BOULDER SUMMER SCHOOL LECTURE: INTRODUCTION TO HYDRODYNAMICS

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1. The two main calculations for today

The last lecture has been more words than equations. Today we are going to do some paradigmatic calculations. We'll start with -one at high Reynolds number and one at low Reynolds number –which represent critical ideas for the subject of fluid mechanics. The first calculation is arguably the most important applied-mathematical idea of the twentieth century (at least I would argue it), and it is Prandtl's discovery of the boundary layer. The second calculation is more prosaic, but it does come up a lot–and illustrations a sort-of application of Prandtl's main mathematical idea at low Reynolds number. We will use the third calculation to present some basic ideas for how to extract meaning (and scaling laws) from the equations themselves.

1.1. Ludwig Prandtl and Boundary Layer Theory.

The basic ideas of boundary layer theory were invented by Ludwig Prandtl, in what was arguably the most significant contribution to applied mathematics in the 20^{th} century. Prandtl presented his ideas in a paper in 1905, though it took many years for the depth and generality of the ideas to be appreciated.

In order to give some historical context for Prandtl's discovery, it must be first understood that at the time *the* technology of the day was man -made flight. The Wright brothers were flying their airplanes, and one can only imagine the excitement in the scientific community for figuring out the mathematical principles underlying these developments.

However, there was a bit of an embarrassment for the theorists: (to paraphrase Lord Rayleigh's **1915** review of a hydrodynamics textbook in *Nature*–"Someday we might hope that theoretical hydrodynamics might be in agreement with experiments." The reason for this negativism was that as of the early 1900's, theories of airplane flight predicted

(1) The only way to get an airplane off the ground is to spin it as it lifts off. However, practically speaking, this is nonsense.
(2) Once it gets off of the ground, the drag should be identically zero. This is also nonsense–fuel is burned!

It turned out that both of these issues were the result of a fundamental theoretical issue: a perfectly reasonable estimate of the dissipation term in the equations of fluid dynamics led to the conclusion that the dissipation was unimportant, and the aforementioned conclusions were made with this in mind. However, Prandtl demonstrated that just because one estimated the dissipation to be small doesn't mean it can be neglected–hence, the idea of boundary layers.

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In what follows, we will go through the historical development to both illustrate Prandtl's discovery, and document how boundary layer theory dramatically simplifies the solution to nonlinear partial differential equations.

1.1.1. *Preliminaries.* To proceed further into airplane flight, we need some fluid mechanical preliminaries.

We introduced before the Navier Stokes equation

$$\rho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla p + \nu\nabla^2\mathbf{u} \tag{1}$$

. We also pointed out that in the airplane that I am now sitting, the Reynolds number is enormous and hence we are safe in neglecting the viscous terms. We thus have

$$\rho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla p. \tag{2}$$

This equation can be written in an interesting form using the vector identity

$$\mathbf{u}\cdot\nabla\mathbf{u} = (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla(\mathbf{u})^2/2 \tag{3}$$

as

$$\rho(\partial_t \mathbf{u} + (\nabla\times\mathbf{u})\times\mathbf{u}) = -\nabla(p/\rho + \mathbf{u}^2/2). \tag{4}$$

If the flow is steady $\partial_t\mathbf{u} = 0$ and irrotational $\omega = \nabla\times\mathbf{u} = 0$ then we have *Bernoulli's law*

$$H = \frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2 \tag{5}$$

is constant. If the flow is not irrotational, then we have $\mathbf{u}\cdot\nabla H = 0$ so that the quantity H is constant along streamlines.

The constancy of H when $\omega = 0$ is a famous result, and has many simple qualitative consequences. It states that the pressure in a fluid is smaller when the velocity is larger. The classical example of this is flow through segments of pipe of varying diameter. Whenever the radius in the pipe drops, the pressure in the pipe increases.

However, for the moment, let's question this theorem: Clearly, the constancy of H only makes sense if it actually happens in practice. For it to happen in practice, it needs to be true that $\omega = 0$ for all time. This leads to the question:

If $\omega = 0$ initially in a flow, does that imply that $\omega = 0$ for all time?

1.2. **Potential Flow.** What are the consequences of $\omega = 0$? The most important is that in this case, it is possible to describe the flow by a velocity potential ϕ. Namely, there exists a scalar function ϕ so that changes in ϕ are given by

$$d\phi = \sum_i u_i dx_i. \tag{6}$$

The velocity $\mathbf{u} = \nabla\phi$. The condition for this to be possible is that

$$\frac{\partial^2\phi}{\partial x_1\partial x_2} = \frac{\partial^2\phi}{\partial x_2\partial x_1}, \tag{7}$$

or in other words $\omega = 0$. Conversely, if $\mathbf{u} = \nabla\phi$ then $\omega = 0$.

Now, the wonderful thing about this is that if the flow is incompressible, then since $\nabla \cdot \mathbf{u} = 0$, we have $\nabla^2\phi = 0$! The potential obeys Laplace's equation. What about the pressure? Well, we can write from the Euler equation

$$\partial_t\phi + (\nabla\phi)^2/2 = -p/\rho, \tag{8}$$

so that at each instant the pressure is given in terms of the velocity potential.

1.3. **Kelvin's Theorem.** This result is clearly only relevant if a fluid which is initially vorticity free remains this way for all time. That this is in fact true was first shown by Lord Kelvin: Consider the *circulation* around a closed loop

$$K(t) = \int \mathbf{u} \cdot d\ell, \tag{9}$$

where $d\ell$ is an element of arc length. By Stokes's theorem, $K(t) = \int \omega dA$. Thus, if $\omega = 0$, $K = 0$. What is the time evolution of K?

Let us introduce the notation

$$\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla.$$

This is called the total derivative and takes into account the rate of change of something due to both its explicit time dependence and its motion (through the laboratory coordinates where we are sitting).

The chain rule implies that

$$\frac{DK}{Dt} = \int (\frac{Du}{Dt} d\ell + u\frac{D\ell}{Dt}), \tag{10}$$

where $Dd\ell/Dt = \partial_\ell u d\ell$ so that

$$\mathbf{u} \cdot \frac{D(d\ell)}{Dt} = \mathbf{u}\partial_\ell \mathbf{u} dl = d(\mathbf{u}^2)/2. \tag{11}$$

Since integrating around the loop leaves u^2 unchanged, the second term is zero. As for the first term, it is easy to argue that it also integrates to zero using the Euler equation itself. thus, we have that

$$\frac{DK}{Dt} = 0. \tag{12}$$

Since this is true for every loop, it shows that a fluid with initially zero vorticity remains zero for all time.

One of the most interesting things about this theorem is its historical origin: Kelvin viewed it as the basis of his vortex theory of the atom!

One historical comment: imagine you were working in the end of the 19th century: you now have the wonderful situation where the most important technological problem of the day, air plane flight, is related to the most beautiful mathematical problem–potential flow, and two dimensional potential flow (conformal invariance and all) at that! Imagine how excited you would be..

1.3.1. *Drag on a wing section.* Lets now make a model for the drag and lift on a wing section. For simplicity lets consider a wing as an infinite two dimensional body with constant cross section. I'm looking out the wing of the airplane to Boulder as I'm writing this, and this is clearly not true-but it is a reasonable model. For simplicity, lets also pretend that the wing is a circle. (we will relax this in a bit). From the approximations given–Euler equation/potential flow, lets compute the lift and the drag.

Initially when the airplane is taking off, there is no vorticity in the flow. The airplane is just sitting at rest. So this must always be true.

Now lets take the wing as circle of radius R. Let's say the flow is uniform (with magnitude V) in the x direction, as $|x| \to \infty$. In polar coordinates, Laplace's equation is

$$\frac{1}{r}\partial_r(r\partial_r\phi) + \frac{1}{r^2}\partial_\theta(\partial_\theta\phi) = 0. \tag{13}$$

A uniform flow has $\phi = Vrcos(\theta)$ At the surface of the circle, we have the boundary condition that the normal component of the velocity is zero: $\partial_r\phi(r = R) = 0$. Let's find the solution of Laplace's equation consistent with these conditions.

To proceed, let's guess a solution to Laplace's equation of the form $\phi = f(r)g(\theta)$. Plugging this into Laplace's equation gives

$$r(rf')'/f + g''/g = 0.$$

The first term in this formula depends solely on r and the second term solely on θ. Hence both terms must be constants: Namely, $r(rf')' = cf$ and $g'' = -cg$. The first equation has solutions of the form $f(r) = r^\alpha$, so that $c = \alpha^2$. This implies that $g(\theta) = Asin(\alpha\theta) + Bcos(\alpha\theta)$. Since g must be single valued, α must be an integer. (Call it n). Hence, the most general solution is

$$\phi = \sum_n (C_n r^n + D_n r^{-n})(A_n cos(n\theta) + B_n sin(n\theta)).$$

We need this to satisfy the boundary conditions. For $\phi \sim x$ as $x \to \infty$ we need the highest power of r in the expansion to be $n = 1$. That is

$$\phi = Vrcos(\theta) + \sum_n r^{-n}((A_n cos(n\theta) + B_n sin(n\theta))).$$

To find the other A_n and B_n we need to satisfy the boundary condition $\partial_r\phi(r = R) = 0$. This implies that

$$Vcos(\theta) - A_1 cos(\theta)/R^2 + \sum_{n>1}(A_n(cos(n\theta) + B_n sin(n\theta)))\frac{-n}{R^{n+1}}.$$

This is satisfied only if $A_1 = VR^2$ and $A_n, B_n = 0$ for $n > 1$. Hence the solution is

$$\phi = (Vr + V\frac{R^2}{r})cos(\theta).$$

Using the solution we just computed, let's figure out the force on the sphere: The velocity components at the sphere are $u_r = 0$ and $u_\theta = -2Vsin(\theta)$. By Bernoulli's law, the pressure is $p = A - 1/2\rho(u_r^2 + u_\theta^2)$. Let's fix the constant $A = V^2\rho/2$ so that the pressure is zero far from the circle. Hence the pressure is $p = 1/2\rho(V^2 - V^2 sin(\theta)^2)$. The pressure is maximal in front and back of the cylinder, and vanishes at the apex.

Now note the interesting result:

The net force on the cylinder vanishes, because the pressure is symmetrical about the equatorial plane of the cylinder. Thus, a cylinder in uniform motion in the fluid experiences no force whatsoever! This result has been known as *D'Alembert's paradox*.

Note something interesting about our solution: At the surface of the wing, the flow does not obey the no-slip boundary condition, since u_θ is not zero at the surface of the cylinder.

1.4. **Solving more arbitrary wing shapes.** There is a beautiful bit of early twentieth century applied mathematics that repeats this analysis and demonstrates that it is true for *arbitrary* wing cross sections, even those that have singular tips (like those on a real wing). The mathematics works using a little bit of complex analysis and conformal mappings. Briefly, we introduce a useful devise for thinking about two dimensional flows. Let's write $(u_x, u_y) = (\partial_y \psi, -\partial_x \psi)$, where ψ is called the *stream function* of the flow. This velocity field automatically satisfies the two dimensional incompressibility condition. Note that

$$\mathbf{u} \cdot \nabla \psi = u_x \partial_x \psi + u_y \partial_y \psi = 0.$$

Thus, ψ is constant along streamlines of the flow.

Now, besides it's physical convenience, another great thing about the stream function is the following: by definition

$$u_x = \partial_x \phi = \partial_y \psi \quad u_y = \partial_y \phi = -\partial_x \psi.$$

The second of the equations in each pair are the well known *Cauchy Riemann* equations of complex analysis! This allows us to use the power of complex analysis to think about two dimensional potential flow problems.

Namely, $w = \phi + i\psi$ is an analytic function of $z = x + iy$ (i.e. depends only on $z = x + iy$.) Using this observation, it is possible to generalize the result for the circular wing to arbitrary shape wings–the lift and drag on the wings simply vanish.

This was a tremendous embarassment for early 20th century theorists. At the dawn of quantum mechanics we couldn't even compute a nonzero drag for an airplane–this despite the fact that fuel was clearly being burned!

1.4.1. *Problems and Resolutions.* We have thus found two crucial problems with our solutions

1 They did not satisfy the no slip boundary condition known to be true at solid boundaries.
2 There was no drag on our objects. Although I didnt have time to show this, one could show that there could be lift but only if circulation was induced around the object initially. However, according to Kelvin's theorem, production of circulation is not possible.

Now we will present solutions to both of these problems, by first, focusing on the first question.

The issue, as we have remarked several times, is our *initial estimate for the importance of the viscosity*: In fact, the beautifully simple calculation that we presented at the beginning of this lecture turns out to be dead wrong. The reason is subtle, but it is one that crops up again and again. Lets review our argument:

We argued that the Reynolds number is enormous, and hence the flow should not be affected by viscosity, and we neglected it; writing down inviscid theories. Although it was by no means obvious at the time, it turns out that all of our troubles can be solved by thinking more carefully about viscosity. Is it really negligible?

The answer to this question, as first given by Prandtl in 1904, is *no*. Prandtl hypothesized that the flow consists of two regions. A region far from the solid body where viscosity really is negligible. And a region close to the body where viscosity is very important. The reason, Prandtl said, that viscosity can be important in a flow with a small viscosity is that it is only important in a layer of very thin thickness. Let's call this thickness δ. Assume the characteristic velocity of the (inviscid) flow is named U_0, and the characteristic scale over which it varies is L. Then the idea is that $\delta \ll L$ in the boundary layer, which makes viscous stresses important.

1.4.2. *The beginning.* For simplicity let's confine ourselves to two dimensional steady boundary layers. Let's suppose that the wall occurs at $y = 0$ and that the variation is along the x coordinate. It is okay if the wall is curved, as long as it isn't so curved that the characteristic curvature is much greater than the characteristic length δ of the boundary layer. We will denote the velocity field by (u, v), where u is going to vary on a scale L in the x direction [1] and v varies across the boundary layer on a scale δ.

The continuity equation for the flow is

$$\partial_x u + \partial_y v = 0. \tag{14}$$

Let's estimate the size of $\partial_x u$. Since u varies on a scale L, this is of size U_0/L. Similarity, $\partial_y v \sim V/\delta$, where V is the size of velocity in the y direction. Hence we have found that

$$V \sim U_0 \frac{\delta}{L}.$$

The y velocity is much smaller than the x velocity!

Onwards to the navier stokes equation. The x component is

$$u\partial_x u + v\partial_y v = -\partial_x p + \nu(\partial_x^2 u + \partial_y^2 u). \tag{15}$$

The y component is

$$u\partial_x v + v\partial_y v = -\partial_y p + \nu\nabla^2 v. \tag{16}$$

Estimate sizes of terms: In the first equation $u\partial_x u \sim U_0^2/L$. $v\partial_y u \sim U_0 V/\delta \sim U^2/L$. The important point is that $\nu\partial_y^2 u \gg \nu\partial_x^2 u$, since the former is of order $\nu U_0/\delta^2$. Thus we neglect that derivatives of u in the x direction in the viscous stresses. What about the pressure gradient? Since it is causing the flow, we expect it will be of order the inertial terms, and so it is not negligible.

Now you should see the point of the boundary layer: it is not legitimate to neglect the viscous term because ν is small because this could be compensated as long as δ is small as

[1] Of course, u will also vary in the y direction. We need to solve for this dependence.

well! In fact, if

$$\nu U_0/\delta^2 \sim U_0^2/L \tag{17}$$

then these two terms are exactly of the same size. Indeed, we will argue that this is exactly what happens.

Before doing this, let's look at the y equations. You can convince yourself that the biggest term in this equation is $\partial_y^2 v$ so to leading order we should just set this equal to zero. This implies that $\partial_y v = f(x)$. By the continuity equation, we know that $f(x) = -\partial_x u$, so that $v = -y\partial_x u$. We will not really use this form in what follows but it is good to keep in mind.

The equations we must solve for the structure of the boundary layer are

$$\mathbf{u}\partial_x u + v\partial_y u = -p(x)'/\rho + \nu\partial_y^2 u, \tag{18}$$

and the continuity equation. That the pressure $p(x)$ depends only on x comes from the following reasoning: far from the boundary layer, the flow is parallel to the plate, and the pressure is just $p(x)$. Near the plate there is indeed a pressure variation induced by the y component of the flow but since the y component of the flow is small this correction to the pressure will be small. And moreover, the pressure in this equation will be given by the pressure variation far away from the plate!

Far from the plate of course, viscosity is not importnat, there are no substantial variations of the flow in the y direction, and the equation for the velocity field $u_0(x)$ is just

$$u_0 u_0' = -p'. \tag{19}$$

So the pressure far away can be represented in terms of the velocity far away. Our goal is now to solve these equation under the conditions that the velocity vanishes at the solid wall $(y = 0)$ and goes to u_0 far away from the wall.

1.4.3. *Back to boundary layers.* It is clear that since the equation for the boundary layer depends on U_0, there will be many different cases for the types of boundary layers that can occur. Roughly speaking, one can divide them into three cases, depending on the pressure gradient:

(1) $dp/dx > 0$. Here the external flow is decelerating.
(2) $dp/dx < 0$. External flow is accelerating.
(3) $dp/dx = 0$ no acceleration.

It turns out that the first situation is rather unstable, and the second is rather stable.

Let's first do the simplest case first, called the Blasius boundary layer in which $p = 0$, and the velocity asympotes to a constant far away. We will consider a situation where in the left half plane there is no solid, and in the right half plane there is an infinite solid at $y = 0$. The question is what does the boundary layer look like in the right side.

We will sketch a few high points of the derivation:

The pressure far from the boundary layer is uniform. Thus, we need to solve the equations

$$u\partial_x u + v\partial_y u = \nu\partial^2 u \tag{20}$$

$$0 = \partial_x u + \partial_y v, \tag{21}$$

with a boundary layer for $x > 0$ (where the plate exists). What will the structure of the boundary layer be? Before going through the derivation let's anticipate. We have already emphasized that the length scale in the x direction is L and the length scale in the y direction is δ, where balancing the viscous term against the others we find that $\delta = \sqrt{\nu L/U_0}$. Now, in the present case, there IS no horizontal length scale L! What does dimensional analysis say?

The quantity u/U_0 must be a function of x/L and y/δ, which is independent of L. The only combination that has this property is $y/\delta/\sqrt{x/L} = y\sqrt{U_0/(\nu x)}$. Hence, we anticipate

$$u = U_0 F(y\sqrt{U_0/(\nu x)}). \tag{22}$$

Let's check that this actually comes out of the calculation. Guess that $u = U_0 F(y/g(x))$. To satisfy continuity, take a stream function $(u, v) = (\partial_y \psi, -\partial_x \psi)$, so that $\psi = Ug(x)\int^\eta F(\eta) + k(x)$. For the plate itself to be a streamline, we need $k = 0$. Let's express $\psi = g(x)f(\eta)$, with $f(0) = 0$. Now plug this into the boundary Layer equations, and proceed.

1.4.4. *Computing Drag.* Now that we have the correct structure near the flat plate, we can return to the strange problems we were having with the drag. The stress on the plate is just

$$t_x = \sigma \cdot (0, 1, 0) = \mu(\partial_y u + \partial_x v)_{y=0} = \mu \partial_y u = \mu U (U/(2\nu x))^{1/2} f''(0).$$

Hence, the total drag on the plate is $2\int t_x = 2\sqrt{2} f''(0)\rho U^2 L/\sqrt{R}$.

Note two features: (a) The drag scales with $\sqrt{L}$, NOT L as you might have guessed. (b) as $\nu \to 0$, the drag is supposed to vanish.

1.5. **A Low Reynolds number Example: Lubrication Theory.** Let's consider a droplet spreading on a solid surface. Assume the liquid is very viscous and that it therefore spreads very slowly. We want to find out how it moves.

In this limit, the fluid obeys the Stokes equations, namely

$$\eta \nabla^2 \mathbf{u} = \nabla p, \tag{23}$$

together with $\nabla \cdot \mathbf{u} = 0$. Lets parameterize the droplet shape by a function $h(x, t)$. Typically in such problems, the thickness of the droplet is much smaller than its extent–ie $h \ll R$, where R is the radius of the droplet. In such a case, we can use an approximation, similar to that of Prandtl, to derive a tractable equation of motion. Indeed, we can use it to derive the equations for droplets as they spread.

Here it goes, as a sketch. Lets call the components of the velocity field (u, v), respectively. The continuity condition then becomes

$$\partial_x u + \partial_y v. \tag{24}$$

If we call the typical horizontal velocity U and the typical vertical velocity V, this equation then implies that

$$V \sim \frac{h}{R} U,$$

so that vertical velocities are smaller than horizontal velocities by a factor h/R.

Now lets write out the two components of the Stokes equations:

$$\eta(\partial_{xx} + \partial_{yy})u = \partial_x p, \tag{25}$$

and

(26) $$\eta(\partial_{xx} + \partial_{yy})v = \partial_y p.$$

Now since $v \ll u$, the second equation implies that $\partial_y p \ll \partial_x p$. We therefore can assume that at first approximation, the pressure in the droplet $p \approx p(x)$; ie , it does not depend on the vertical (y) direction.

Now lets turn to 25: because of the scale separation, the $\partial_{xx} \ll \partial_{yy}$. Thus we have that

(27) $$\eta \partial_{yy} u = \frac{dp}{dx}.$$

Since dp/dx is only dependent on x we can integrate the equation twice with respect to y, to arrive at

(28) $$u = \frac{1}{\eta}\frac{dp}{dx}\left(\frac{y^2}{2} + Ay + B\right).$$

Here A and B are constants. We can determine then using boundary conditions. No slip at the solid surface ($y = 0$) implies that $B = 0$. If we require there is no tangential stress at the top surface $y = h$, this implies that $\partial_y u(y = h) = 0$, which means that $A = -h$. We thus have

(29) $$u = \frac{1}{\eta}\frac{dp}{dx}\left(\frac{y^2}{2} - hy\right).$$

Now to proceed and arrive at an equation, we compute the total flux in the x direction, given by

(30) $$Q = \int_0^h dy\ u = -\frac{h^3}{3\eta}\frac{dp}{dx}.$$

Mass conservation

(31) $$\partial_t h = -\partial_x Q$$

then implies the equation of motion

(32) $$\partial_t h = \partial_x\left(\frac{h^3}{3\eta}\frac{dp}{dx}\right).$$

Other remarks:

(1) What sets the pressure? For droplets this is generally a combination of surface tension and gravity. For surface tension, there is a boundary condition across the fluid interface that says the pressure jump is the product of the surface tension and mean curvature. This gives that $p = \gamma \partial_{xx} h$.
(2) Including gravity, assuming that the droplet is spreading on a horizontal substrate, means that the pressure is $p = \gamma \partial_x x h + \rho g h$.

Including these two effects we arrive at the equation

(33) $$\partial_t h = \partial_x\left(\frac{h^3}{3\eta}(\gamma h''' + \rho g h')\right).$$

1.6. **Analyzing this in detail.** Let's analyze this equation in detail. It might not seem like such a great simplification–in that we are now left with a nonlinear partial differential equation to solve. However, using much the same philosophy we have been discussing one can understand it in essentially quantitative detail. This was first done, masterfully, by de Gennes, and what we are about to do is essentially a rehashing of some of his work.

1.6.1. *Dominant Balance.* The equation we have derived has three terms–the time derivative on the left hand side, a term representing capillarity on the right and a term representing gravity. Let's compare the sizes of the gravity and capillary terms. We need to estimate the ratio

$$\frac{\gamma h'''}{\rho g h'}.$$

If we say that L is the horizontal length scale governing the droplet, this ratio is just

$$\frac{\gamma}{\rho g L^2}.$$

This means that if $L > \ell_{cap}$, where

$$\ell_{cap} = \sqrt{\frac{\gamma}{\rho g}}$$

is the so-called capillary length, then gravity dominates the spreading behavior of the droplet (and we can, at least subject to the caveats we have discussed, delete surface tension from the equation). If L is smaller than ℓ_{cap} surface tension dominates and we can delete gravity. The spreading of small droplets is governed by surface tension and the spreading of large droplets is governed by gravity.

1.6.2. *Small Droplets.* Let's analyze the behavior when the droplet is small and surface tension dominates. We will just delete the gravitational term so our equation is

$$\partial_t h = \partial_x \left(\frac{h^3}{3\eta} (\gamma h''') \right). \tag{34}$$

What is the spreading law? Since the equation here is two dimensional and real droplets are three dimensional, we may as well write down the radially symetric version of the equation that governs three dimensional spreading. This is

$$\partial_t h = \frac{1}{r} \partial_r \left(r \frac{h^3}{3\eta} (\gamma h''') \right). \tag{35}$$

Now lets assume that the droplet spreads with a maximum height $H(t)$, a typical radius $R(t)$. If we then use this in the equation we find

$$\frac{dH}{dt} \sim -\frac{\gamma}{\eta} \frac{H^4}{R^4}. \tag{36}$$

I've put in a minus sign because this equation is 'diffusive' and hence the droplet will clearly spread outwards.

To close the equation, we need to use one more equation, that of volume conservation. Namely we know that the equation preserves

$$\int_0^\infty r dr h(r)$$

This means that

$$\pi H R^2 = V.$$

Here V is the droplet volume. Using this in our equation gives

$$\frac{dH}{dt} \sim -\frac{\gamma}{\eta}\frac{H^6}{V^2}. \tag{37}$$

Thus,

$$H(t)^{-5} \sim \frac{\gamma}{\eta V^2} t,$$

Or, $H(t) \sim t^{-1/5}$. And the radius spreads like $R(t) \sim t^{1/10}$. These are the so-called *Tanner's laws*

Things we will do:

(1) Show how to construct similarity solution for this solution. Discuss the fact that the solution does not allow compactly supported droplets.
(2) Discuss gravity, in same spirit.
(3) Discuss small time and large time asymptotics. Discuss validity of scaling laws in different regimes.

2. Point force electrostatics for low Reynolds number flow

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