Quantum Hall effect with electrons and cold atoms

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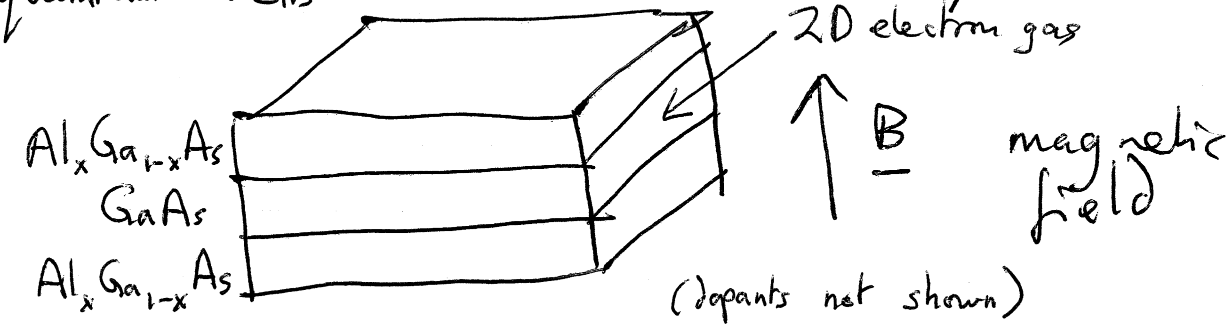
Overview:

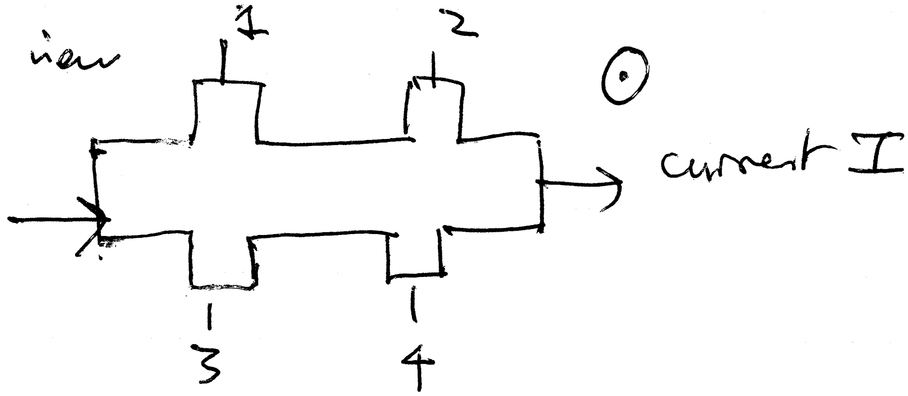
* Experimental setting: semiconductor heterostructures
  trapped atoms
* Laughlin states & excitations
* Composite particles, Fermi liquids, pairing
* Nonabelian states; bosons, breakdown of BEC

Electronic QHE typically in semiconductor heterostructures & quantum wells: schematically:



(dopants not shown)

Plane view



Resistance: $R_{xx} = \dfrac{V_{12}}{I}$, $R_{xy} = \dfrac{V_{13}}{I}$

Classically, $R_{xy} = \rho_{xy}$ (resistivity) $= \dfrac{B}{\bar{n}ec}$

$\bar{n} = N/A =$ electron density $\sim 10^{11}\,\mathrm{cm}^{-2}$

Qu. Mech'y (at low $T \sim {}^\circ K$), plateaus (with zero in ρ_{xx} or R_{xx}) when

$$\sigma_{xy} = \frac{1}{\rho_{xy}} = \nu \frac{e^2}{h}$$

$\nu \leftrightarrow \dfrac{\bar{n}}{B}\dfrac{hc}{e}$ (dimensionless density or <u>filling factor</u>) is <u>integer</u> or <u>rational fraction</u>.

Landau Levels

One Charged particle in two-dims in a magnetic field

$$H_1 = \frac{1}{2m_e}\left(-i\hbar\nabla - \frac{e}{c}\underline{A}\right)^2$$

Energy eigenvalues $\boxed{E_n = (n+\tfrac{1}{2})\hbar\omega_c}$ Landau levels

(cyclotron frequency) $\omega_c = \dfrac{eB}{m_e c} > 0$

$n = 0, 1, 2, \ldots$

In "symmetric gauge", $\underline{A} = \frac{1}{2}\underline{r}\times\underline{B}$, eigenfunctions for $n=0$ are

$$\boxed{u_m(z) = \frac{z^m e^{-\frac{1}{4}|z|^2}}{\sqrt{2\pi\, 2^m\, m!}}}, \quad m = 0, 1, 2, \ldots \quad z = x+iy$$

(We set $l_B^2 = \dfrac{\hbar c}{eB} = 1$.)

u_m peaked at $|z| = \sqrt{2m}$, so no of states per unit area is $\dfrac{1}{2\pi}$.

(m_{max} states inside circle radius is $R^2 = 2m_{max}$)

(Same for other LLs, but $u_{m,n}$ differs)

Many non-interacting fermions (electrons), occupy lowest levels. If occupy $n = 0, \ldots, \nu - 1$, then density is

$$\bar{n} = \frac{\nu}{2\pi l_B^2}$$

ie $\nu = 2\pi l_B^2 \bar{n} = \frac{\bar{n}}{B} \frac{hc}{e}$ again.

ν = fractional number of levels occupied or filling factor.

Without any translational symmetry-breaking terms in Hamiltonian,

Hall conductivity $\frac{1}{\rho_{xy}} = \sigma_{xy} = \nu \frac{e^2}{h}$, $\rho_{xx} = \sigma_{xx} = 0$

for <u>all</u> ν.

But we need trans symm. breaking (by disorder) to explain plateaus at <u>integer</u> $\sigma_{xy}/(e^2/h)$.

These values are special (for non-interacting electrons) because $(T=0)$ chemical potential μ jumps at ν = integer – it jumps from $\mu = (\nu - \frac{1}{2})\hbar\omega_c$ to $(\nu + \frac{1}{2})\hbar\omega_c$, the next LL.

So $\frac{d\mu}{d\bar{n}}$ has δ-function spikes at these values – <u>incompressibility</u>, $\kappa = \frac{d\bar{n}}{d\mu}$.

Atoms in a simple-harmonic trap

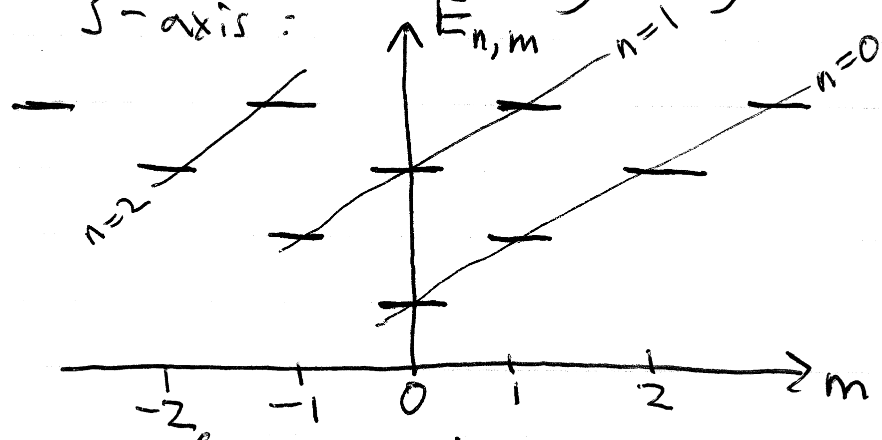
Single-particle Ham is axially-symmetric SHO:

$$H_1 = -\frac{\hbar^2}{2m_b}\nabla^2 + \frac{1}{2} m_b \omega_\perp^2 (x_1^2 + x_2^2) + \frac{1}{2} m_b \omega_{\parallel}^2 x_3^2$$

Separate into oscillator in 3-direction, $E_{n_3} = (n_3 + \frac{1}{2})\hbar\omega_{\parallel}$
and in 1-2 plane,

$$E_{n_1, n_2} = (n_1 + n_2 + 1)\hbar\omega_\perp$$

Latter can be classified (in different basis) using angular momentum
L_3 about 3-axis :



$$E_{m,n} = (m + 2n + 1)\hbar\omega_\perp$$

$n = 0, 1, 2, \ldots$
$m = -n, -n+1, \ldots, -1, 0, 1, 2, \ldots$ for each m.

Total energy eigenvalues $E_{m,n,n_3} = E_{m,n} + E_{n_3}$

with corres product eigenfunctions

If we look at $n=0$, the 1-2 eigenfunctions are

$$u_{m,0}(z) = \frac{z^m e^{-\frac{1}{2}|z|^2/l_\perp^2}}{l_\perp^{m+1}\sqrt{\pi\, m!}}$$

$(z = x_1 + ix_2)$

<u>same</u> as in LLL in QHE, with

$$l_B = l_\perp/\sqrt{2}$$

$$l_\perp = \sqrt{\frac{\hbar}{2m_b\omega_\perp}}$$ "quantum oscillator length"

Notice these have lowest single-particle energy $E_{m,n}$ for each value $m \geq 0$.

<u>N Rotating atoms</u> about 3-axis

To find ground states (or low-lying states) that represent this situation, two approaches

1) find lowest energy for given

$$L_{3tot} = \sum_{i=1}^{N} L_{3i}$$

2) find lowest energy in frame rotating at ω, ie minimise

$$H' = H - \omega L_{3tot}$$

These related by a Legendre trans^n.

In rotating frame,

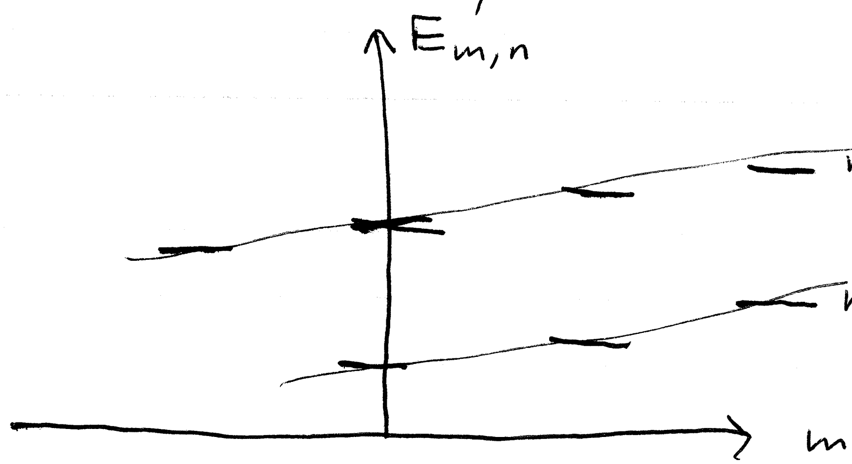
$$H_1' = \frac{1}{2m_b}\Big(-i\hbar\nabla - m_b\omega\,\underbrace{\hat{\underline{x}}_3}_{\text{unit vector in 3-direction}} \times \underbrace{\underline{r}}_{\text{produces Coriolis } (\equiv \text{Lorentz}) \text{ force}}\Big)^2$$

$$+ \tfrac{1}{2} m_b \omega_{\parallel}^2 x_3^2 + \tfrac{1}{2} m_b \omega_{\perp}^2 (x_1^2 + x_2^2)$$

$$- \tfrac{1}{2} m_b \omega^2 (x_1^2 + x_2^2) \leftarrow \text{centrifugal force}$$

In the rotating frame, the centrifugal forces reduces trapping potential in 1-2 plane, while Coriolis force produces LL-like behavior. These combine to give <u>same eigenfunctions</u> as those of H, while eigenvalues of H' are

$$E'_{m,n} = E_{m,n} - \omega m \hbar$$



For $\omega = \omega_\perp$, pot cancels and E' values degenerate. For $\omega > \omega_\perp$, system is "unstable" ($|z| \to \infty$) – but orbits are still stable.

Interactions in many-particle system

Many particles in LLL: wavefunctions

$$\Psi(z_1 \dots z_N) = f(z_1 \dots z_N)\, e^{-\frac{1}{4}\sum_i |z_i|^2}$$

↑ complex, analytic in each z_i

Interaction (in 3 dims)

$$H_{int} = \frac{1}{2}\sum_{i \neq j} V(\underline{r}_i - \underline{r}_j)$$

$$V(\underline{r}) = \begin{cases} \dfrac{e^2}{|\underline{r}|} & \text{Coulomb (electron)} \\[2ex] \dfrac{4\pi\hbar^2 a}{M}\,\delta(\underline{r}) & \text{pseudopotential (atoms)} \end{cases}$$

$a =$ s-wave scattering length

"Trap" Ham

$$H_0 = \sum_i H_{1i}$$

$$= \omega_\perp L_{3tot} + N\hbar\omega_\perp \quad \text{when all ptcles in LLL.}$$

and L_{3tot}, N commute with H_{int}.
So sufficient to solve H_{int} for fixed N, L_{3tot}.

Laughlin states

Assume weak interactions:

$$e^2 \bar{n}^{1/2} \ll \hbar\omega_c$$

or $\frac{4\pi\hbar a}{M}\frac{\bar{n}}{\ell_{\parallel}} \ll 2\hbar\omega_{\perp}, \hbar\omega_{\parallel}$

($\bar{n}$ = no. per unit area)

Then at $T=0$ can assume all ptcles in LLL states (and in lowest state of potential in x_3 direction).

Laughlin trial wavefunction:

$$\Psi(z_1 \ldots z_N) = \prod_{i<j}(z_i - z_j)^q \, e^{-\frac{1}{4}\sum_i |z_i|^2}$$

q = integer:
symmetric if q even $\leftrightarrow$ bosons
antisymmetric if q odd $\leftrightarrow$ fermions

Highest orbital occupied is $m_{max} = q(N-1) = N_\phi$, corres radius $R = \sqrt{2m_{max}}$

Then if the density is uniform, the filling factor must be

$$\boxed{\nu = 2\pi\bar{n} = \frac{N}{N_\phi} = \frac{1}{q}}$$

as $N \to \infty$.

How are particles distributed?

$$|\Psi|^2 = \prod_{i<j} |z_i - z_j|^{2q} e^{-\frac{1}{2}\sum_i |z_i|^2}$$

$\rightarrow 0$ as $z_i \rightarrow z_j$, so low energy for short-range interaction

$$|\Psi|^2 = e^{-\beta_{plas} H_{plas}}$$

$$H_{plasma} = -\sum_{i<j} \ln|z_i - z_j| + \frac{1}{4q}\sum_i |z_i|^2$$

$$\beta_{plas} = 2q$$

is Boltzmann weight of 2D plasma at temperature $1/2q$ (with uniform background charge).

At sufficiently high temp (small q), the plasma is in a screening phase, and particles neutralize background with density $\bar{n}$ inside radius R.

Uniform on scales larger than screening length.

Screens for $q \lesssim 70$.

Special case $q=1$,

$$\prod_{i<j}(z_i - z_j) = \det \begin{vmatrix} 1 & z_1 & z_1^2 & \cdots & \\ 1 & z_2 & z_2^2 & \cdots & \\ 1 & z_3 & z_3^2 & \cdots & \\ \vdots & \vdots & \vdots^2 & & \\ 1 & z_N & z_N^2 & \cdots & z_N^{N-1} \end{vmatrix}$$

is Vandermonde det, = Slater det of filled LLL.

<u>Interaction for which Laughlin states are exact</u> (Haldane)

$(z_i+z_j)^M (z_i - z_j)^{m_{ij}} e^{-\frac{1}{4}|z_i|^2 - \frac{1}{4}|z_j|^2}$ is eigenstate of relative ang mom m_{ij}. Let $P_m(ij)$ be projector onto this subspace with $m_{ij} = m$.

Then

$$H_{int} = \sum_{m=0}^{q-1} V_m P_m(ij)$$

annihilates Laughlin state with exponent q. — they are zero-energy eigenstates.

Note $P_0(ij) \propto \delta^2(\underline{r}_i - \underline{r}_j)$ — $\nu = \frac{1}{2}$ Laughlin is exact in boson problem!

Any H_{int} in LLL trans, rotationally 2-body can be written in form $\sum_m V_m P_m(ij)$

Excitations

Laughlin quasihole: mult Ψ_L by

$$\prod_i (z_i - w)$$

In plasma, charge $1/q$ at w, so ptcles avoid w: there is a net charge $1/q$ missing from vicinity of w.

$$\boxed{\text{Fractionally charged excitations}}$$

Quasielectrons: mult by $\prod_i \left(2\frac{\partial}{\partial z_i} - w \right)$, (adjoint of $\prod_i (z_i - w)$)
$\rightarrow$ extra charge $1/q$ at w.

Variationally, a state with a well-separated quasihole and quasielectron has energy $\Delta > 0$ compared with ground state (as $N \to \infty$).

Coulomb or δ-fn ints.

So μ jumps by $q\Delta$ at $\nu = 1/q$, and

$$\boxed{\text{Laughlin state is incompressible}}$$

This remains true even when it is not the exact ground state (e.g. for Coulomb int). It represents incompressible phase of matter

Statistics of quasiparticles

Bosons, fermions, ... ?

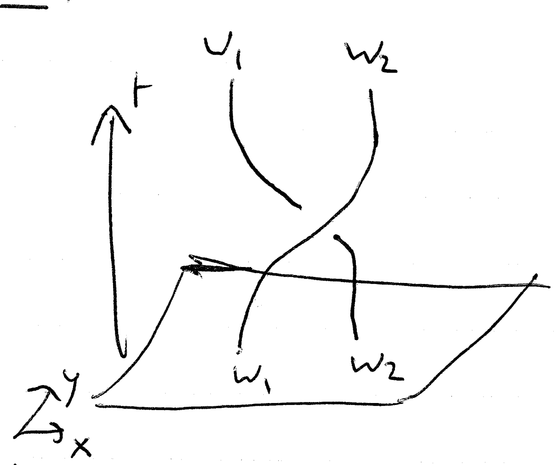
In two dims, easy to change symmetry of wavefunction by multiplying it by any power of

$$\frac{\prod_{i<j}(w_i - w_j)}{\prod_{i<j}|w_i - w_j|}$$

(a "singular gauge transformation").

So need a gauge-inv definition. One way is by adiabatic transport.

Drag quasiparticles slowly



and calculate (Berry) phase change in state.

First one quasihole. Use a Ham that has it as zero-energy ground state (and a gap), with qhole at location w we can control (say by an attractive potential. Then we calculate for $w = w(s)$

$$\frac{d\gamma}{ds} = i \left\langle \Psi(s) \middle| \frac{d\Psi(s)}{ds} \right\rangle$$

(normalized)

If at $s = S$, $|\Psi(S)\rangle = |\Psi(0)\rangle$, then $\gamma(S) - \gamma(0)$ is desired phase.

$$\Psi_L^{+w} = \prod (z_i - w)\, \Psi_L$$

$$\Rightarrow \frac{d\Psi_L^{+w}}{ds} = -\frac{dw}{ds} \sum_i \frac{1}{z_i - w}\, \Psi_L^{+w}$$

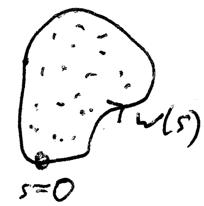
$$= -\frac{dw}{ds} \left(\int d^2z' \, \frac{n(z')}{z' - w} \right) \Psi_L^{+w}$$

Here $n(z) = \sum_i \delta^2(z_i - z)$ is ptcle density

$$\Rightarrow \frac{d\gamma}{ds} = -i \int d^2z' \, \frac{1}{2} \left(\frac{1}{z' - w} \frac{dw}{ds} - \frac{1}{\bar{z}' - \bar{w}} \frac{d\bar{w}}{ds} \right) \langle n(z') \rangle$$

$$\Rightarrow \gamma(S) - \gamma(0) = -i \oint dw \int d^2z' \, \frac{1}{2} \left(\frac{1}{z' - w} - \frac{1}{\bar{z}' - \bar{w}} \right) \langle n(z') \rangle$$

$$\Rightarrow \gamma(S) - \gamma(0) = -2\pi \int_{\text{interior}} d^2z' \; \langle n(z) \rangle$$



$$= -\oint d\underline{w} \cdot \underline{a}(\underline{w}) / q$$

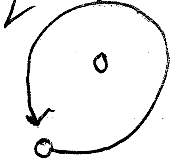
where $\nabla \times \underline{a} = 2\pi q \langle n \rangle$.

$\Rightarrow$ **As with vortices in a (super)-fluid, quasiholes experience particle density as an "Aharonov-Bohm" flux.**

$\underline{a}$ = 2D vector potential representing this flux.

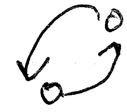
* In uniform fluid, density $\langle n \rangle = \bar{n} = \dfrac{\nu}{2\pi}$, quasihole experiences $\dfrac{1}{q}$ times mag field on ptcles.
  — consistent with fractional charge.

* If path encloses another qhole, we get in addition



$$\Delta\gamma = \frac{2\pi}{q}$$

Or for counterclockwise exchange, phase $\boxed{e^{i\theta} = e^{i\pi/q}}$



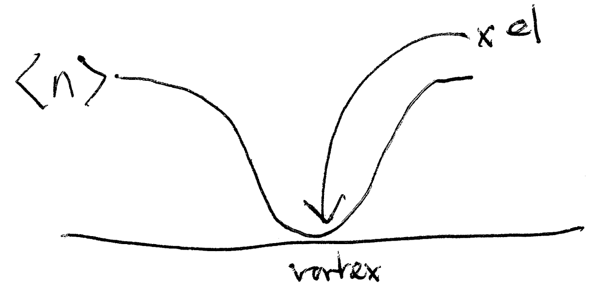
— <u>fractional statistics</u> if $q > 1$
(fermions if $q = 1$ ✓)

Composite Particles

Girvin + MacDonald 1987
N.R.
Jain

Start from beginning again

An electron and a vortex (quasi hole) are attracted and can bind.



in "any" fluid background

Similarly an electron and q vortices $(q = 1, 2, \ldots)$

Adiabatic transport calc" only used the av density so we still have:

* el + q vortices experiences mag field

$$1 - q\nu$$ times that for el.

* q vortices exchanged with q vortices gives phase

$$e^{i q^2 \pi / q} = e^{i \pi q}$$

So el + q vortices has stats $e^{i\theta}$

$$\frac{\theta}{\pi} = (q+1)\pi \pmod{2\pi}$$ (or $q\pi$ for bosons in place of els)

Hence:

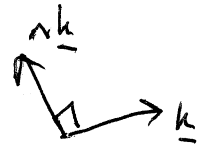
1) at $\nu = 1/q$, net effective mag field on composite is $\underline{\text{zero}}$.

2) composites are $\underline{\text{bosons}}$ (q odd)
$\underline{\text{fermions}}$ (q even)
(reverse if particles are bosons)

Energetics: if electron is displaced from center of (q-fold) vortices, energy increases.

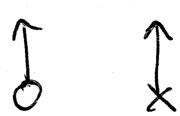
But in LLL, displacing el is done by $e^{i\underline{k}.\underline{R}}$ ($\underline{R}$ = el coord op) which displaces it by

$\wedge \underline{k}$,



($\because$ x and y are canonically conj in LLL)

El and vortices $E \times B$ drift due to potential int:



in parallel straight lines if $\nu = 1/q$.

Ie since effectively neutral, behaves like zero mag field!

Pot energy $V(\wedge \underline{k}) \simeq V_0 + \frac{k^2}{2m^*} + \ldots \to$ effective mass from ints.

Composite Bosons $\left(\nu = \frac{1}{q},\ q \text{ even}\right)$ (N.R., 1989)

Composites want to bind and have $\underline{k}=0$ to minimize energy. Therefore, form composite Bose condensate.

This is exactly the Laughlin state:

$\psi^+(z)$ = el creation op in LLL, then

$$|\Psi_L\rangle = \left(\int d^2 z\ \psi^+(z)\, U(z)^q \right)^N |0\rangle$$

$|0\rangle$: no particles

where $U(z)^q = \prod_i (z_i - z)^q e^{-\frac{1}{4}|z|^2}$ in first quantization

Like $\left(b^+_{\underline{k}=0}\right)^N |0\rangle = \left(\int d^2 r\ b^+(r) \right)^N |0\rangle$ — BEC.

$\psi^+ U(z)^q$ is Bose operator, has long-range order in Laughlin state.

Quasiparticles are vortices in this condensate. (phase of condensate changes by $\pm 2\pi$ around elementary vortex = Laughlin quasiparticle).

Finite energy $\Leftrightarrow$ Meissner effect $\Leftrightarrow$ incompressibility

Relation $\nabla \times a = 2\pi q \langle n \rangle$ implies fractional charge on vortices (Girvin).

Composite Fermions

At $\nu = \frac{1}{q}$, q odd (q even if particles are bosons)

composite fermions in zero effective magnetic field.

Two possibilities:

i) Fermi sea

2) pairing as in BCS superfluidity.

1) Fermi sea state (Halperin, Lee, N.R.) 1993

- no Meissner effect, so ~~compressible~~ fluid. No Hall plateau will be ~~observed~~.

- gapless fermion excitations at Fermi surface, k_F determined by (2D) density as usual (spinless particles here)

- gauge field crucial in response functions.

(surface acoustic wave expts: Willett)

2) return to paired state after $\nu \neq 1/q$.

Composite Fermions when $\nu \neq 1/q$: Jain states

(q even for els) (1989)

The effective mag field not zero. Suppose composite fermions occupy p LLs of the <u>effective</u> magnetic field

$$B_{eff} = B - B_{1/2} = \nabla \times (A - a)$$

(this has <u>nothing to do</u> with els in "true" higher LLs, which are at "high" energies).

Then state appears to be incompressible again.

These occur when $\quad \dfrac{1}{\nu_{eff}} = \dfrac{1}{\nu} - q = \dfrac{1}{p}$

$$\Rightarrow \quad \boxed{\nu = \frac{p}{qp+1}} \; , \quad p = \ldots, -1, \; 1, 2, \ldots$$

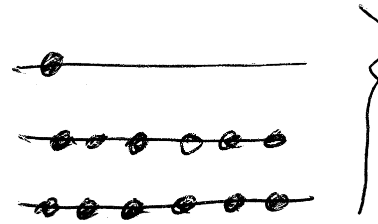
— Jain "leading sequence"

(odd denominators for el case)

(p negative means B_{eff} has opposite sign to B).

These states are essentially same as corres fraction in "hierarchy" theory, which can be related to comp bosons at $\nu \neq 1/q$ (q odd for els).

Excitations: leave a hole in one of p LLs or occupy an empty state



LLs of B_{eff}

– expect $\Delta \sim \frac{\hbar^2 \bar{n}}{m^*} \frac{1}{p} \to 0$ as $p \to \infty$ (q fixed)

(a little too naive)

However, while we viewed them "formally" as fermions, the <u>actual</u> excitations have

charge $\pm(1 - q\nu) = \frac{\pm 1}{qp+1}$

adiabatic statistics $\frac{\Theta}{\pi} = (q^2\nu - 1) = \frac{p'}{qp+1}$

where $p'(qp+1) = 1 \pmod{qp+1}$
$p' =$ odd.

so are <u>not</u> fermions except when $\nu = \frac{1}{2}$.

Paired QH states

A simple idea (Halperin 1983): if els form pairs, pairs are bosons, and can make Laughlin state of these charge 2 bosons

$\Rightarrow$ $\nu = \dfrac{2}{m}$ electron filling factor. $m = 1, 2, \ldots$

Quasiptle Excitations have charge $\frac{1}{m}$, fractional stats.
m = multiple of 4, violates "odd denominator rule".

Composite fermions at $\nu = \frac{1}{q}$, q even:

if they Cooper-pair like in BCS theory, the Meissner effect $\Rightarrow$ incompressible fluid.

Flux quantum halved by pairing

$\Rightarrow$ charge of quasiparticle $= \pm \dfrac{1}{2q}$.

as in Halperin type state.

Simple trial wavefunction for spin-polarized electrons based on this idea (Moore + N.R. 1991)
(inspired by Haldane-Rezayi (1988) spin-singlet)

$$\boxed{\Psi_{MR} = Pf\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^q e^{-\frac{1}{4}\sum_i |z_i|^2}}$$

q = even (fermions)
q = odd (bosons)

where Pfaffian is

$$Pf(M_{ij}) = M_{12} M_{34} M_{56} \cdots \pm \text{perms}$$

$$= \sqrt{\det M_{ij}} \quad \text{for } M_{ij} \text{ antisymmetric matrix}$$

Real-space form of BCS state is generally a Pfaffian (including spinors if necessary).

Two-Quasihole trial state (M+R 1991)

$$\Psi_{MR}^{+w_1, w_2} = Pf\left(\frac{(z_i - w_1)(z_j - w_2) + (w_1 \leftrightarrow w_2)}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^2$$

– each qhole acts on one member of pair $\leftrightarrow \frac{1}{2}$ flux quantum (in agreement with total N_ϕ).

For $q=1$, ground & quasihole states are zero-energy eigenstates of three-body Ham:

$$H_3 = V \sum_{i<j<k} \delta^2(z_i - z_j)\,\delta^2(z_j - z_k)$$

Greiter, Wen, Wilczek
1991

which generalizes to $q>1$ also.

Numerical calc^{ns} on these Hams strongly indicate incompressibility (N.R. + Rezayi 1996)

Numerical work on Coulomb interaction strongly suggests the ground state at $\nu = \frac{5}{2}$ (LLL filled with both spins) is in this phase

Morf (1998)
Rezayi + Haldane (2000)

NonAbelian statistics

Within BCS mean-field theory of pairing (applied to composite fermions), for complex p-wave gap function in two dim

BCS gap function $\Delta_{\underline{k}} \sim (k_x - ik_y)$ as $|\underline{k}| \to 0$

one finds (N.R. + Green, 2000)

* two phases — "weak" and "strong" pairing — with a transition between

* in weak-pairing phase, real-space ground state wavefunction has pairing function

$$g(z_i - z_j) \sim \frac{1}{z_i - z_j} \quad \text{for } |z_i - z_j| \to \infty$$

as in MR state

(exp' in strong-pairing phase

* well-separated vortices carry a fermion zero-mode in weak-pairing, but not in strong pairing phase.

These zero-modes can be described by Majorana (real) fermion ops, one localized at each vortex ie operators γ_i, action localized in packet at vortex w_i, with

$$\{\gamma_i, \gamma_j\} = \delta_{ij}$$

$\Rightarrow$ n well-separated vortices have degeneracy of $2^{n/2-1}$ states (for fixed N)

(form $\psi_i = \gamma_{2i} + i\gamma_{2i-1}$
$\psi_i^+ = \gamma_{2i} - i\gamma_{2i-1}$,
canonical anticomm rel^{ns})

(Note n even if we fix b.c.'s at ∞)

Same degeneracy is found in trial states, exact for three-body int. Nayak + Wilczek 1996

N.R. + Rezayi 1996

Adiabatic transport leads to a matrix operation in this space — predicted by Moore + N.R. 1991

see also Nayak + Wilczek
Gurarie, Flohr, Nayak
Tserkovnyak + Simon

— non Abelian stats