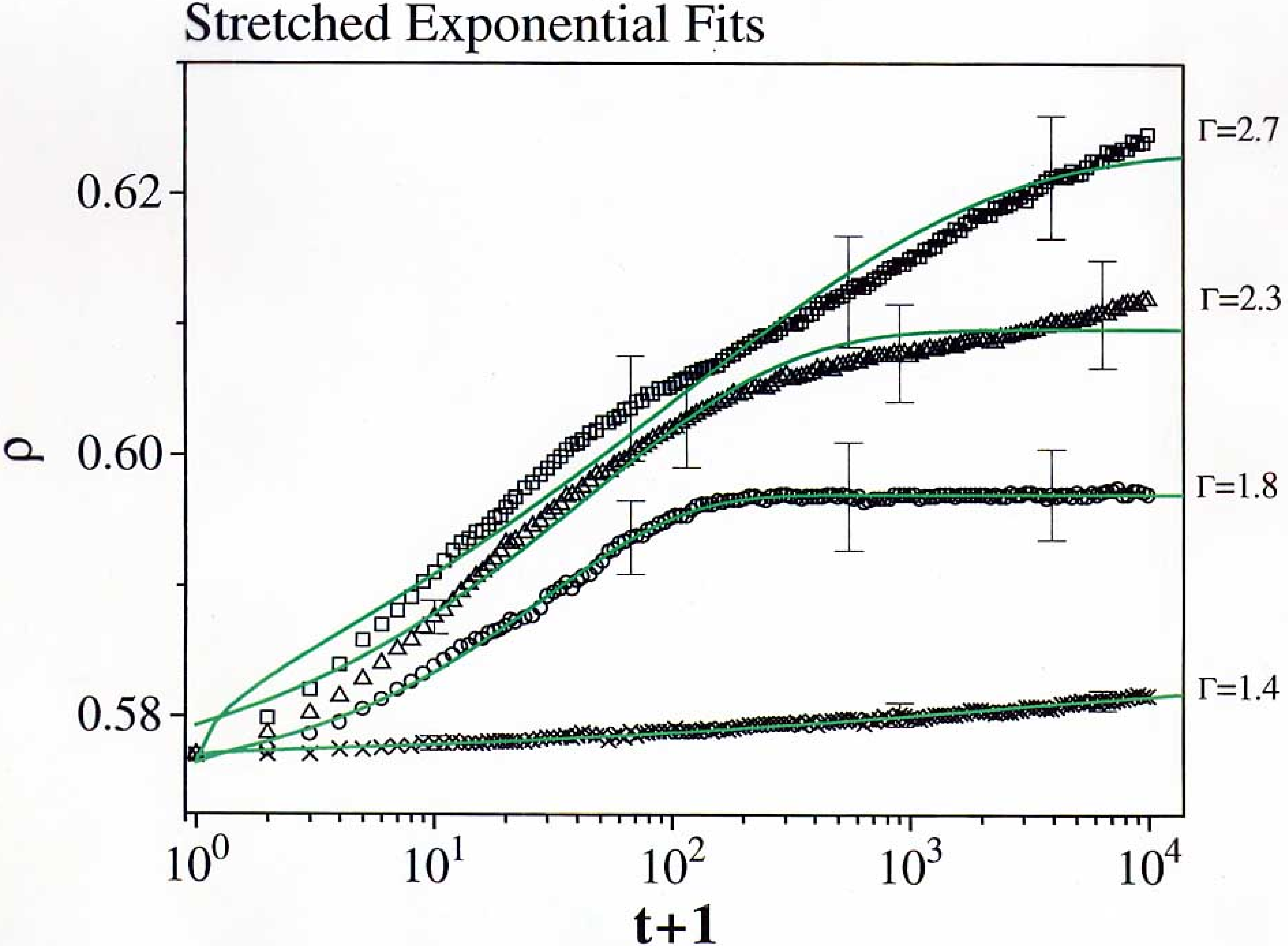
Power Law

diffusing voids
(Hong, Yue, Rudra, Choi, & Kim PRE **50**, 4123 1995)

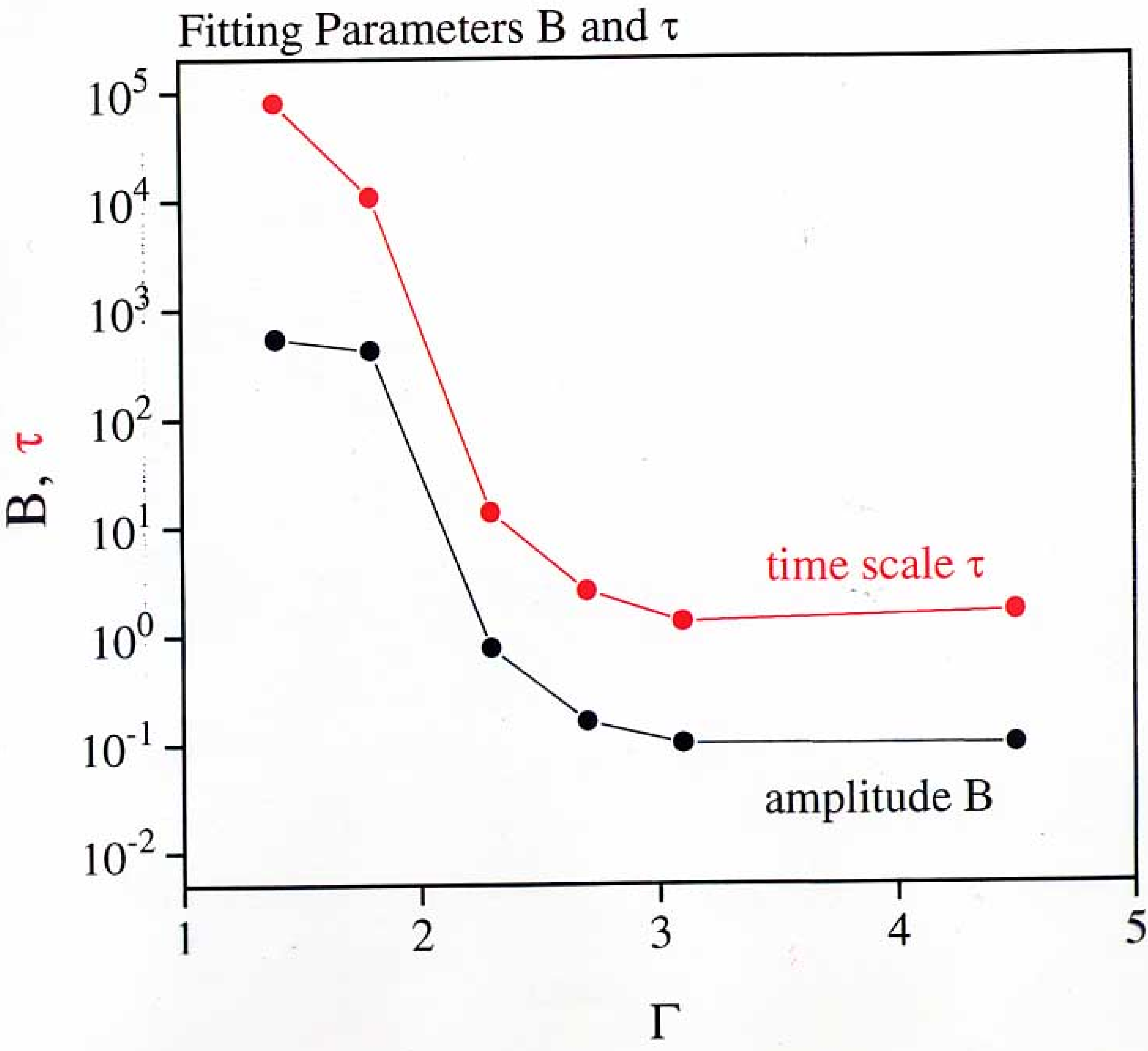
$$\Delta h \propto t^z \rightarrow \rho(t) = \frac{\rho_0}{1 - ct^z}$$

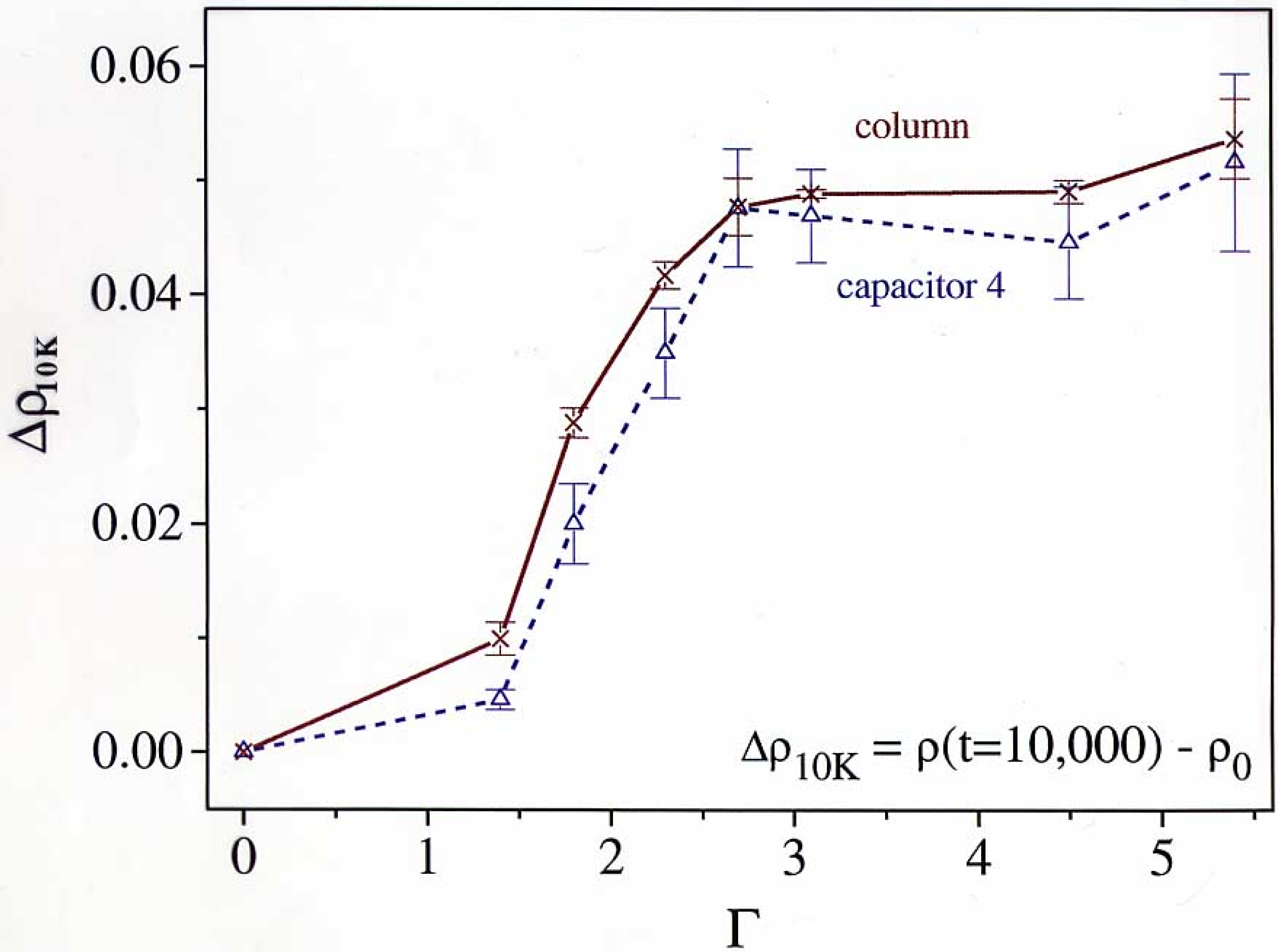


Double Exponential Fits
ρ
0.62
0.60
0.58
Γ=2.7
Γ=2.3
Γ=1.8
Γ=1.4
10^0
10^1
10^2
10^3
10^4
t+1



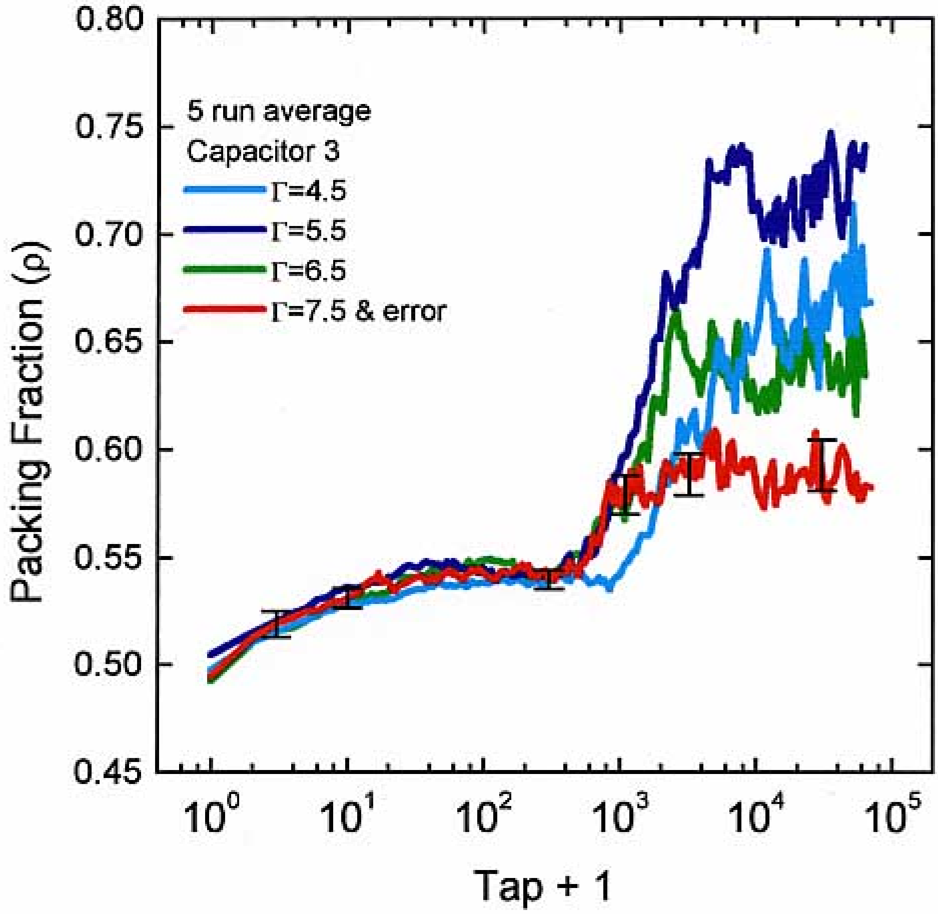
Stretched Exponential Fits
Γ=2.7
Γ=2.3
Γ=1.8
Γ=1.4
0.62
0.60
0.58
ρ
10^0
10^1
10^2
10^3
10^4
t+1


Fitting Parameters B and τ
B, τ
10^5
10^4
10^3
10^2
10^1
10^0
10^-1
10^-2
time scale τ
amplitude B
1
2
3
4
5
Γ

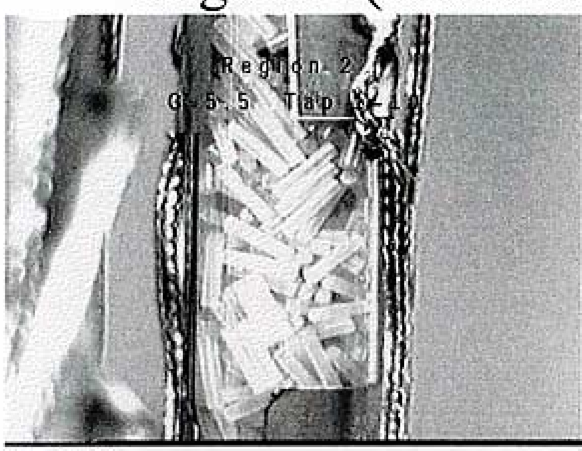


0.06
0.04
0.02
0.00
0
1
2
3
4
5
Γ
Δρ10K
column
capacitor 4
Δρ10K = ρ(t=10,000) - ρ0

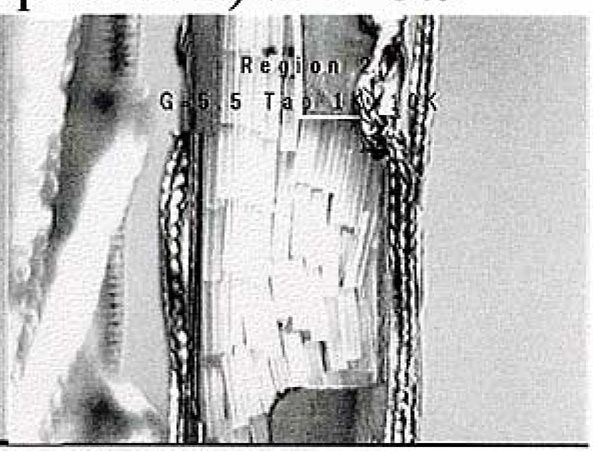
Compaction of a prolate sample



Region 2 (above capacitor 3) at Γ=5.5



Tap 0 - 10

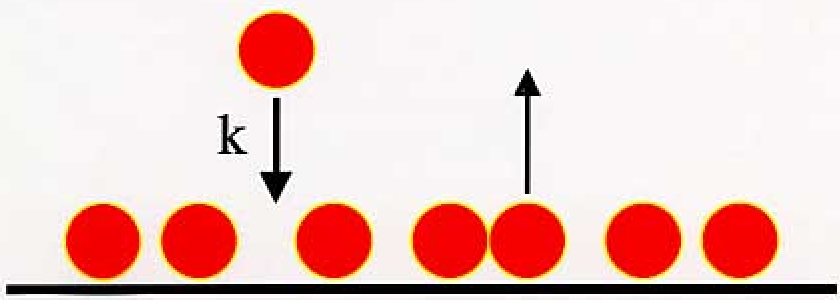


Tap 1000 - 10,000

Model for Compaction

Parking-lot model:

Cars enter and leave a parking lot without assigned spaces. Can another car fit in?

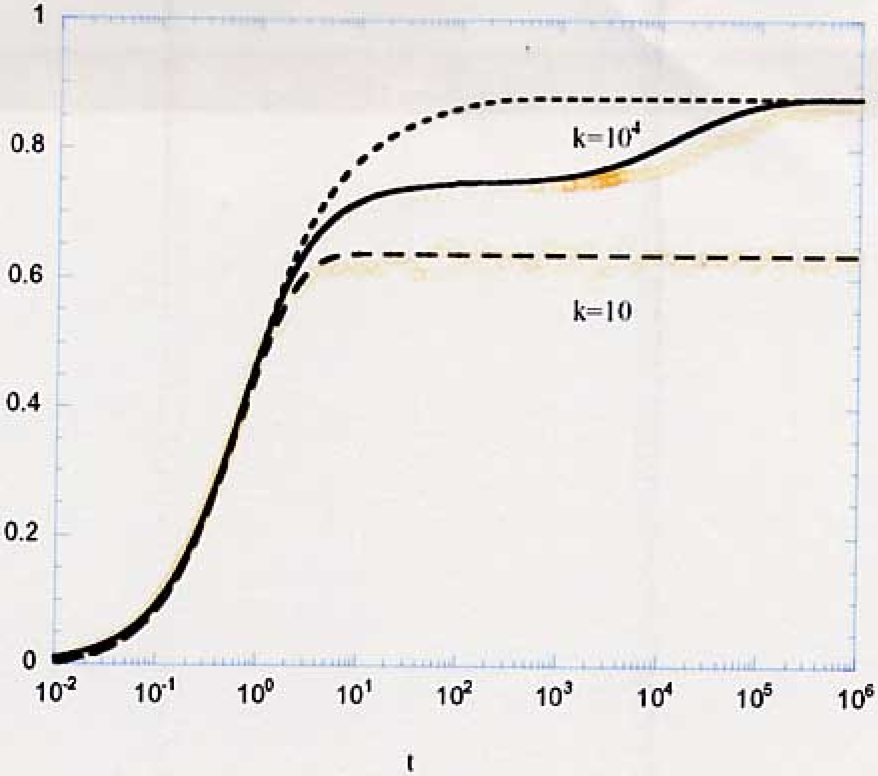


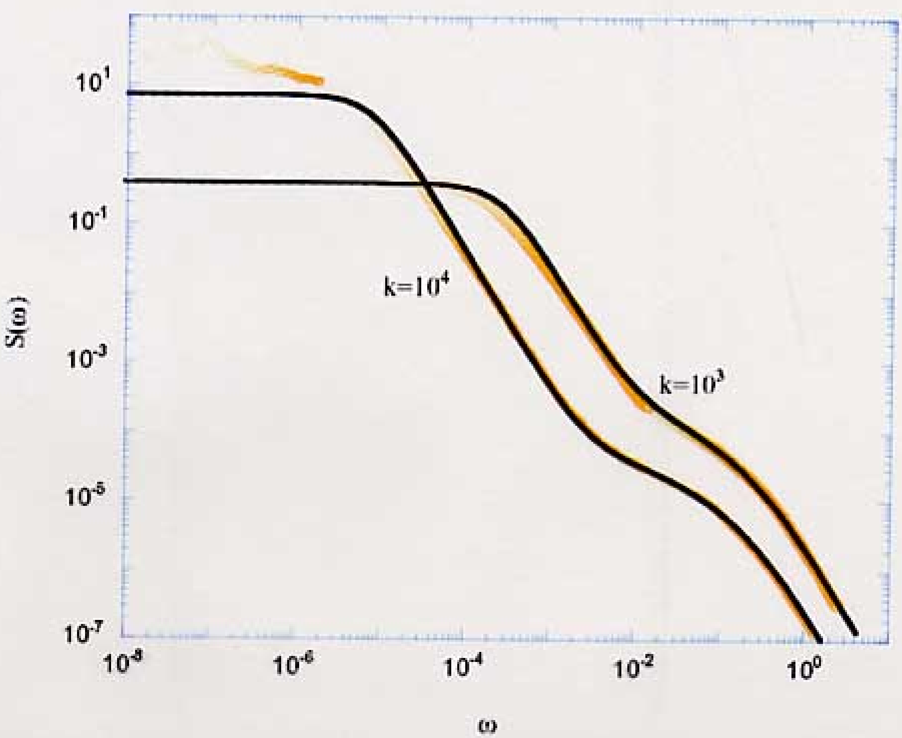
Density evolves, at long time:

$$\rho(t) \approx \rho_\infty - [\ln (kt)]^{-1}.$$

As density increases, must move many more "cars" to make a space large enough for one more to enter.

See: Nowak, Knight, Ben-Naim, Jaeger, SRN (PRE, 1998)
Kolan, Nowak , Tkachenko (PRE, 1999)



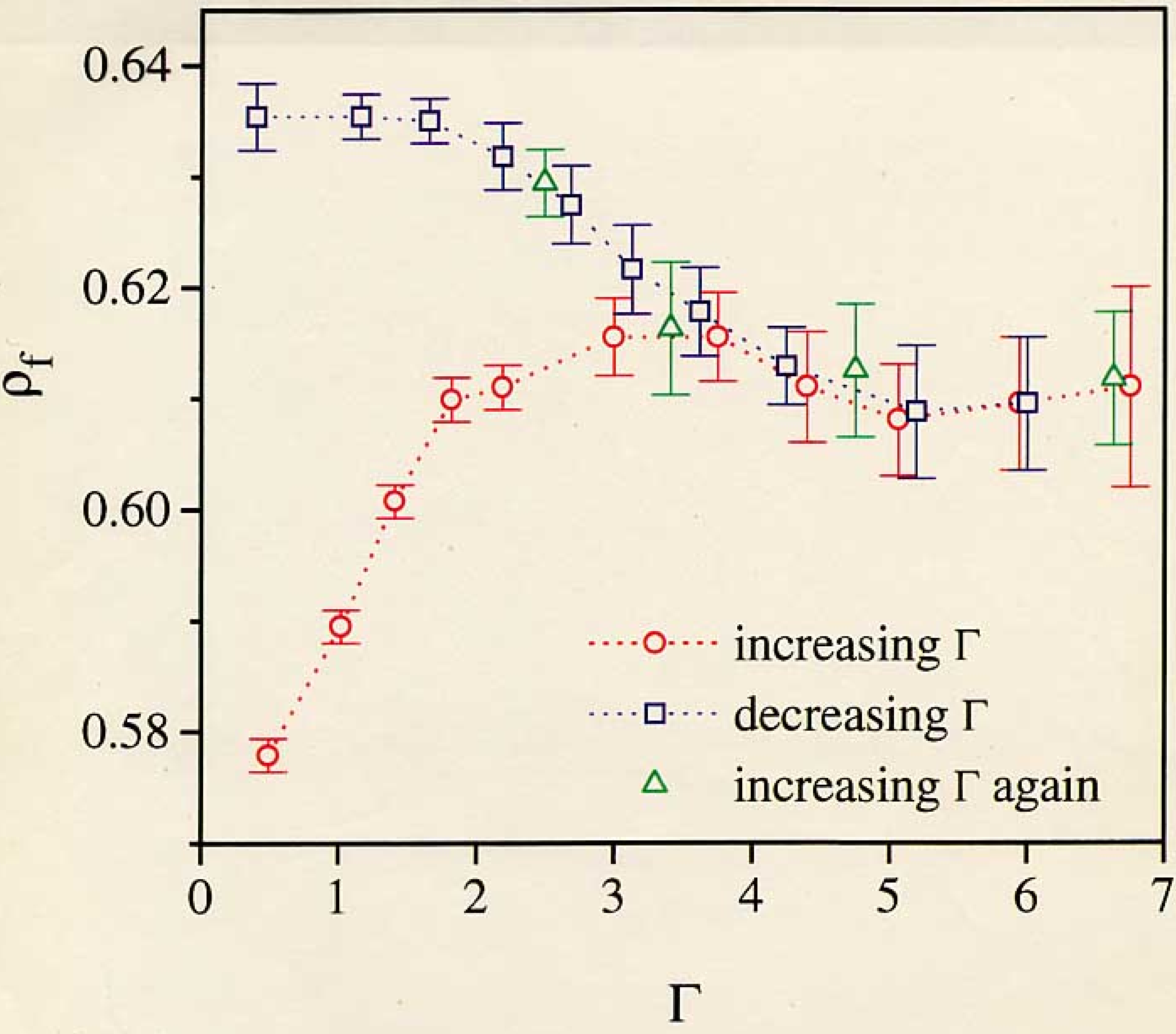


Glassy behavior of the parking lot model

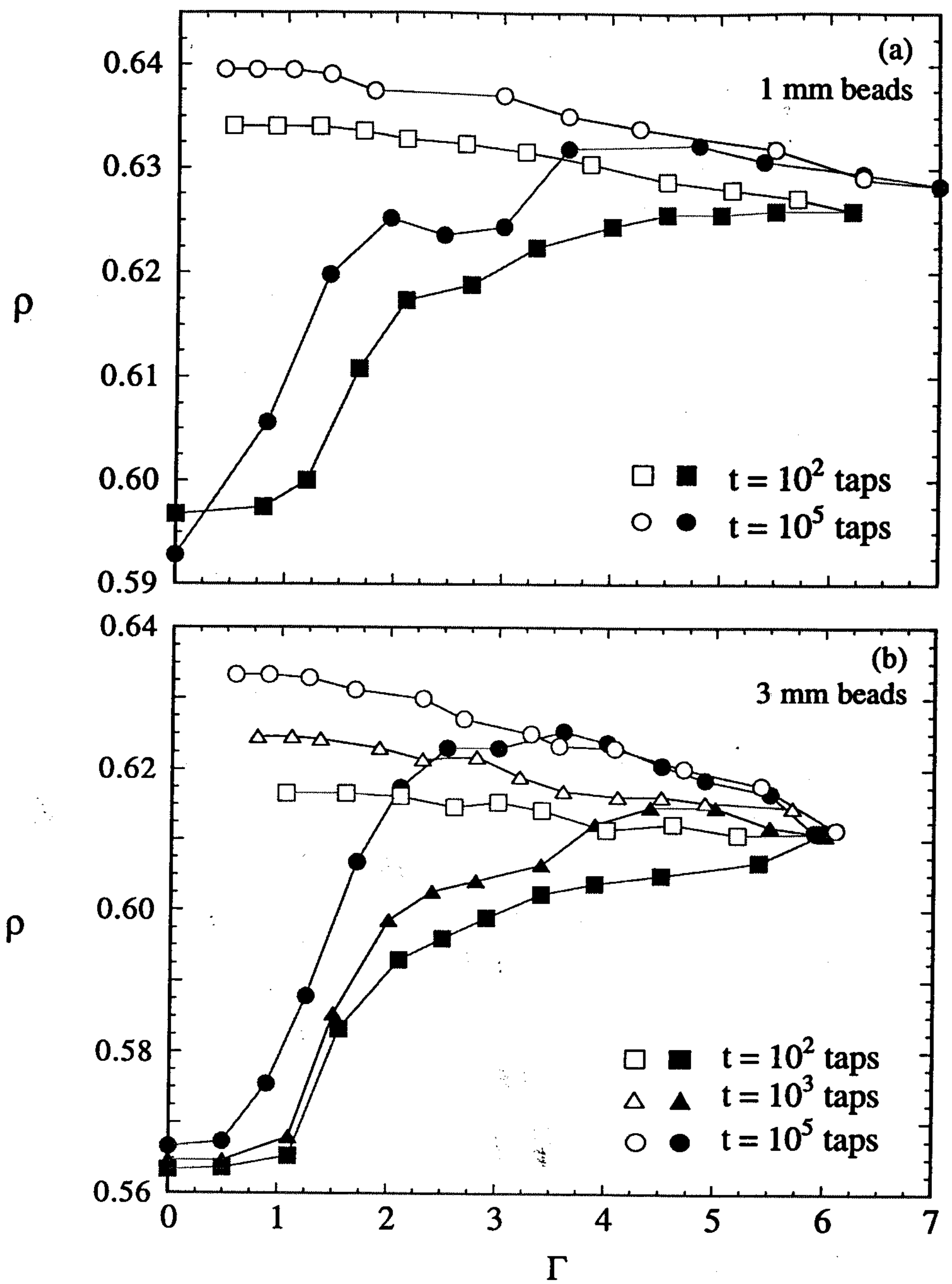
A. J. Kolan, E. R. Nowak, and A. V. Tkachenko

Phys. Rev. E 59, 3094-3099 (1999)

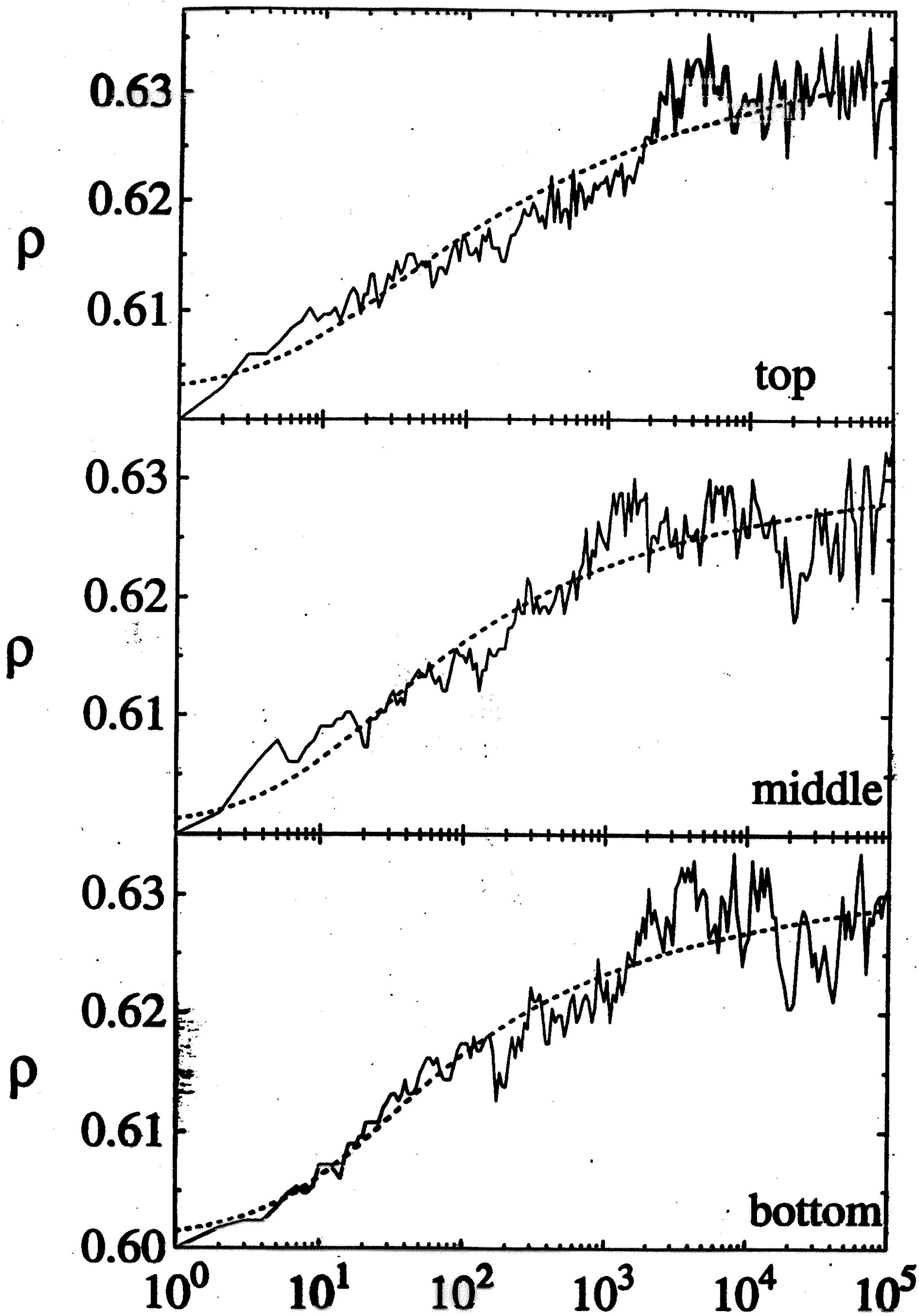
Final Density $\rho_f = \rho(t=100,000)-\rho_0$



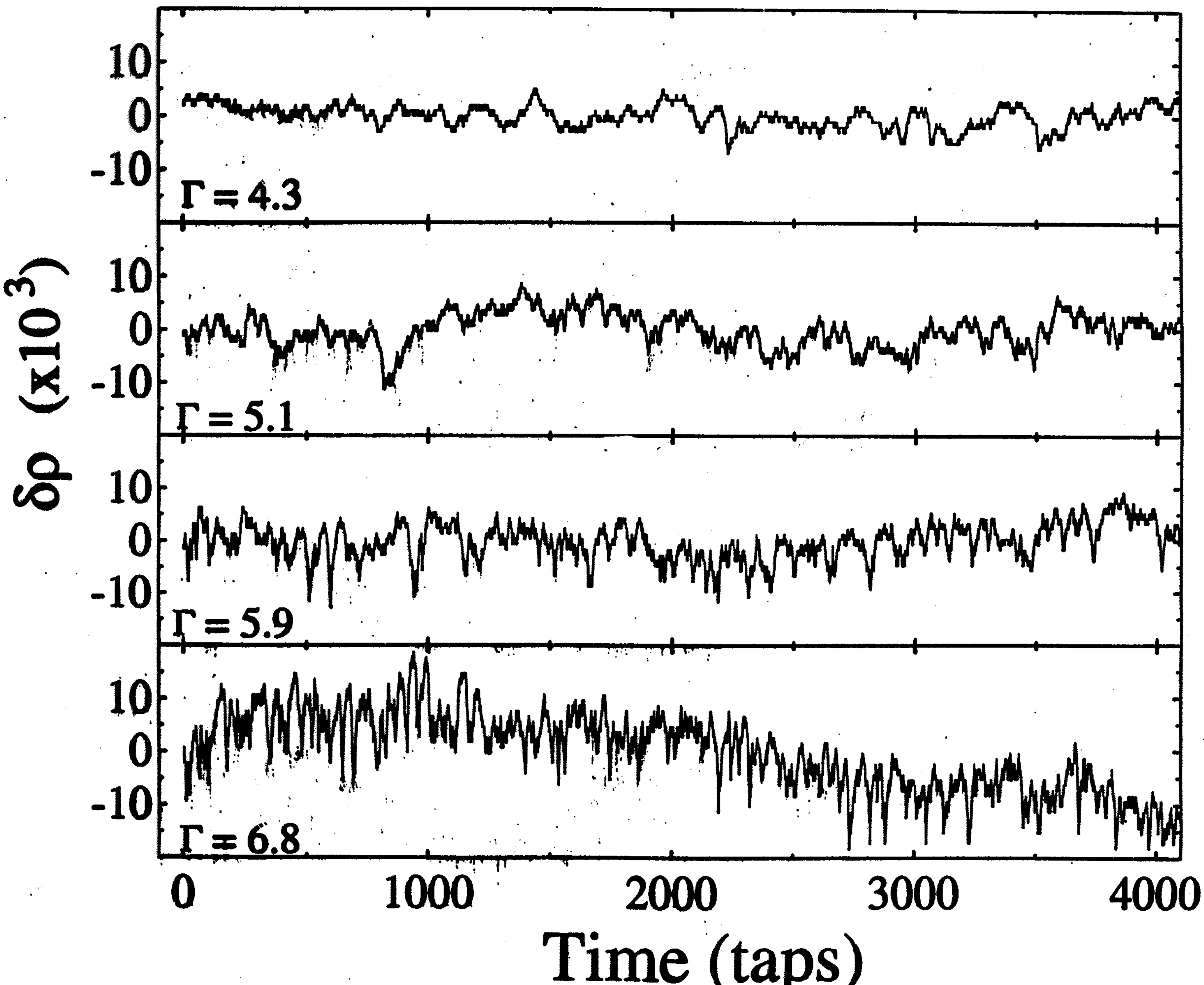
REVERSIBLE LINE → IN "STEADY STATE"

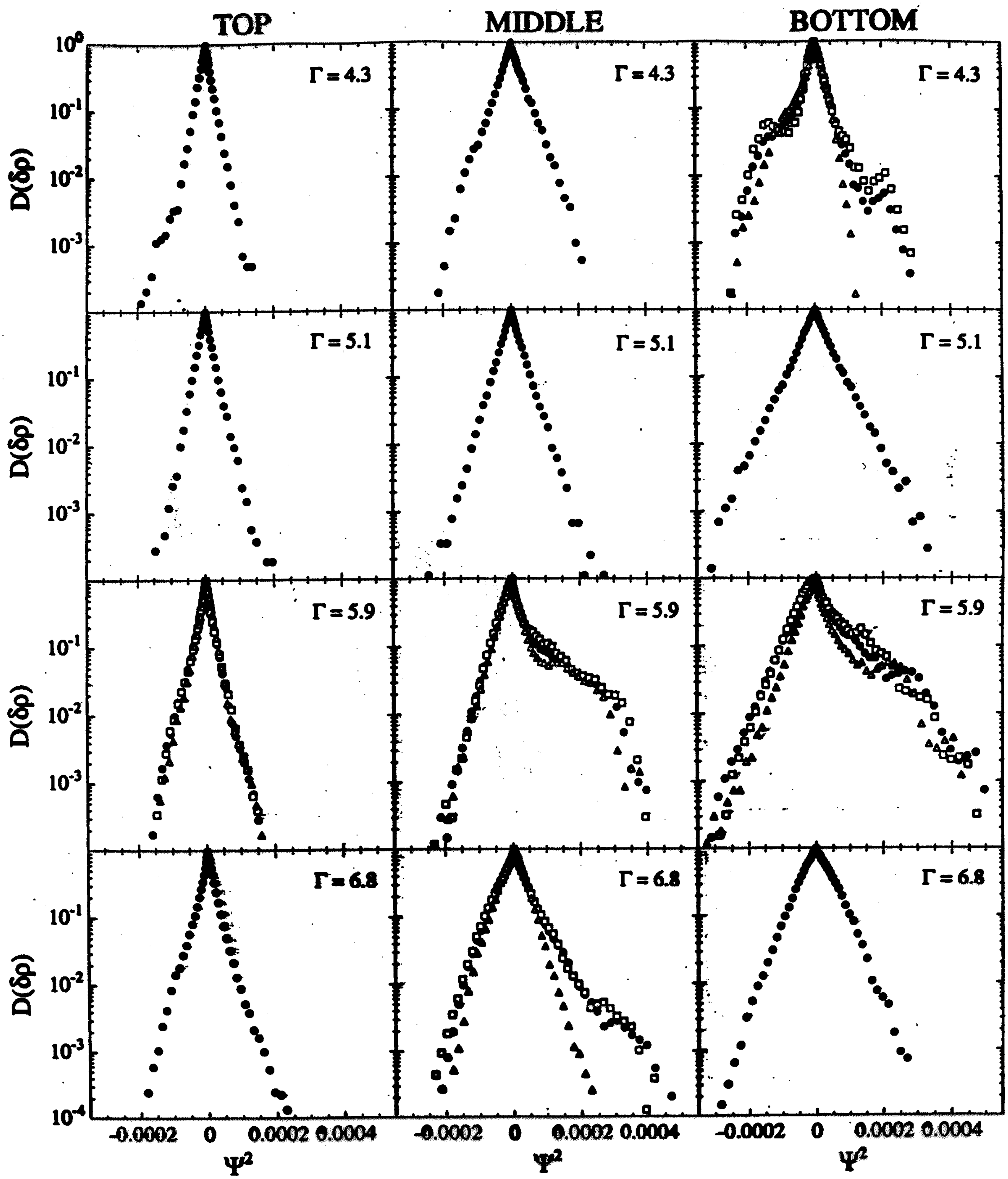


(a)
1 mm beads
ρ
0.64
0.63
0.62
0.61
0.60
0.59
t = 10^2 taps
t = 10^5 taps
(b)
3 mm beads
ρ
0.64
0.62
0.60
0.58
0.56
t = 10^2 taps
t = 10^3 taps
t = 10^5 taps
0
1
2
3
4
5
6
7
Γ


0.63
0.62
0.61
0.60
ρ
top
middle
bottom
10^0
10^1
10^2
10^3
10^4
10^5
Time (taps)

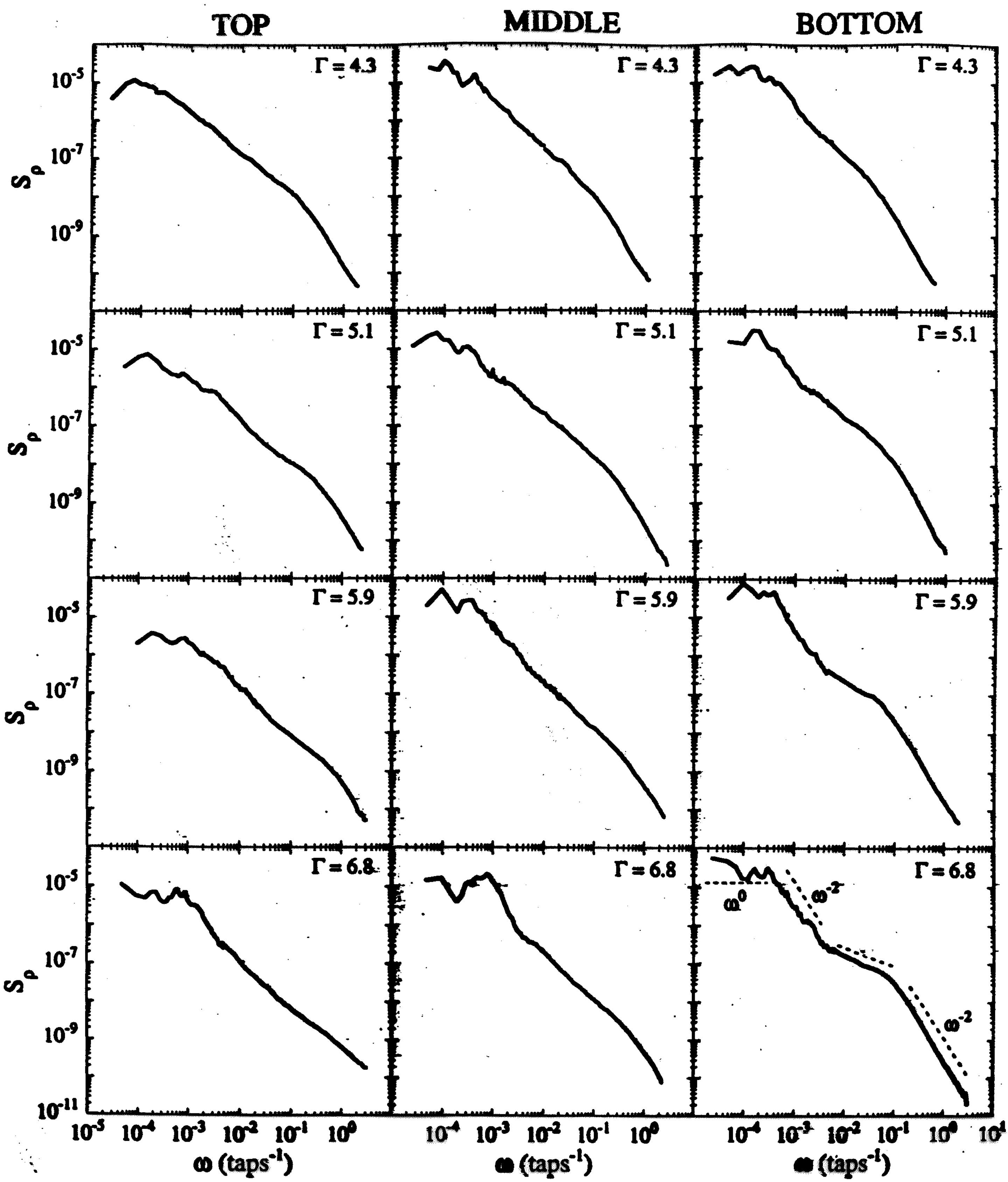
$\Gamma = 6.8$

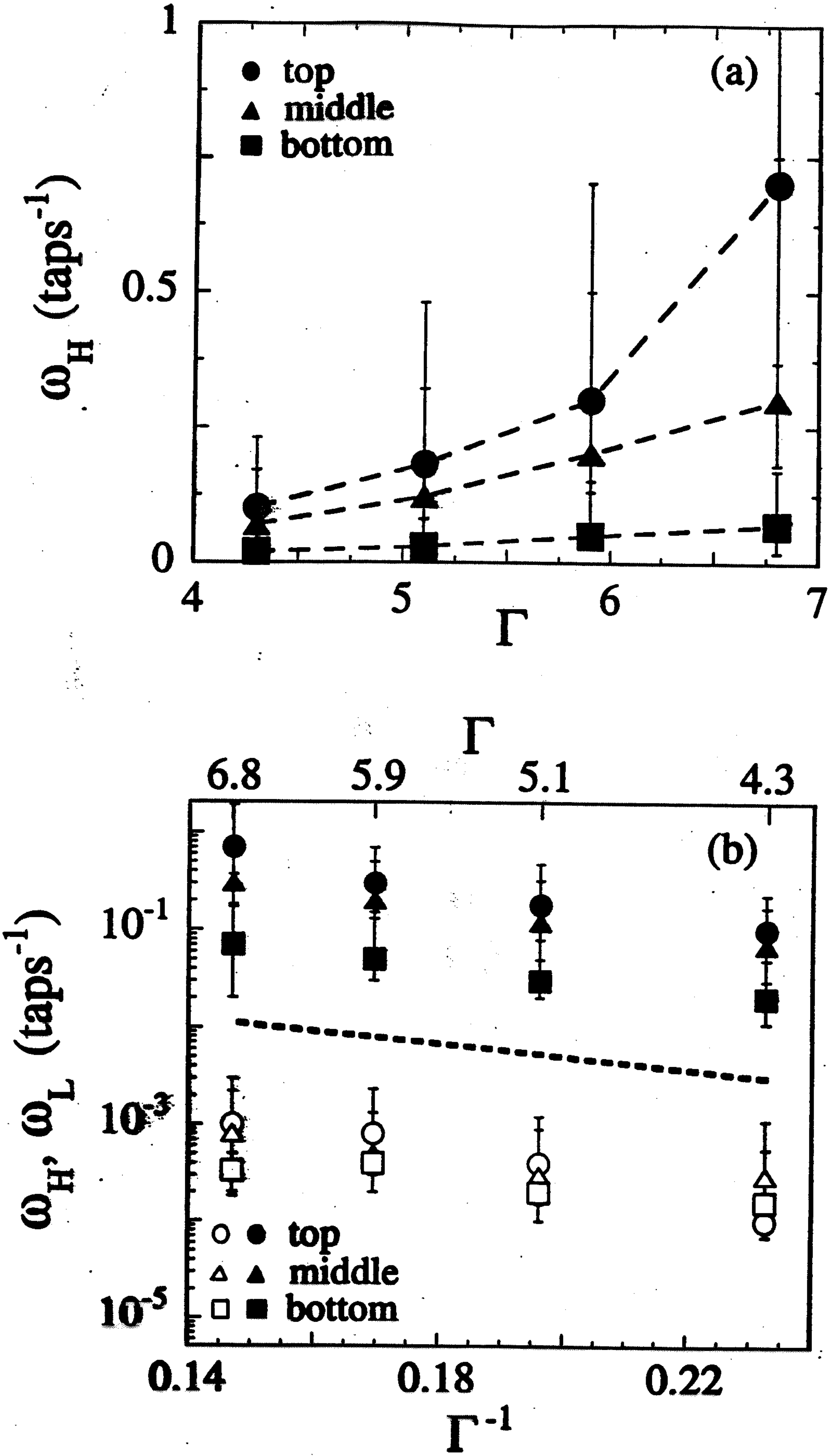

δρ (x10^3)
10
0
-10
Γ = 4.3
Γ = 5.1
Γ = 5.9
Γ = 6.8
0
1000
2000
3000
4000
Time (taps)



$$\Psi^2 = \delta\rho^2 \, \mathrm{sign}(\delta\rho)$$

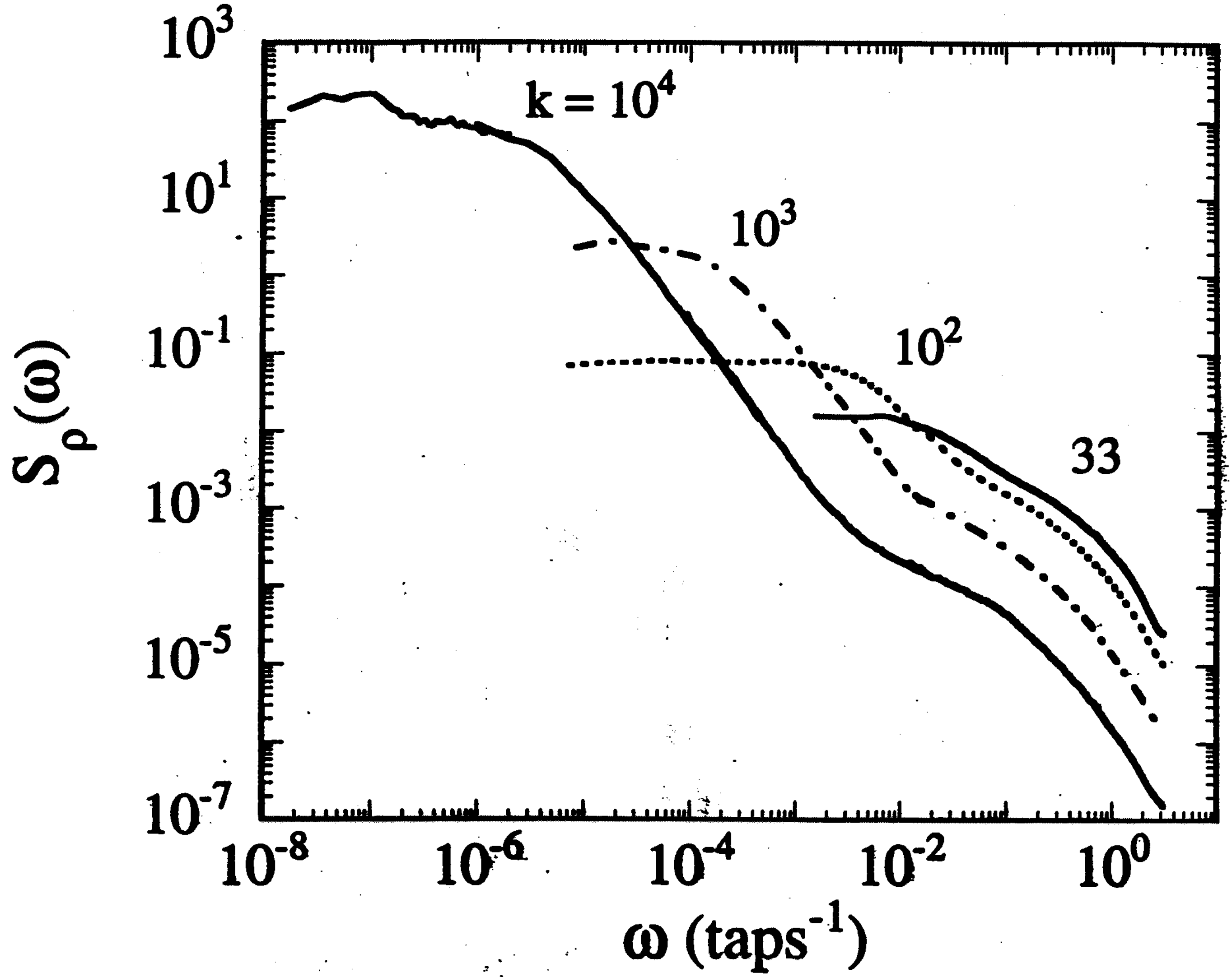
POWER SPECTRA





(a)
top
middle
bottom
ω_H (taps^{-1})
1
0.5
0
4
5
6
7
Γ
(b)
Γ
6.8
5.9
5.1
4.3
ω_H, ω_L (taps^{-1})
10^{-1}
10^{-3}
10^{-5}
top
middle
bottom
0.14
0.18
0.22
Γ^{-1}

SIMULATION OF "PARKING LOT" MODEL.

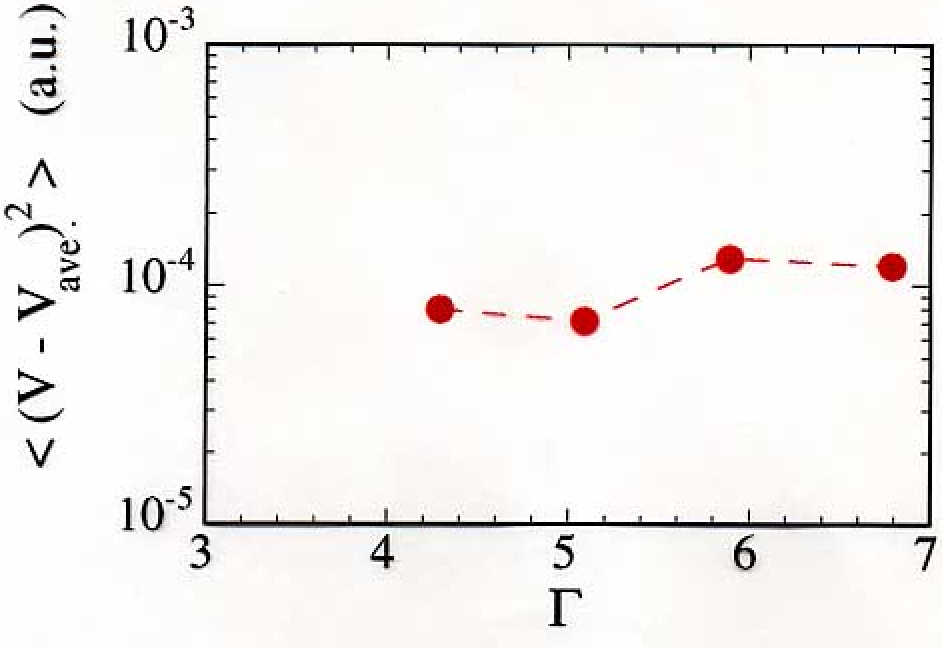


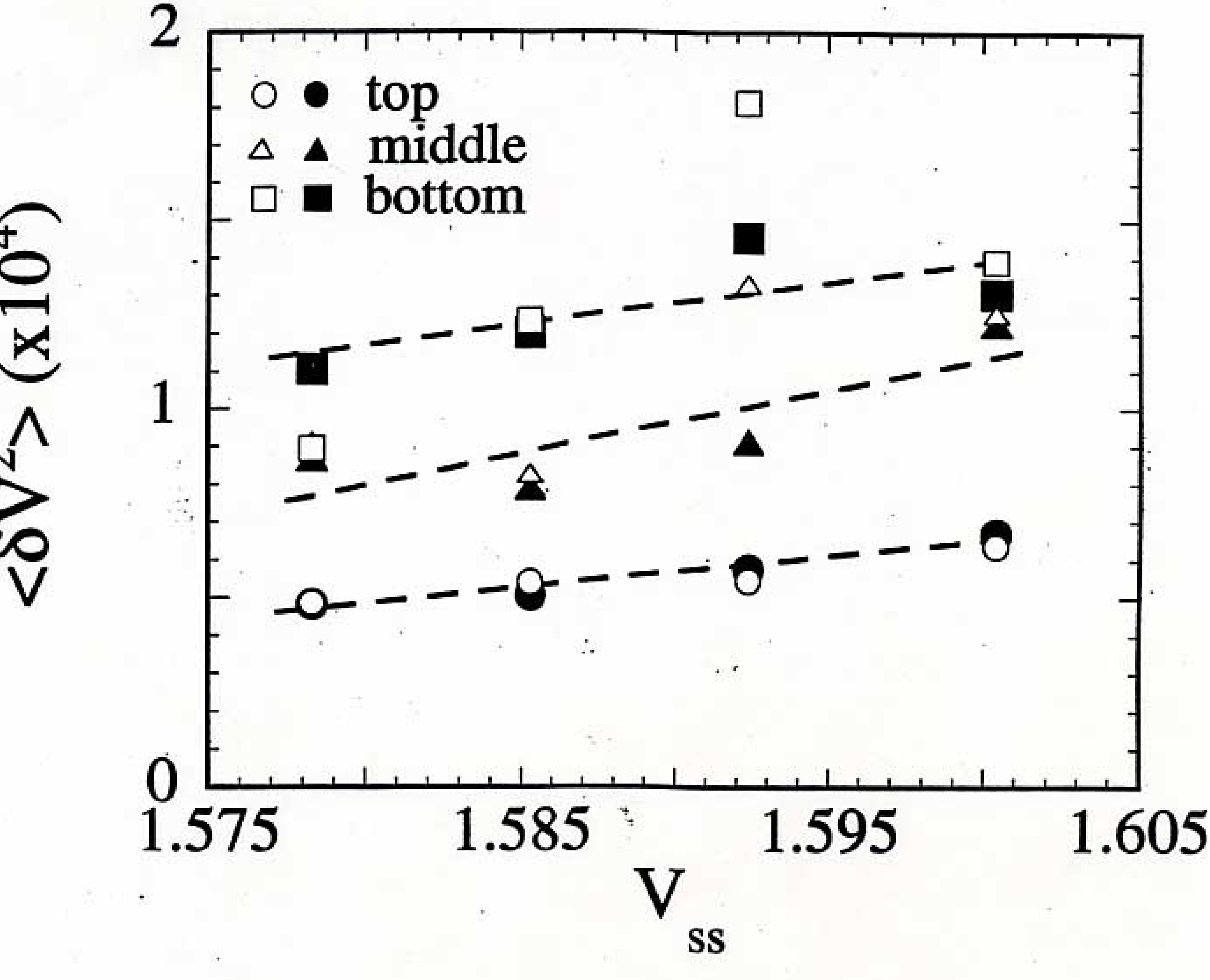
Fluctuations and

Granular "Specific Heat"

Thermal System	Granular System
$c_v = d\bar{E}/dT\big|_V$	$c = d\bar{V}/d\chi$
$= \langle(E - \bar{E})^2\rangle/k_B T^2$	$= \langle(V - \bar{V})^2\rangle/\lambda\chi^2$

$$\int d\bar{V}/\langle(V - \bar{V})^2\rangle = \int_{\chi_1}^{\chi_2} d\chi/\lambda\chi^2$$
$$= 1/\lambda\chi_1 - 1/\lambda\chi_2$$





$<\delta V^2> = a + b V_{ss}.$

$1/\lambda\chi \propto \log(a + b V_{ss})$

Fluctuations and Vibrations

T = 0 ⇒ vibrations important

(sound, shape change, density change, excitations)

Fluctuations ⇒ Response Functions

(a true suseptibility of steady state)

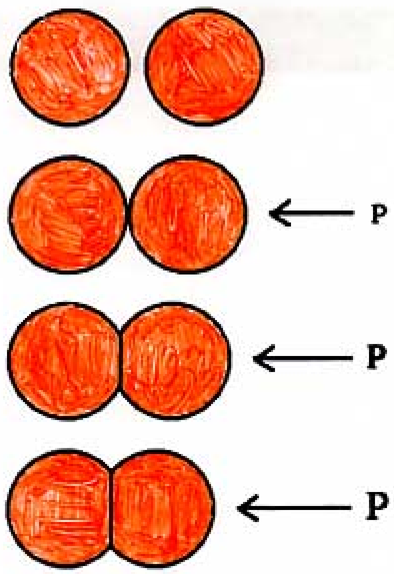
Look at *approach* to steady state and dynamics of that state.

New time scales appear

(knee frequencies)

Do fluctuations measure the thermodynamics?

HERTZIAN CONTACTS



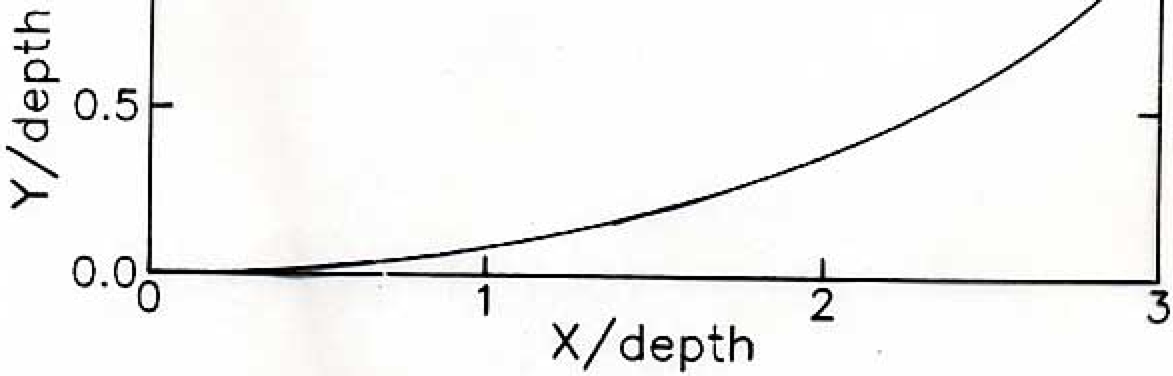
AS PRESSURE INCREASES SO DOES THE RESTORING FORCE AND VELOCITY OF SOUND.

HERTZIAN CONTACTS $\Rightarrow v \propto P^{1/6}$

$\therefore\ v \propto (\text{DEPTH})^{1/6}$

NO HORIZONTAL SOUND !!!

"MIRAGE" EFFECT

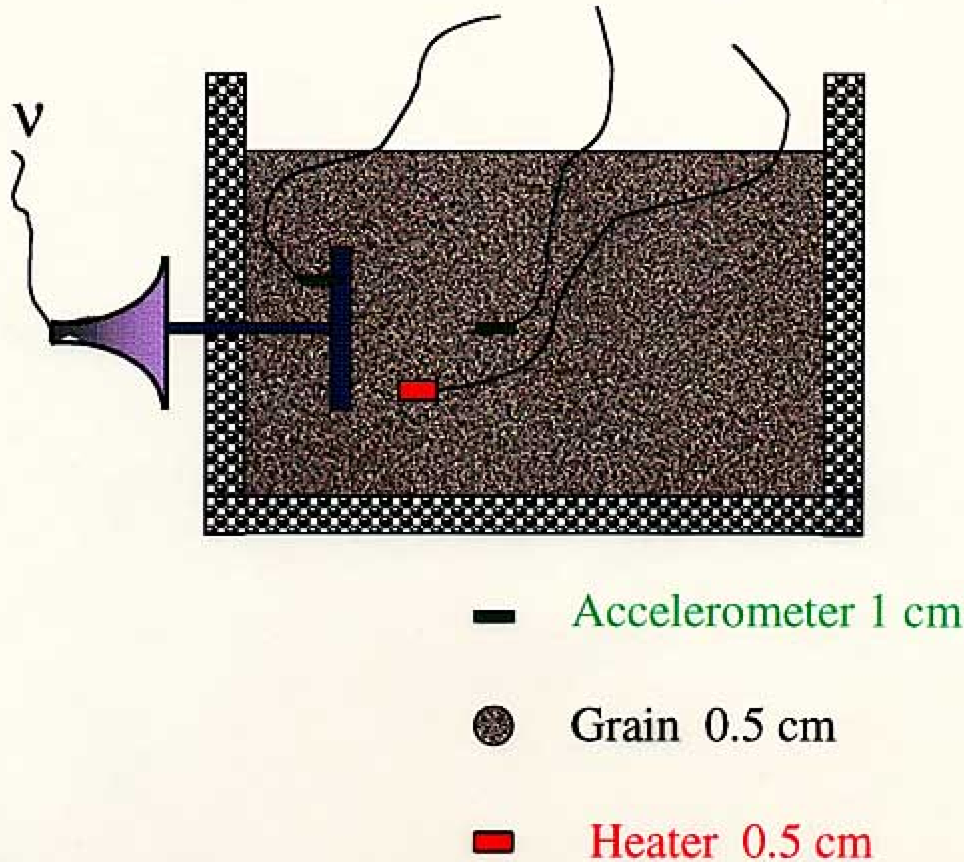


D.L. JOHNSON

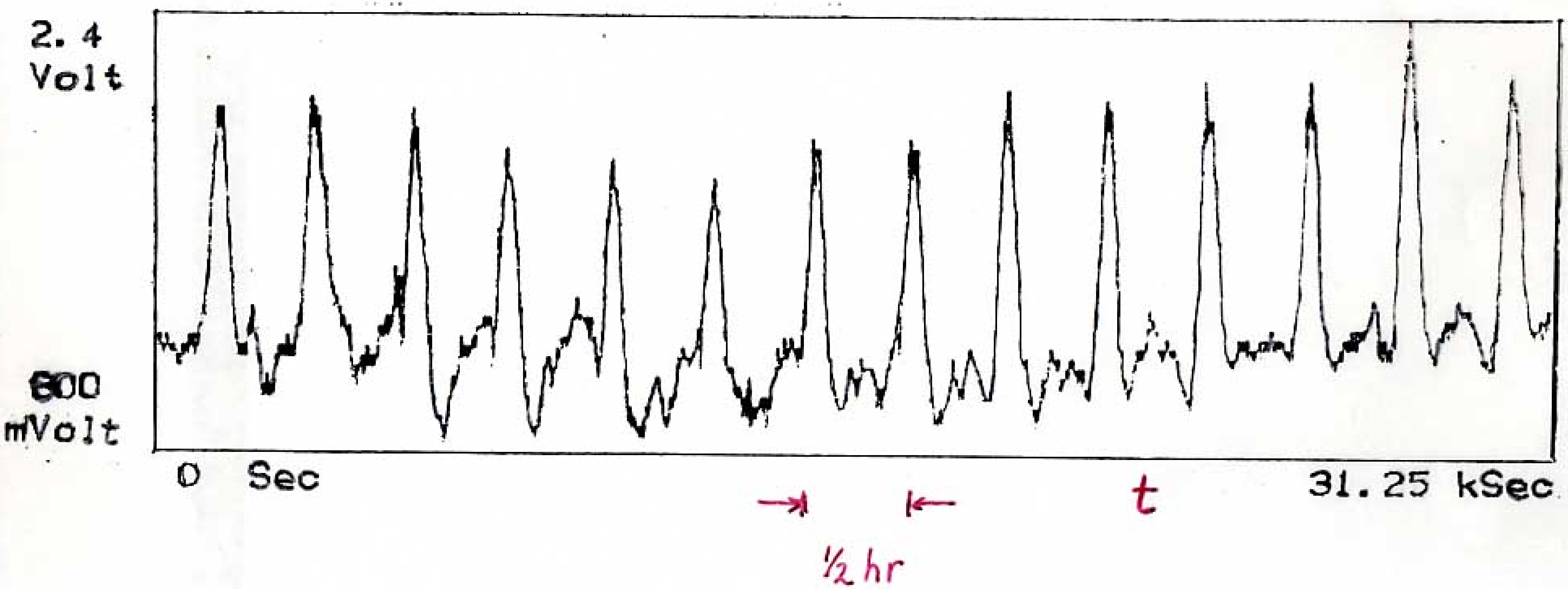
“Sound” Propagation

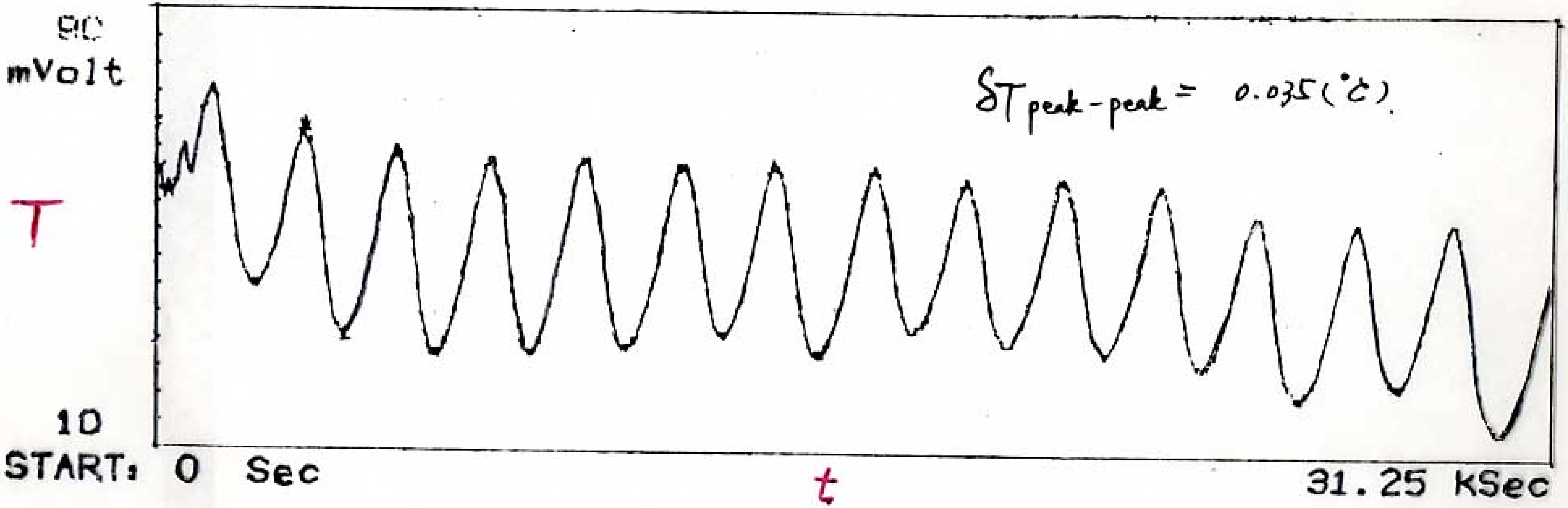
“sound” because in near field regime

Extremely sensitive to *minute* details of granular packing.



LOW AMPLITUDE DRIVING SIGNAL



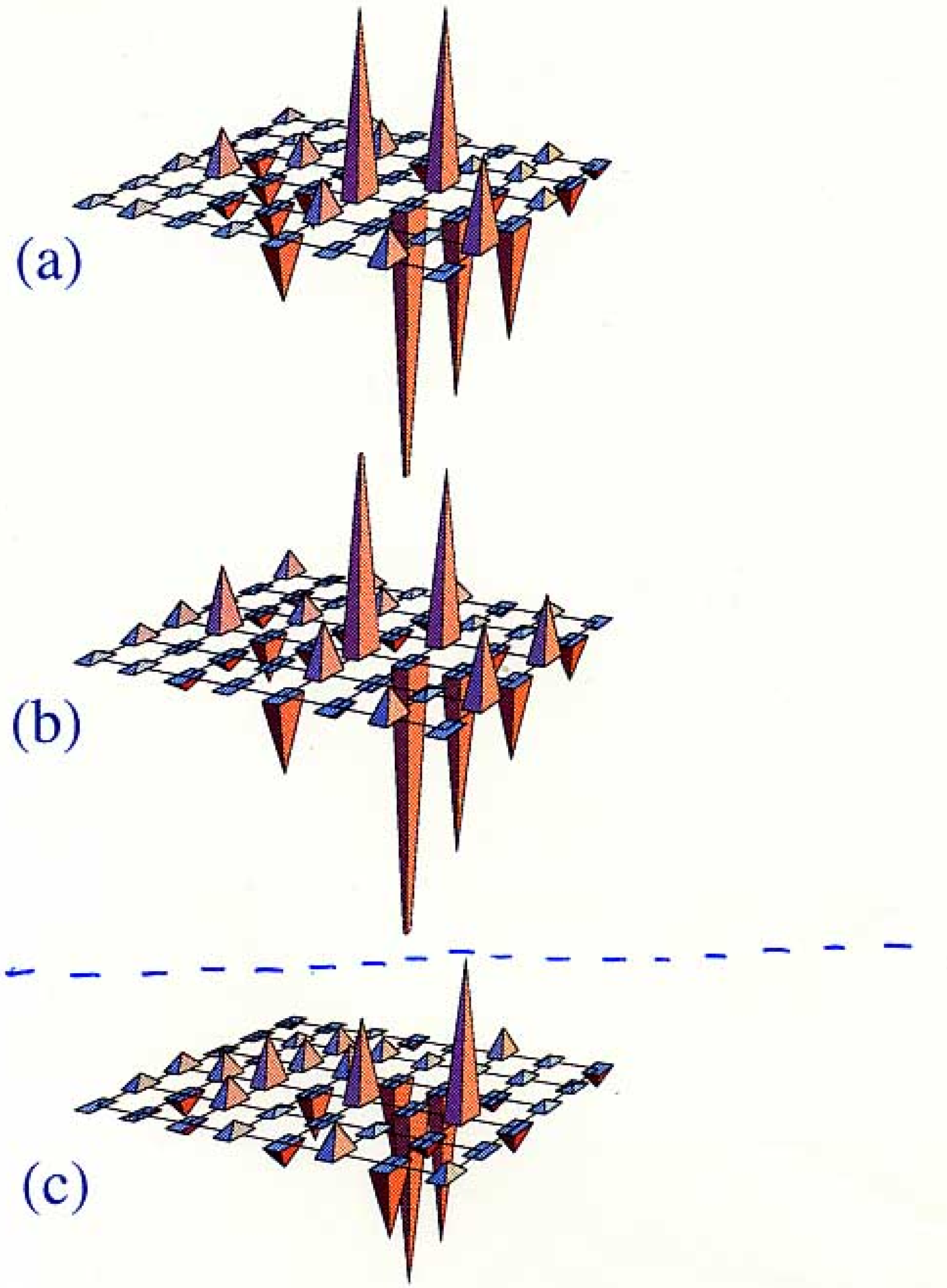


TEMPERATURE OSCILLATION

$\Delta T \sim 0.035\ °K$

$\frac{\Delta T}{T} \sim 10^{-4}$!

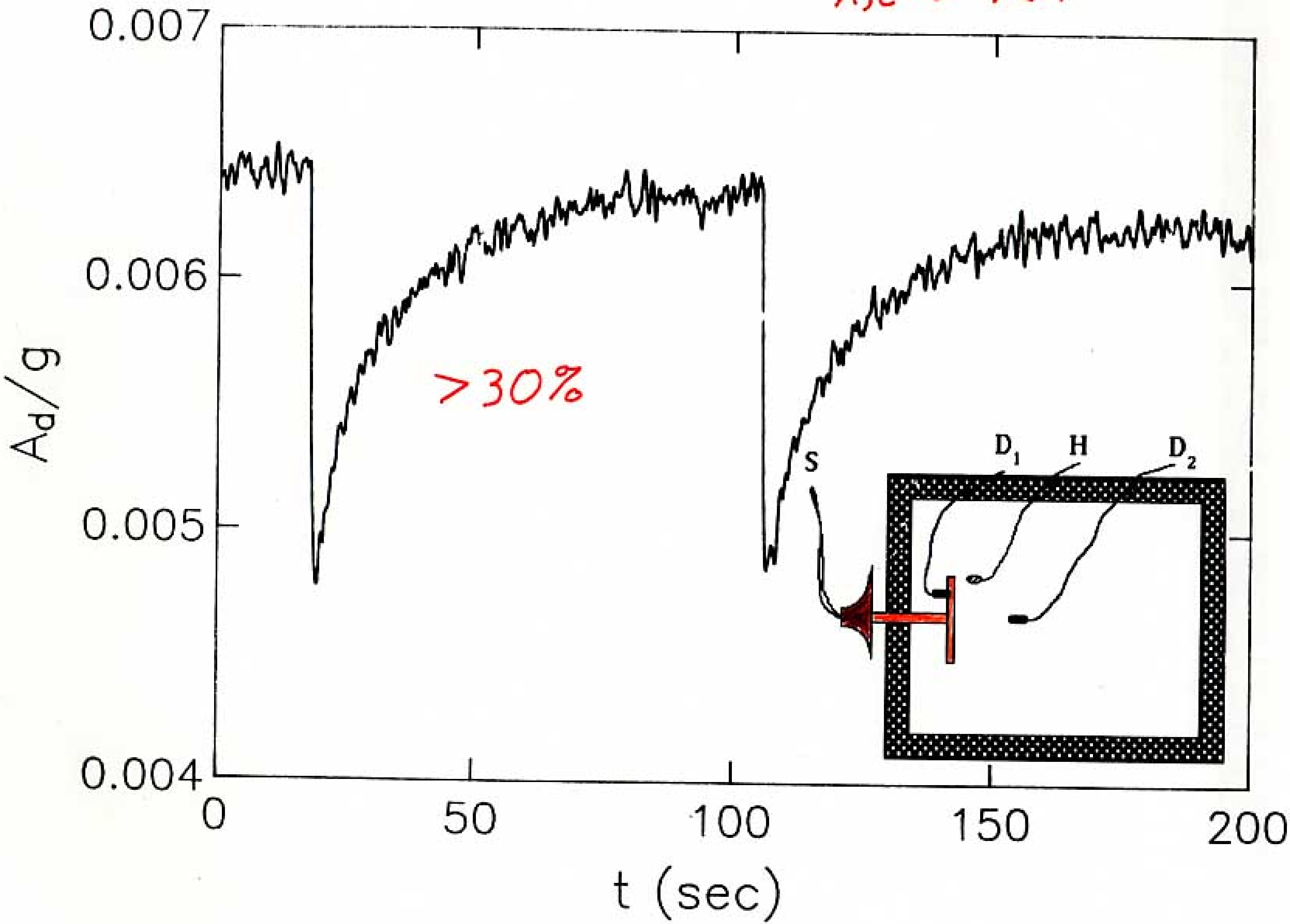
RESPONSE OF HEATER ARRAY

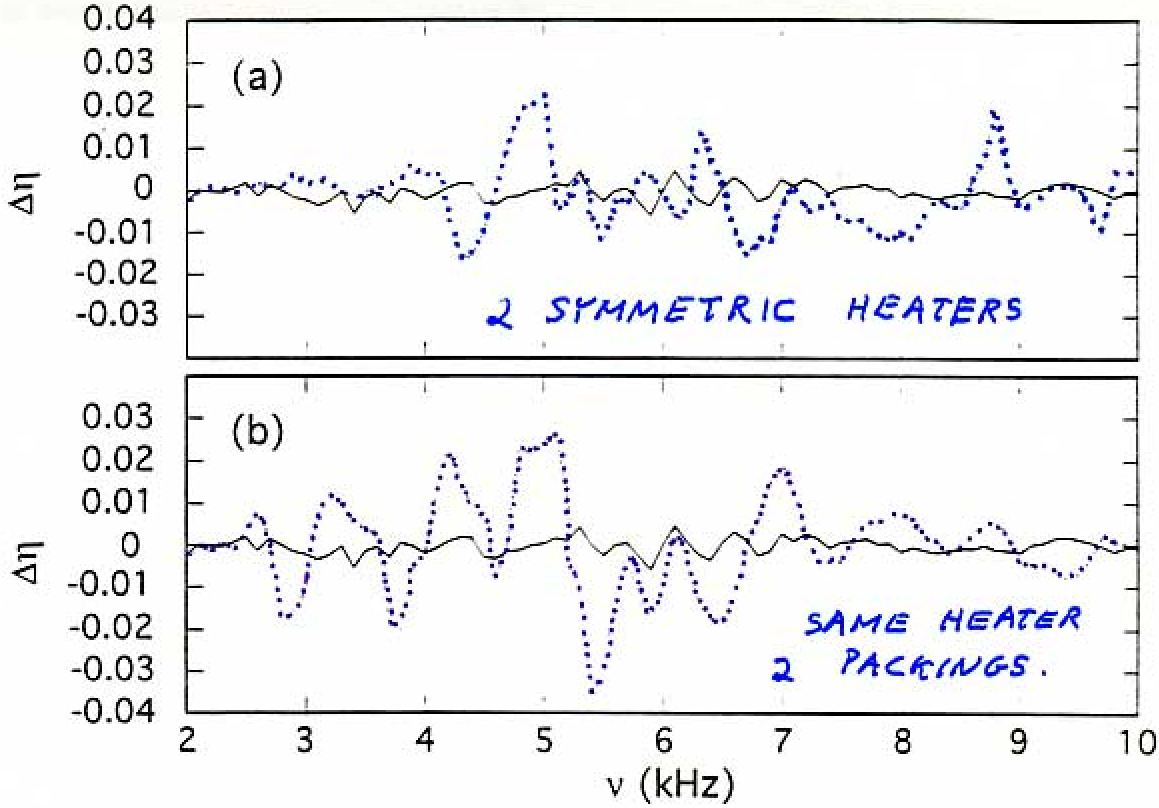


PULSED HEATER: ΔT ≈ 1K

Δl ≃ 1000 Å

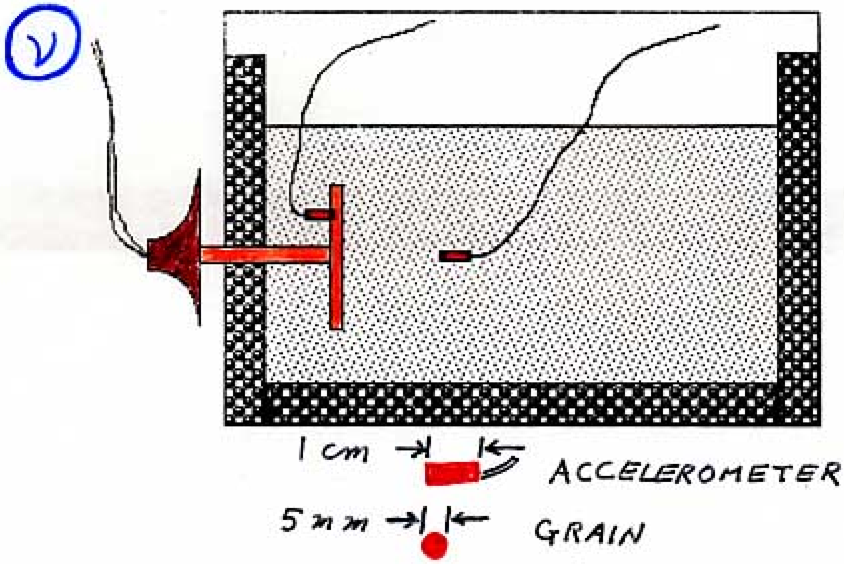
λ,d ~ 1 CM

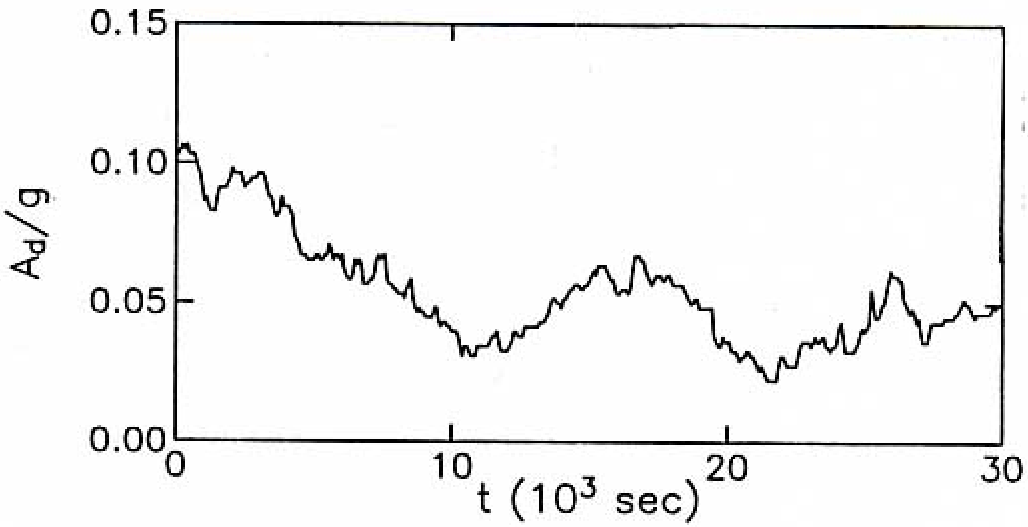




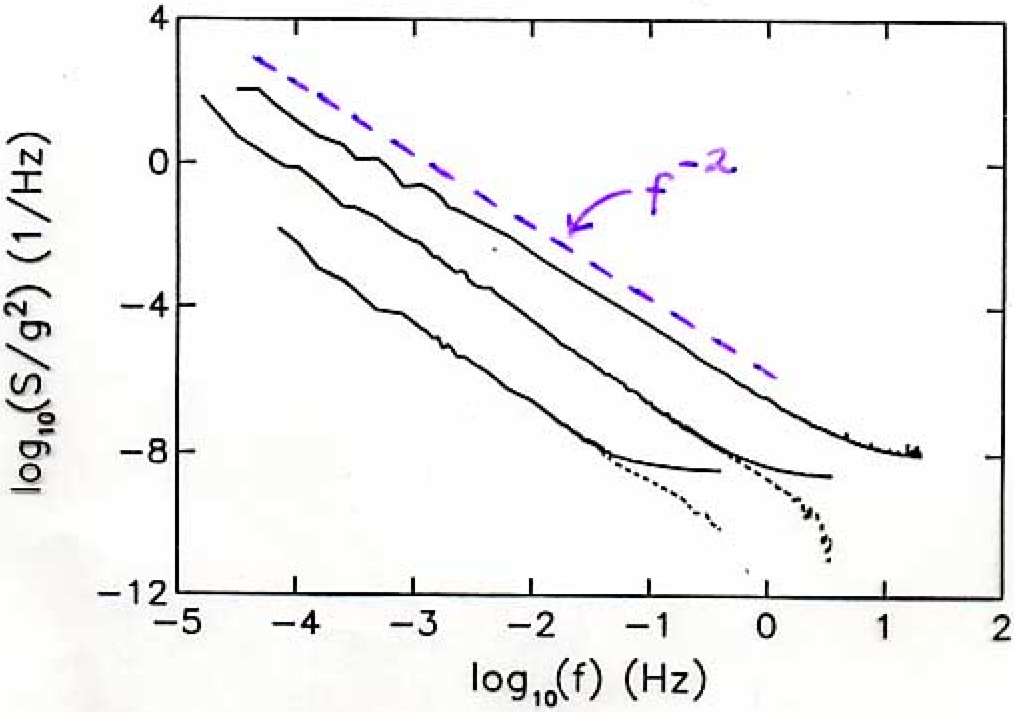
⟹ EXISTANCE OF FORCE CHAINS ?

DIFFUSION ON A MORE COMPLICATED TOPOLOGY

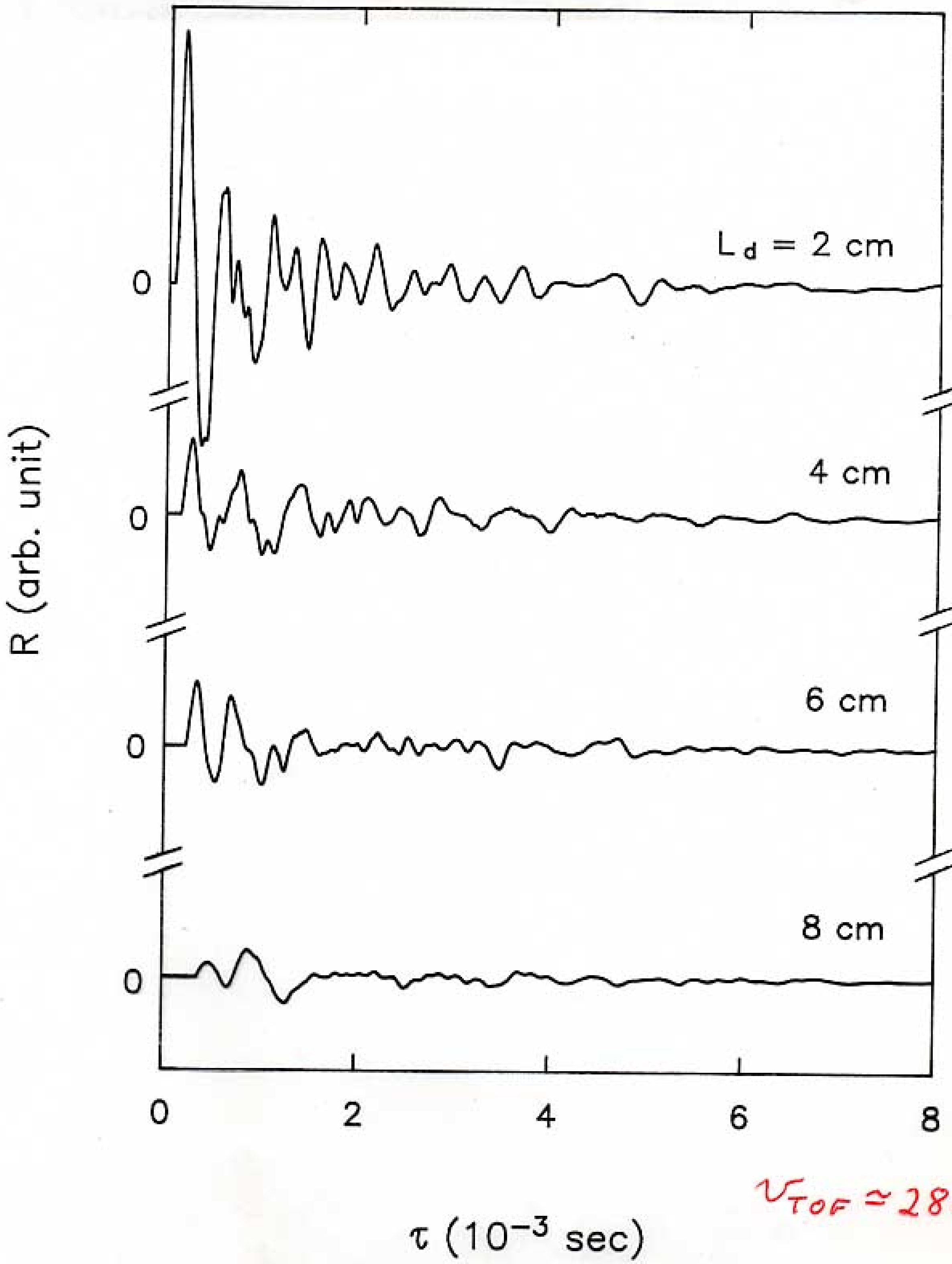




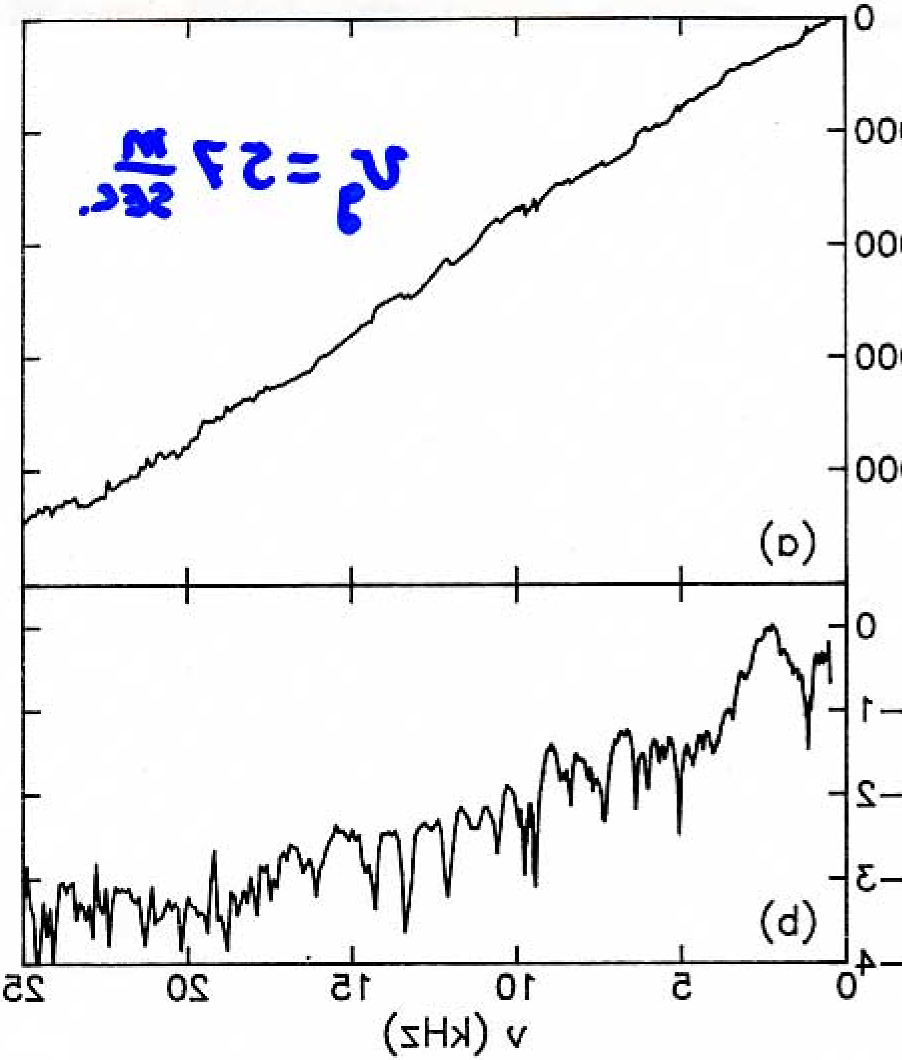
POWER SPECTRUM.



RESPONSE TO A SHARP PULSE



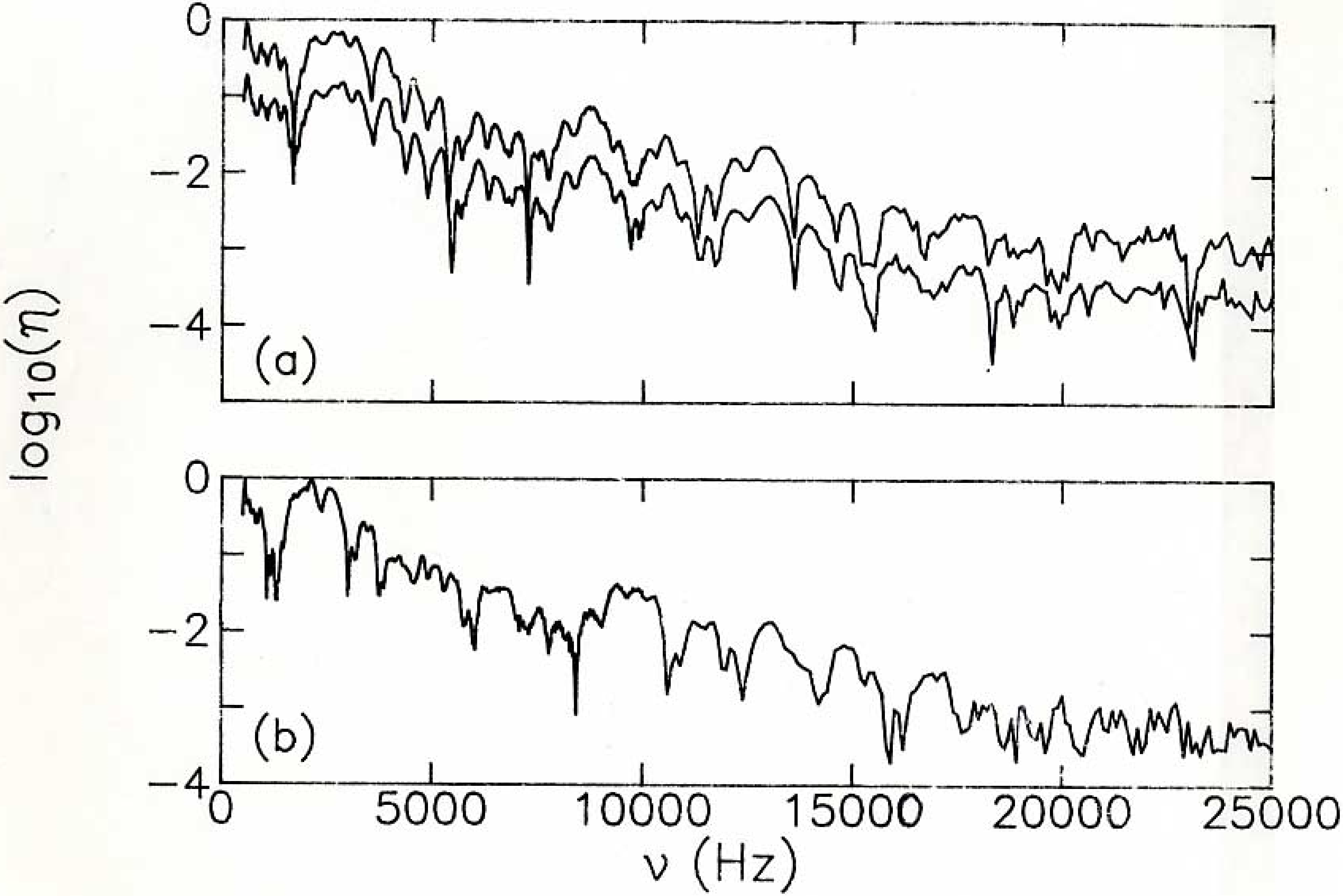
$v_{TOF} \simeq 280 \frac{m}{sec}$



v_g = 57 m/sec

BUT: v_{TOF} = 280 m/sec.

FREQUENCY RESPONSE



$\eta \equiv A_d / A_s$

Does a Diffusion Picture Apply?

from Feng and Sornette

$$\Phi = \frac{2\pi\nu}{c}\frac{L^2}{l^*}$$

$$l^* \approx \frac{L}{5} \approx 0.9\ cm$$ FROM $v_{TOF}/v_g \approx 5$

$$\Delta\nu = \frac{D}{L^2} = \frac{cl^*}{3L^2} \approx 530\ \mathrm{Hz}$$

At $\nu = 4$ kHz, $k = \frac{2\pi\nu}{c} \approx 0.9\ \mathrm{cm}^{-1}$

So $kl^* \approx 0.8$

Is it close to localization?