Boulder notes by Victor V. Albert

I. NICK READ

I.1. What is an equilibrium phase of matter?

In order to be able to distinguish different quantum phases of matter, we want a characterization that is independent of Hamiltonian details, thermodynamic quantities, or even constituents. In such a characterization, phases should remain unchanged if we do not change parameters too much.

- Ex: From this point of view, liquid and gas are the same phase (fluid) since you can go between them continuously without going through a phase transition.

We assume that we are dealing with systems with many local degrees of freedom (e.g., many particles on sites in a lattice or many spins). As a result, the Hamiltonians will consist of sums of local terms (short range) and will often be translation invariant.

Suppose we have a Hamiltonian with a symmetry holding throughout parameter space. This symmetry may spontaneously break in different ways in different regions in parameter space. If there is a symmetry breaking, then such regions are separated by phase transitions.

- Ex: When going from a liquid to a solid, translational and rotational symmetries spontaneously break.

It used to be thought that the converse is true: if phases cannot be continuously connected, then there is some symmetry that is spontaneously broken. This is wrong, as evidenced by several challenges: the BKT transition in 2D XY model, spin glasses, metal-insulator transition, and I/FQHE. So we come to the notion of topological phases.

I.2. Topological phases

A **topological phase** is any quantum system at $T = 0$ for which there is an energy gap for creating bulk excitations above the ground state in the thermodynamic limit.[1] It is generally believed that sufficiently small perturbations to such a phase do not close the gap. A **topological property** is any property of the system which is the same throughout a topological phase. Some example properties include:

1. Existence of bulk gap.

2. Multiplicity of ground states of Hamiltonian in phase on space of non-trivial topology (**topological order**; Wen/Niu 1990).

3. Quasiparticle excitations with non-trivial statistics (Moore/Read 1991).

4. Robust gapless edge excitations (Wen 1990).

5. Quantized transport properties (Laughlin 1980).

6. "Non-trivial" entanglement (ground state is non-deformable to a product state using local unitaries).

A convenient/useful way of studying topological phases is using an **effective field theory** (i.e., integrating out high-energy degrees of freedom). In bulk, such a field theory consists of local terms and invariance of topological properties follows since the gap is preserved under RG. Often the field theories that emerge are **topological field theories** (Witten, 1989).

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[1] This definition does not include gapless phases — systems for which there are low-energy excitations (e.g., Fermi liquids or systems with spin wave excitations).

I.3. Basic notions

Local operators at region x,

$$O_x = ... \otimes I \otimes I \otimes O_x \otimes I \otimes I \otimes ... \,,$$

commute if they are well-separated (i.e., they are bosonic). Take a torus, for which a non-trivial topological phase has degenerate states $|\alpha\rangle$. For such phases,

$$\langle\alpha|O_x|\beta\rangle \propto \delta_{\alpha\beta} \,. \tag{1.1}$$

If this were not true, adding the term $\lambda \int d^d x O_x$ to the Hamiltonian splits the degeneracy between ground states. But we assumed that any splittings that could be split have already been split, so in a topological phase, you cannot tell whether you are in $|\alpha\rangle$ or $|\beta\rangle$ by looking at local operators.

- In the symmetry-breaking paradigm, symmetry-broken ground states are not in a topological phase because a local order parameter can distinguish them.

This is true for finite systems, up to corrections exponential in system size. The gap property implies that correlations between local operators decay exponentially in separation between them.

I.4. Quasiparticles

Assume 2D space. "Twist" ground state about a point (using non-local twist operator), making a linear combination of eigenstates such that it locally looks like the ground state far away from the point α. Such a procedure makes a **quasiparticle** α.

- You can detect the quasiparticle by a local operator at α and also move it around. Quasiparticles were not created by local operators, however, so they are robust.
- **Quasiparticles of the same type** are those which are mapped to each other by local operators, and can be distinguished by measuring local operators. A trivial twist (doing nothing) is a trivial quasiparticle.
- Since the twist operator is invertible, there exists an antiparticle $\bar{\alpha}$ which is just action of the inverse twist on the ground state. A particle and antiparticle can come together and annihilate to trivial particle $\alpha = 0$. Conversely, a local operator can create a particle and quasiparticle at a point, which can then be moved away from each other.
- Two quasiparticles α and β look like one single object from far away (they are viewed as one composite type of twist you perform on the ground state), i.e., are equivalent to a single quasiparticle of some type γ (**fusion**).
- **The state of the system is not fully determined by specifying the number and position of particles.** This state cannot be determined by local measurements.

II. NICK READ 2

II.1. Fusion rules and degeneracy

As mentioned above, quasiparticles α and β can fuse to γ in $N_{\alpha\beta} \in \{0, 1, 2...\}$. Let's represent particles with symbols ϕ_α, and represent the product by the **fusion rule**

$$\phi_\alpha \times \phi_\beta = \sum_\gamma N^\gamma_{\alpha\beta} \phi_\gamma = \phi_\beta \times \phi_\alpha \,.$$

This gives us the structure of a commutative ring with identity being the trivial particle $\alpha = 0$ with $\phi_0 = I$:

$$\begin{aligned} 1 \times \phi_\alpha &= \phi_\alpha \text{ with } N^\beta_{\alpha 0} = \delta_{\alpha\beta} \\ \phi_\alpha \times (\phi_\beta \times \phi_\gamma) &= (\phi_\alpha \times \phi_\beta) \times \phi_\gamma \\ \phi_\alpha \times \phi_{\bar{\alpha}} &= I + \sum_{\gamma \neq 0} N^\gamma_{\alpha\bar{\alpha}} \phi_\gamma \end{aligned}$$

If $\sum_\gamma N^\gamma_{\alpha\beta} > 1$ for some α, β, then we get non-Abelian statistics. If $\sum_\gamma N^\gamma_{\alpha\beta} = 1$, then statistics is Abelian.

The state at infinity should be unchanged, so we need to make sure that fusing all anyons should give the trivial particle. We can draw a diagram to represent this fusion and use the diagram to calculate the degeneracy (going from right to left in F1):

$$\begin{aligned}
\text{fusing all particles to } \alpha{=}0 \text{ (the BC at infinity)} &= \phi_{\alpha_n} \times \phi_{\alpha_{n-1}} \times ... \times \phi_{\alpha_1} \times \phi_{\alpha_0} \\
&\quad \longrightarrow \text{starts and ends with vacuum} \\
&= \phi_{\bar{\alpha}_{n-1}} \times \phi_{\alpha_{n-1}} \times ... \times \phi_{\alpha_1} \times \phi_0 \\
&\quad \longrightarrow \text{fusion rules} \\
&= \sum_{\gamma_2,\ldots,\gamma_{n-1}} N^0_{\gamma_{n-1}\alpha_n} N^{\gamma_{n-1}}_{\gamma_{n-2}\alpha_{n-1}} \cdots N^{\gamma_2}_{\alpha_1\alpha_2} \\
&= \text{degeneracy of all states}
\end{aligned}$$

II.2. Dragging quasiparticles

We can now define statistics by adiabatically dragging well-separated qparticles around, exchanging them up to permutations of qparticles of the same type, and looking at the Berry phase/matrix. The Berry phase can in general depend on the path (i.e., background fields, which may not be distributed uniformly). However, a small variation of the path will only bring about a change in the phase which depends on the background only and **not** on any other qparticles. We can mod out the effects of such variations and organize paths into **isotopy classes of braids**. Isotopy classes are just (homotopy) classes of paths which can be smoothly deformed into each other, but with the restriction that the paths cannot cross.

Suppose now that all qparticles are of the same type. Then we can define braiding neighboring particles, $\tau_{i,i+1}$ and $\tau^{-1}_{i,i+1}$ (F2). These braids form a group, the **Artin braid group** B_n, which is generated by the $\tau_{i,i+1}$'s with the additional relations

$$\begin{aligned}
\tau_{12}\tau_{34} &\cong \tau_{34}\tau_{12} \\
\tau_{23}\tau_{12}\tau_{23} &\cong \tau_{12}\tau_{23}\tau_{12} \, .
\end{aligned}$$

If we stop caring about which line is on top in the diagrams, then $\tau_i^2 = 1$ and we just get the permutation group S_n.

Going back to quantum mechanics, to determine the effect of braiding, we just need to determine the unitary representation of $\tau_{i,i+1}$. In the simplest scalar irrep, $\tau_{i,i+1} = e^{i\theta}$ and these are the braiding rules of **Abelian anyons** (e.g., $\theta = 0, \pi$ gives bosons/fermions). For a finite number of qparticles, you wind up with rational θ.

For distinct quasiparticles, we also need to introduce an additional operator σ which represents one particle going around another of a different type.

Combining braiding rules with fusion rules needs to be consistent. For example, we need fusion and braiding to commute since they are isotopic (F3; see also Droplicher, Roberts, Frohlich 1980). The same structure occurs in rational conformal field theory (Moore, Seiberg 1987), in quantum groups, in modular tensor category theory (Resh, Tur 1990), in Chern-Simons gauge theory (Witten 1989), and in isotopy invariance of knots and links (Jones 1984).

II.3. Quantum Hall effect

The Hall effect occurs in 2DEGs (F4). One applies a current and measures voltage between two of the four ends, defining resistances

$$R_{xx} = \frac{V_{12}}{I} \qquad\qquad R_{xy} = \frac{V_{13}}{I} \, .$$

Classically, letting $\bar{n}$ be the density,

$$R_{xy} = \rho_{xy} = \frac{B}{\bar{n}ec} = \frac{B}{\bar{n}} \cdot \frac{e}{hc} \cdot \frac{h}{e^2} = \frac{B}{\bar{n}} \cdot \frac{1}{\Phi_0} \cdot \frac{h}{e^2}$$

Quantumly, there are plateaus for which $R_{xx} = 0$ and

$$\sigma_{xy} = \frac{1}{\rho_{xy}} = \nu \cdot \frac{e^2}{h} = \frac{\bar{n}\Phi_0}{B} \cdot \frac{e}{h} = \frac{N}{(AB/\Phi_0)} \cdot \frac{e}{h} = \frac{N}{N_\Phi} \cdot \frac{e}{h} \, .$$

II.4. Continuum Landau level problem

This is a single particle problem of an electron in a magnetic field:

$$H_1 = \frac{1}{2m_e}\left(\vec{p}-\vec{A}\right)^2 \qquad\qquad \vec{\nabla}\times\vec{A}=\vec{B}\,.$$

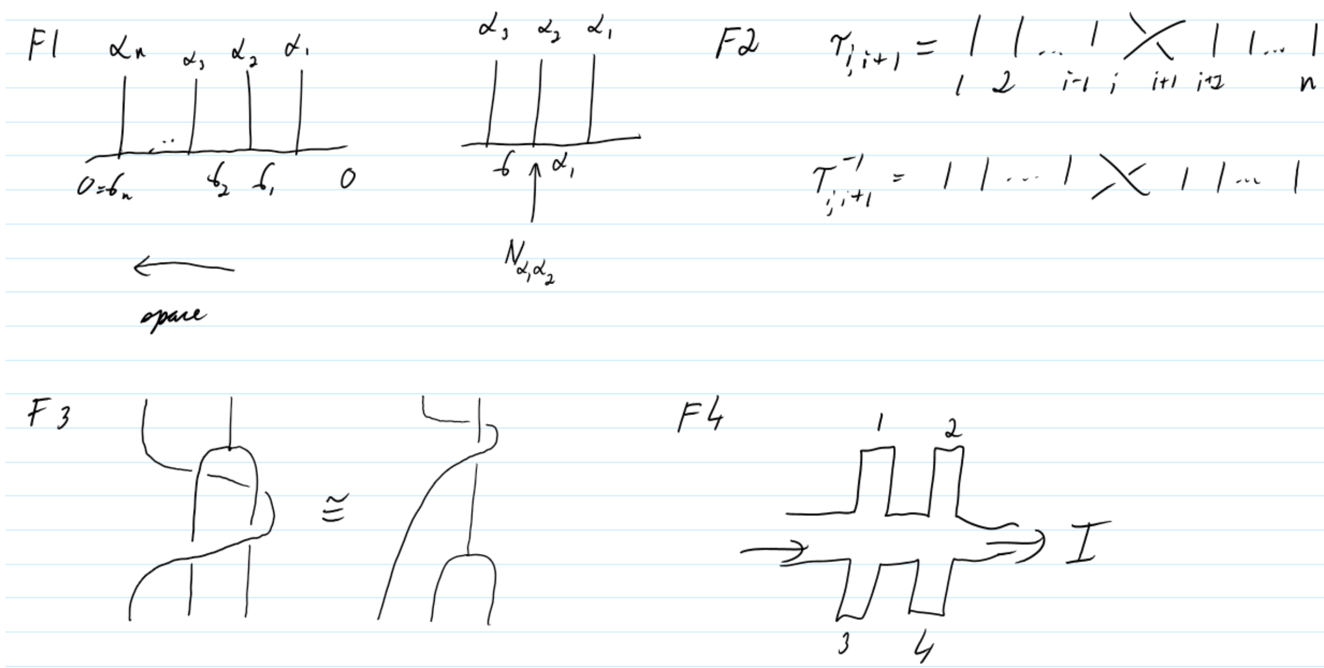
The eigenvalues of this are $E_n = \omega\left(n+\frac{1}{2}\right)$, same as a harmonic oscillator. In the symmetric gauge, $\vec{A}=\frac{1}{2}\vec{r}\times\vec{B}$ and each n has an infinite degeneracy with the $n=0$ case (the LLL) spanned by

$$u_m\left(z\right) = \frac{z^m}{\sqrt{2\pi 2^m m!}}e^{-\frac{1}{2}|z|^2}\,.$$

This $u_m\left(z\right)$ is peaked at $|z|=\sqrt{2m}$, so the number of states in the LLL per unit interval is $1/2\pi$. If we occupy the first ν Landau levels, then the density is uniform,

$$\bar{n} = \frac{\nu}{2\pi}\,,$$

and the filling factor ν from before is an integer.



III. NICK READ 3

III.1. Laughlin state and its excitations

Continuing with fractional QHE (FQHE), the most general LLL wavefunction is

$$\Psi\left(z_1,...,z_N\right) = f\left(z_1,...,z_N\right)e^{-\frac{1}{4}\sum_j|z_j|^2}\,,$$

where f is (anti-)symmetric for (fermions) bosons. Laughlin in 1983 proposed his wavefunction

$$\Psi_L\left(z_1,...,z_N\right) = \prod_{i<j}\left(z_i-z_j\right)^Q e^{-\frac{1}{4}\sum_j|z_j|^2}\,,$$

where Q is inverse of the filling factor (even for bosons, odd for fermions). We define

$$m_{max} = Q\left(N-1\right)\equiv N_\phi \qquad\qquad R=\sqrt{2N_\phi} \qquad\qquad \nu = \lim_{N\to\infty}\frac{N}{N_\phi+1}$$

and density

$$\bar{n} = \frac{\nu}{2\pi} = \frac{1}{2\pi Q}$$

is uniform. The normalization can be interpreted as a 2D plasma in a screening phase (which occurs for $Q \le 70$):

$$|\Psi_L|^2 = \exp Q \left[\sum_{i<j} \ln |z_i - z_j|^2 - \frac{1}{2Q} \sum_j |z_j|^2 \right] .$$

Since $0 \le m \le m_{max}$, the states form a droplet with uniform density.

- For $Q = 1$, we obtain a Slater determinant in the Ψ_L.
- There is a parent Hamiltonian in the LLL for which Ψ_L is its unique zero energy eigenstate.

Now let's consider excitations — fractionally charged quasihole quasiparticles:

$$\Psi_w = \prod_j (z_j - w) \cdot \Psi_L .$$

There are no particles at w. Taking the log of the normalization,

$$\ln \langle \Psi_w | \Psi_w \rangle = \sum_{i<j} \ln |z_i - z_j|^2 - \frac{1}{2Q} \sum_j |z_j|^2 + \frac{1}{Q} \sum_j \ln |z_j - w| ,$$

w acts as an impurity charge of $+\frac{1}{Q}$. Therefore, there is net deficiency of charge around this impurity of $-\frac{1}{Q}$. The droplet becomes a bit larger, and now the impurity charge will be transported to the edge of the droplet.

There are also quasielectrons, $\Psi_e = \prod_j \left(\frac{\partial}{\partial z_j} - w \right) \cdot \Psi_L$, which is not as easily interpretable using the plasma analogy. There are also other trial wavefunctions one could write down which are not the LLL.

Returning to quasiholes, one can put Q quasiholes in one place,

$$\Psi_w = \prod_j (z_j - w)^Q \cdot \Psi_L ,$$

which is equivalent to a trivial quasiparticle. So there are Q types of quasiholes $q \in \{0, 1, ..., Q-1\}$ with charges q/Q. These excitations are gapped, which can be shown by showing that the change of interaction $\langle \Psi_{w,e} | V | \Psi_{w,e} \rangle$ when one quais-hole/electron pair is created is nonzero.

III.2. Statistics of quasiholes (Arovas/Schrieffer/Wilczek)

Let's assume that w is far from the edge of the droplet. We can calculate the Berry phase of exchanging two quasiholes, but first let's look at what the Aharonov-Bohm phase will be.

$$\frac{d\Psi_w}{ds} = -\frac{dw}{ds} \sum_i \frac{1}{z_i - w} \Psi_L = -\frac{dw}{ds} \int d^2 z \frac{n(z)}{z - w} \Psi_L ,$$

where $n(z) = \sum_j \delta(z - z_j)$. If we include normalization of the state,

$$\frac{d\gamma}{ds} = -i \int d^2 z \frac{1}{z} \left(\frac{1}{z + w} \frac{dw}{ds} - \frac{1}{z - w} \frac{dw}{ds} \right) \langle n(z) \rangle_w .$$

Now let's write expectation of density in terms of ground state Ψ_L:

$$\langle n(z) \rangle_w = \langle n(z) \rangle_L + \delta n_1(z) + \delta n_2(z) ,$$

where δn_1 is the density deficit (nonzero near w) and δn_2 is nonzero near the edge such that $\int (\delta n_1 + \delta n_2) d^2 z = 0$. Neither of these contribute to the Berry phase.

1. Since δn_1 is rotationally invariant, $\int d^2 z \frac{\delta n_1(z)}{z - w} = 0$.

2. For the other one, $\left|\int d^2z \frac{\delta n_2(z)}{z-w}\right| \leq \int d^2z \frac{|\delta n_2(z)|}{R} = O\left(\frac{1}{R}\right) \to 0$. Although this is not an exponential correction.

The Berry phase is then

$$\begin{aligned}\gamma(1) - \gamma(0) &= -i \int d^2z \frac{1}{z}\left(\frac{1}{z+w}\frac{dw}{ds} - \frac{1}{z-w}\frac{dw}{ds}\right)\langle n(z)\rangle_L + O\left(\frac{1}{R}\right) \\ &= -i \int_{\text{interior of loop}} d^2z \langle n(z)\rangle_L \\ &= -2\pi \frac{\nu}{2\pi}\text{area} \\ &= -\frac{\text{area}}{Q}\,.\end{aligned}$$

Generalizing to charge $q > 1$ adds a prefactor of q.

Now let's consider statistics: rotating w_1 around a second quasihole w_2, both of charges 1. I this case,

$$\begin{aligned}\gamma(1) - \gamma(0) &= -i \int_{\text{interior of loop}} d^2z \langle n(z)\rangle_{w_2} \\ &= -\frac{\text{area}}{Q} - \frac{2\pi}{Q}\,,\end{aligned}$$

and one gets an extra phase due to the charge deficiency. By either calculating the exchange or just halving the phase above, we get the statistical phase $\theta = -\frac{\pi}{Q}$. With a different charge, we generalize to

$$\theta = -\frac{\pi q_1^2}{Q} + O\left(\frac{\ell}{R}\right)\,,$$

where ℓ is the magnetic length. The correction can be improved to $O\left(\frac{\ell^2}{R^2}\right)$ (stated in the Arovas et al. paper).

III.3. Conformal field theory constructions (Moore/Read 1991)

Switching gears, consider a massless scalar field in 2D with action

$$S = \int \frac{1}{2}(\nabla\phi)^2 d^2x \qquad\qquad z = x + iy$$

Solutions to the EOM, which is a Laplace equation, can be split into holomorphic ("right") and antiholomorphic ("left") parts

$$\phi(z,\bar{z}) = \phi(z) + \phi(\bar{z})\,.$$

The correction function is $\langle\phi(z)\phi(0)\rangle = -\ln z$. Consider more general case

$$\left\langle O\prod_{i=1}^{N} a(z_i)\right\rangle \qquad a(z) = e^{i\phi(z)/\sqrt{\nu}} \qquad O = e^{-i\frac{\nu}{2\pi}\int d^2z\phi(z)}\,.$$

As an example, we can expand a simpler correlation function to obtain

$$\begin{aligned}\left\langle a^\dagger(z)\, a(0)\right\rangle &= \left\langle e^{-i\phi(z)/\sqrt{\nu}} e^{i\phi(0)/\sqrt{\nu}}\right\rangle \\ &= \left\langle\left(1 - i\frac{\nu}{2\pi}\int d^2z\phi(z) + ...\right)\left(1 + \frac{i\phi(0)}{\sqrt{\nu}} + ...\right)\right\rangle \\ &\longrightarrow \text{Wick's theorem and normal ordering of } O, a \text{ to remove infinities} \\ &= 1 - \frac{1}{\nu}\ln z + \frac{2}{4\nu^2}\ln^2 z - ... \\ &= \frac{1}{z^{1/\nu}}\,.\end{aligned}$$

More generally, we need to be "charge neutral": $\left\langle\prod_j e^{-iq_j\phi(z_j)/\sqrt{\nu}}\right\rangle = 0$ unless $\sum_j q_j = 0$. Returning to the case with O, we obtain

$$\left\langle O\prod_{i=1}^{N} a\left(z_i\right)\right\rangle = \prod_{i<j}\left(z_i - z_j\right)^{1/\nu} e^{-\frac{1}{4}\sum_j |z_j|^2} \times \text{singular phase}\,.$$

The Laughlin wavefunction is reconstructed from the massless scalar theory. The singular phase makes the function multivalued and represents the area-dependent AB phase obtained by the Laughlin wavefunction. Since it's not a topological phase, we do not focus on it.

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If you want to now construct quasiholes, you have to include $Z\left(w\right) = e^{i\sqrt{\nu}\phi(w)}$ in the correlation function, yielding

$$\left\langle OZ\left(w\right)\prod_{i=1}^{N} a\left(z_i\right)\right\rangle = \prod_{k<l}\left(w_k - w_l\right)^{1/Q}\prod_{i<j}\left(w_i - z_j\right)\prod_{i<j}\left(z_i - z_j\right)^{Q} e^{-\frac{1}{4}|z_j|^2 - \frac{1}{4Q}\sum_k |w_k|^2}$$

The phase in front makes the function multivalued and is not removed because it is related to the statistical phase θ calculated above. Upon exchanging w's, one has to perform a gauge transformation to go back to the same sheet (**monodromy**), which obtains the same θ as above.

Taking another CFT primary field $\psi\left(z\right)$, let

$$a\left(z\right) = e^{i\phi(z)}\psi\left(z\right)\,.$$

For example, let ψ be a chiral Majorana field, obtaining

$$\left\langle\psi\left(z\right)\psi\left(0\right)\right\rangle = \frac{1}{z}\,.$$

Extending this yields the Moore-Read state (1991) (for even N)

$$\begin{aligned}\left\langle O\prod_{i} a\left(z_i\right)\right\rangle &= \prod_{i<j}\left(z_i - z_j\right)^{1/\nu}\left(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\frac{1}{z_{N-1} - z_N} + \text{perms.}\right) e^{-\frac{1}{4}\sum_j |z_j|^2}\\ &= \prod_{i<j}\left(z_i - z_j\right)^{1/\nu}\,\text{Pf}\left\{\frac{1}{z_i - z_j}\right\} e^{-\frac{1}{4}\sum_j |z_j|^2}\,.\end{aligned}$$

Above we defined the Pfaffian

$$\text{Pf}\left\{M_{ij}\right\} = \frac{1}{2^{N/2}\left(N/2\right)!}\sum_{\sigma\in S_n}\text{sgn}\sigma\prod_{j=1}^{N/2} M_{\sigma(2j-1),\sigma(2j)}\,.$$

The Pfaffian $g\left(z_i - z_j\right)$ is also present in real-space BCS pairing of spinless fermions (see Sec. IV.2). The $1/\nu \equiv Q$ is even for fermions and odd for bosons (different from Laughlin case. This allows a fermionic construction for the $\nu = \frac{1}{2}$ LLL state and also a $\nu = \frac{5}{2}$ state if you go into the second LL. Adding quasiholes to this state is equivalent to adding a field $\tau\left(w\right) = e^{iq\phi(w)/\sqrt{\nu}}\sigma\left(w\right)$ with σ being a chical vertex operator. This Majorana fermion model is part of the CFT of the Ising model at criticality, and σ is the chiral part of the Ising spin field. Since $\left\langle\sigma\left(w\right)\sigma\left(0\right)\right\rangle \sim \frac{1}{w^{1/8}}\sigma\left(0\right)$ as $z \to 0$ and since we need to be single-valued in the z's, then we need to adjust Q accordingly. The wavefunction is then

$$\left\langle O\prod_{i}\tau\left(w_i\right)\prod_{i} a\left(z_i\right)\right\rangle = \frac{1}{\left(w_1 - w_2\right)^{1/8}}\text{Pf}\left\{\frac{\left(z_i - w_1\right)\left(z_j - w_2\right) + \left\langle w_1 \leftrightarrow w_2\right\rangle}{z_i - z_j}\right\}\cdots.$$

These wavefunctions also have a monodromy, but thise one non-Abelian (Nayak/Wilczek 1996). In other words, Moore-Read quasiholes are related to correlation functions of the Ising CFT.

IV.1. Fusion rules

Retuning to the two-point correlation function, we look at higher-order terms:

$$\langle \sigma(w)\,\sigma(0)\rangle \sim \frac{1}{w^{1/8}}\sigma(0) + \frac{1}{w^{\frac{1}{8}-\frac{1}{2}}}\psi(w)\ .$$

This is related to the third fusion rule below. Taking $Q = 1$, we have three qparticle types: $1, \psi, \sigma$ (different from the CFT operators) with charges $0, 0, \frac{1}{2}$ mod 1. The fusion rules are

$$\begin{aligned} \psi \times \psi &= 1 \\ \psi \times \sigma &= \sigma \\ \sigma \times \sigma &= 1 + \psi\,. \end{aligned}$$

If we take four σ's, then

$$\sigma \times (\sigma \times [\sigma \times \sigma]) = 2I + 2\psi\,.$$

But if we now remember that all particles fuse to the trivial type (BC at infinity), we project out 2ψ to get just $2I$. More generally, for an even number n of fields,

$$\left\{\bigotimes^{n}\sigma\right\}_I = 2^{\frac{n}{2}-1}.$$

To summarize, Moore-Read wavefunctions have the fusion rules of Ising anyons.

IV.2. $p_x + ip_y$ paired (superconducting) states (Read/Green 2000)

Assume we have spinless or spin-polarized fermions with $B = 0$. The standard pairing Hamiltonian reads

$$K_{eff} = \sum_k \left[h_k c_k^\dagger c_k + \frac{1}{2}\left(\Delta_k^\star c_{-k}c_k + \Delta_k c_k^\dagger c_{-k}^\dagger\right)\right] = \sum_k \begin{pmatrix} c_k^\dagger & c_{-k}\end{pmatrix}\begin{pmatrix} h_k & \Delta_k \\ -\Delta_{-k}^\star & -h_{-k}^\star\end{pmatrix}\begin{pmatrix} c_k \\ c_{-k}^\dagger\end{pmatrix}$$

with $h_k = \frac{k^2}{2m} - \mu$, anticommuting c_k's, and

$$\Delta_k = -\Delta_{-k} \to \begin{cases} 0 & k \to 0 \\ 0 & k \to \infty \end{cases}\ .$$

Performing the Bogoluibov transformation: choose $w_k = \begin{pmatrix} u_k \\ v_k \end{pmatrix}$ and $\alpha_k = u_k c_k + v_k c_{-k}^\dagger$ such that

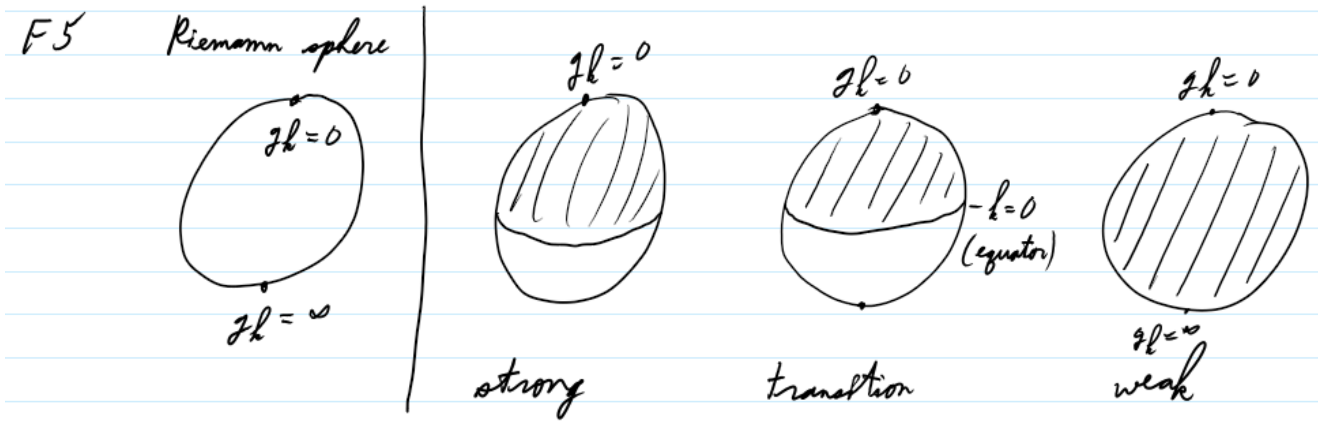
$$H = \sum_k E_k \alpha_k^\dagger \alpha_k \qquad\qquad E_k = \pm\sqrt{h_k^2 + |\Delta_k|^2}$$

and $|u_k|^2 = \frac{1}{2}\left(1 + \frac{h_k}{E_k}\right)$, $|v_k|^2 = \frac{1}{2}\left(1 - \frac{h_k}{E_k}\right)$, and $g_k = \frac{v_k}{u_k}$. In all cases, $|u_k|, v_k \to 1, 0$ as $k \to \infty$. For the other limit, we have several phases:

1. $h_k > 0$ as $k \to 0$, the strong pairing phase: $|u_k|, v_k, g_k \to 1, 0, 0$ as $k \to 0$.

2. $h_k < 0$ as $k \to 0$, the weak pairing phase: $|u_k|, v_k, g_k \to 1, 0, \infty$ as $k \to 0$.

3. $h_k \to 0$ as $k \to 0$, the (phase) transition point: $g_k \to$constant and $E_k \to 0$ as $k \to 0$.



We can then look at the wrapping of the vector w_k (F5), which is a map (not necessarily continuous) from k to CP1 (space of a two component wavefunction). In the weak pairing phase, w_k is not continuous and covers the sphere once, but doesn't do so in the strong pairing case. There is a topological phase transition between the two.

The wavefunction is then the standard BCS pairing $|\Omega\rangle$, which we look at in real space to make contact with the Moore-Read state. We can then obtain the Pfaffian mentioned above:

$$\Psi\left(r_1, ..., r_N\right) = \langle r_1, ..., r_N|\Omega\rangle = \langle 0|c_{r_1}...c_{r_N}|\Omega\rangle = \mathrm{Pf}\left\{g\left(r_i - r_j\right)\right\}$$

where $g\left(r\right) = \int \frac{d^2k}{(2\pi)^2} e^{ikr} g_k$.

1. At strong pairing, g_k is real analytic at small k (since it approaches zero). Then correlations decay exponentially:

$$g\left(\vec{r}\right) \propto p - \text{wave function} \times e^{-r/3}\,.$$

2. At weak pairing, $g_k \propto \frac{1}{k_x + ik_y}$ as $k \to 0$ and is not analytic. Then, $g\left(\vec{r}\right) \propto \frac{1}{x+iy}$ as $r \to \infty$:

$$g\left(\vec{r}_i - \vec{r}_j\right) = \frac{1}{z_i - z_j}\,,$$

just like in the Moore-Read state (!). Also, long-distance behavior of the real-space wavefunctions can provide details about the topological structure in phase space.

IV.3. Edges and vertices

Let's now look at small k and small chemical potential μ:

$$E_k \approx \sqrt{\mu^2 + \left|\Delta_k\right|^2 k^2} \qquad h_k = -\mu \qquad \Delta_k = \Delta\left(k_x - ik_y\right)\,.$$

Then we can, analogous to Kane's talk, form a Dirac equation from h_k expanded around $k = 0$:

$$i\frac{\partial u}{\partial t} = -\mu u + i\Delta^\star\left(\partial_x + i\partial_y\right) v$$
$$i\frac{\partial v}{\partial t} = -\mu v + i\Delta\left(\partial_x - i\partial_y\right) u$$

It is possible to make a representation of the Dirac matrices in which everything is real (Majorana). This is compatible with $u = v^\star$, the Majorana condition. All particles, and not just the zero-energy states, are self-conjugate in real space and can therefore be called Majorana fermions.

If we now look at a domain wall between strong and weak pairing, we can use the same Jackiw-Rebby analysis of Kane's talk and obtain a chiral Majorana field of energy $E = -\left|\Delta\right| k_y$.

Now let's look at a vortex in a small circular region of strong pairing in a sea of weak pairing (roughly, a domain wall wrapped in a circle). Around this, Δ winds by 2π and we get excitations

$$E_m = \frac{1}{R_{vortex}} m$$

with the zero-energy mode E_0 with $u = v^{-1}$. The corresponding momentum-space mode is the "Majorana zero mode"

$$\alpha_k \to \gamma = uc + u^\star c^\dagger = \gamma\,.$$

For a system with an edge, we also get a zero mode on the edge. There is an exponentially small splitting between the zero modes on the edge and on the vortex. For even n, we need $2^{n/2}$ modes, which obey the Ising fusion rules.