Boulder notes by Victor V. Albert

I. CHETAN NAYAK

After first studying an example of a topological phase and its underlying structures, we study effective field theories for 2D topological phases.

I.1. Example of a topological phase: toric code

Start with the toric code (F1) for $J_{1,2} \geq 0$:

$$H_{TC} = -J_1 \sum_{\text{vertices } v} \underbrace{\prod_{i \in N(v)} \sigma_i^z}_{A_v} - J_2 \sum_{\text{plaquettes } p} \underbrace{\prod_{i \in p} \sigma_i^x}_{B_p} \,.$$

All terms of the same type involve σ_z's and terms of opposite types give two minus signs, so everything commutes (frustration-free):

$$[A_v, A_{v'}] = [B_p, B_{p'}] = [A_v, B_p] = 0 \,.$$

Moreover, $A_v^2 = B_p^2 = 1$, so e-vals are all ± 1. Ground state is common e-vec of all operators with $+1$ e-val:

$$A_v |0\rangle = B_p |0\rangle = +1 |0\rangle \;\; \forall v, p \,.$$

- Vertex terms (F2): If it happens that $A_v = +1$, then that vertex has either 0, 2, or 4 up spins and so curves of up spins do not terminate for all vertices with $A_v = +1$. When $A_v = -1$, a curve of up-spins terminates, which costs energy. So the subspace $\{|C\rangle\}_C$ of all closed loops C (with $A_v = +1$) is a low-energy subspace out of which we can construct the ground state $|\Psi\rangle$:

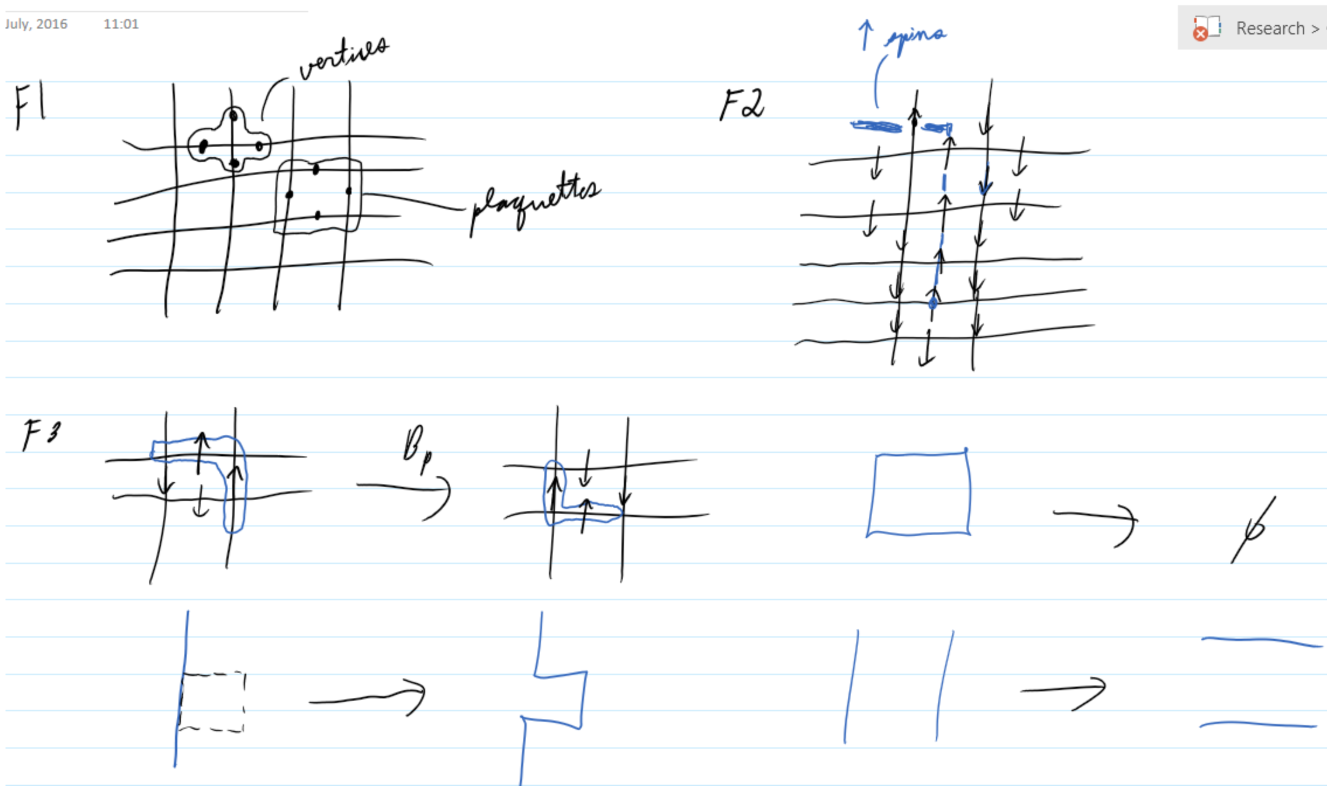
$$|\Psi\rangle = \sum_C \langle C|\Psi\rangle |C\rangle \,.$$

- Plaquette terms (F3): these flip all spins on the plaquette, deforming a string of up spins or killing a square (more generally, any closed loop) of squares.

In the ground state $|\Psi\rangle$, overlaps with all states which are obtainable from each other by application of plaquette terms should be the same (so that B_p's just re-shuffle terms and leave the full state invariant):

$$\langle \text{line}|\Psi\rangle = \langle \text{deformed line}|\Psi\rangle \tag{1.1}$$
$$\langle \| \, |\Psi\rangle = \langle = |\Psi\rangle \tag{1.2}$$
$$\langle \square|\Psi\rangle = \langle \;\; |\Psi\rangle \tag{1.3}$$

Therefore, any contractible loop can be deformed away (using "surgery") by application of plaquette operators (eq. 1.3).



I.2. Toroidal BC

Let's now assume we are on a torus. Even numbers of non-contractible loops can also be deformed away, so those amplitudes are also equal (F4):

$$\langle \text{pair of non-contractible loops} | \Psi \rangle = \langle \ | \Psi \rangle \,.$$

However, odd numbers of loops give their own amplitude equivalence, and we have to worry about four cases:

$$\begin{pmatrix} \langle \ | \Psi \rangle \\ \langle \text{vertical non-contractible loop} | \Psi \rangle \\ \langle \text{horizontal non-contractible loop} | \Psi \rangle \\ \langle \text{vertical and horizontal non-contractible loops} | \Psi \rangle \end{pmatrix}$$

Therefore, on the torus, we have four ground states, each corresponding to one class of equivalent amplitudes. These are exactly degenerate. Each ground state is an identical superposition of one of the four representative elements above with all possible deformations of it via plaquettes. For more complicated BC, genus determines ground state degeneracy.

I.3. Excited states

For an excitation, A_v or B_p will have e-val -1.

- $A_v = -1$: Breaking a loop (acting with σ_x) will give a string with two end points (each of which is called e). So an excitation is an identical superposition of a broken loop with all possible deformations of it via plaquettes.
- $B_p = -1$: Acting with σ_z creates two neighboring frustrated plaquettes (each of which is called m). One can construct excitations out of the basis of frustrated plaquettes and closed loops. In that case, permuting a loop (via a plaquette operation) and a frustrated plaquette gives a minus sign (F5).
- One can also have combinations of the above two (e.g., a frustrated plaquette with a terminating string on one of its ends; this is called $em \equiv \psi$), including strings winding around plaquettes.

Equivalently, one can understand excitations by studying the ground states on an annulus:

types of bulk excitations $\longleftrightarrow$ degenerate ground state subspace on punctured surface

As you take the hole in the annulus to zero, some of the ground states (since we started out with a non-trivial topology) will go into bulk excitations (since ground state becomes unique). See F6.

I.4. 2D topological phase

A **topological phase** consists of a system size L, a local Hamiltonian H, a bulk gap Δ, and correlation length ξ such that

1. Gap remains open:

$$\lim_{L\to\infty} \Delta(L) > 0$$

2. Correlations do not blow up:

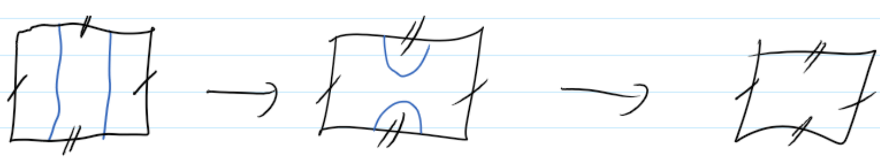
$$\lim_{L\to\infty} \xi(L) < \infty$$

3. The number of states $\{|a\rangle\}$ with energy $< \Delta$ is finite and only depends on topology of surface.

4. Any local operator X does not distinguish those states:

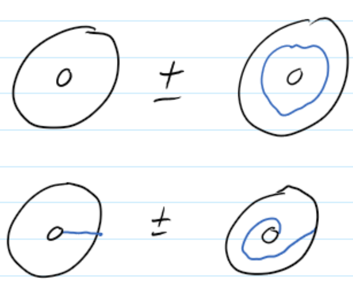
$$\langle a|X|b\rangle = C\delta_{ab} + O\left(e^{-L/\xi}\right) \tag{1.4}$$

F4



F5

F6



← diff excitations under rotation of annulus

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Let's return to studying quasiparticles using degenerate ground states on surfaces with multiple punctures. For example, a positive superposition of the trivial path class and the class of paths deformable to one non-contractible loop around the center of an annulus corresponds to the ground state in the planar geometry system:

$$|\text{ground}\rangle = \frac{1}{\sqrt{2}} \left(| \ \rangle + |\text{one loop around center}\rangle \right) .$$

This is because such a superposition is invariant under expansion of the annulus.

We can generalize this by considering a three-punctured sphere. For large spheres, the set of states consisting of classes of homotopically equivalent[1] paths are the quasi-ground states of the system. These form the vector space V_{ab}^{c} with dimension

$$N_{ab}^{c} = \dim V_{ab}^{c} .$$

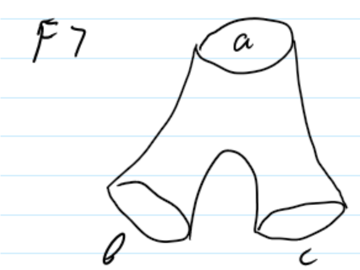
The sphere with three punctures can be deformed to a **pair of pants** (F7), where the punctures are now the waist and two legs of the pants. One can now graphically represent fusion rules by gluing two pairs of pants (F8). In F8, the fusion rule tells us that

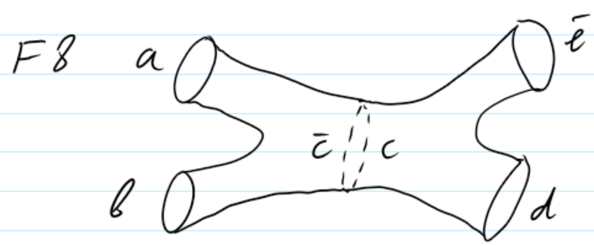
$$V_{abd}^{e} = \bigoplus_{c} V_{ab}^{c} \otimes V_{cd}^{e} .$$

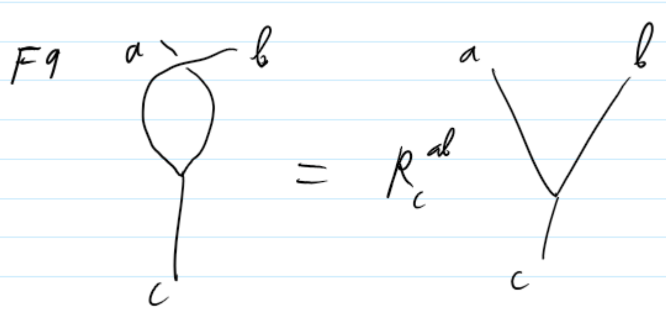
Generalizing to an n-punctured sphere Σ, we can look at the mapping class group — all diffeomorphisms of the punctures modded out by the trivial ones. This turn out to be "essentially" the Braid group of $n-1$ particles (since the last one is taken to be at infinity):

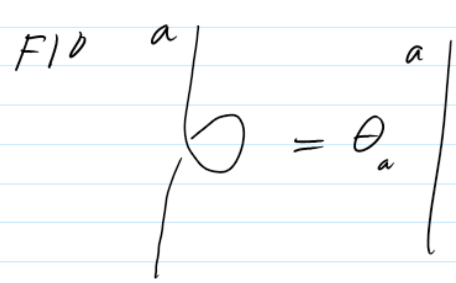
$$\text{Mapping class group} = \frac{\text{diff}\Sigma}{(\text{diff}\Sigma)_0} \approx B_{n-1} .$$

- Ex: N_{ab}^{c} is either 0 or 1, and moving b around a with c spectator picks up only a phase R_{c}^{ab} (F9). For higher N_{ab}^{c}, this phase becomes an R-matrix.









[1] Chetan said it is better to think of the paths as **isotopically equivalent** (i.e., paths cannot cross), but the additional conditions that contractible paths can be removed and two vertical paths can be changed to two horizontal ones (see right-most panels in F3). However, technically, blue paths can cross and, for this case, homotopically also works.

Now let's go back to the toric code, which has particles $1, e, m, \psi$. The fusion rules are

$$\begin{aligned} e \times e &= 1 \\ m \times m &= 1 \\ e \times m &= \psi \\ \psi \times \psi &= 1 \\ e \times \psi &= m \\ m \times \psi &= e\,. \end{aligned}$$

which tell us that $N_{ee}^{1} = N_{mm}^{1} = N_{em}^{\psi} = 1$ and $N_{ee}^{m} = 0$. Since the N's are either 0 or 1, these are Abelian. One can think of these as quasiparticle excitations of a unique ground state in the plane **or** a set of quasi-degenerate ground states on a sphere with three punctures. The braiding rules for exchanging like particles are

$$R_1^{ee} = R_1^{mm} = -R_1^{\psi\psi} = 1\,.$$

Exchanging an e and an m gives $R_\psi^{em} R_\psi^{me} = -1$ and, by convention, we let

$$R_\psi^{me} = -R_\psi^{em} = 1\,.$$

Adding in Dehn twists (F10), we also have the twist factor $\theta_a = e^{i\cdots}$ and

$$\theta_1 = \theta_e = \theta_m = -\theta_\psi = 1\,.$$

The twist factors are related to the exchanges by the diagram in F11.

Now we can generalize the toric code and consider a general model with a Hamiltonian on a manifold Σ or excited states of a ground state on a plane. Equivalently, we can study the algebraic structure (modular tensor category) with data $a, \theta_a, N_{ab}^{c}, R_c^{ab}, \ldots$, which turns out to correspond to a topological quantum field theory (TQFT; F12).

II.1. TQFTs

TQFT's model the topological properties of the qparticles (listed in the definition of the topological phase) of a concrete Hamiltonian system, but have very little to do with the Hamiltonian of the system itself. For Abelian theories, we can use Abelian Chern-Simons theory with action

$$S = \int d^2x dt \left[\frac{m}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + j^\mu a_\mu\right]\,,$$

where a_μ is the dynamical field and j^μ is the static source. The **sources** j^μ [with $j^0 \propto \delta(\vec{r})$] represent the particles and **fields** a_μ represent the effect of the particles which are enclosed in a path of a particle. The EOM is

$$\frac{m}{2\pi}\epsilon^{ij}\partial_i a_j = j^0\,,$$

so a particle gets a phase of $2\pi/m$ upon going around another one.

Extending to the toric code requires introducing more types of a's, with a matrix $K = 2\sigma_x$, $I, J = 1, 2$, and action

$$S = \int d^2x dt \left[\frac{K_{IJ}}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu^I \partial_\nu a_\lambda^J + j_I^\mu a_\mu^I\right]\,.$$

Now the EOM couple a_1 to density j_2 of the type 2 particle and visa versa:

$$\epsilon^{ij}\partial_i a_j^1 = \pi j_2^0\,.$$

Density j_2^0 represents the m particle and j_1^0 represents the e particle. Therefore, e particles feel m and visa versa.

- *Relation to the real world:* One could try to realize anyonic statistics using ensembles of spins (relevant to, e.g., the toric code), 2D superfluids, or s-wave superconductors. While the latter do have Berry phases, the theories are gapless due to, e.g., Goldstone modes.

II.2. Example of a topological phase: Ising anyons

This is the simplest non-Abelian theory. The particle types here are $1, \sigma, \psi$, fusion rules are

$$\begin{aligned}
\psi \times \psi &= 1 \\
\sigma \times \sigma &= 1 + \psi \\
\sigma \times \psi &= \sigma\,.
\end{aligned}$$

This is represented physically by four Majorana operators (which could correspond to four vortices in a superconductor) $\gamma_{i=1,2,3,4}$ which are self-adjoint and anticommute. The operator $i\gamma_1\gamma_2 = \sigma_z$ is the fermion parity; this 2-by-2 matrix is related to the 2D dimension of the Hilbert space of $\sigma \times \sigma$.

Consider a system with four σ's. One can fuse them in several different ways (F13), but since the resulting Hilbert space is the same, those ways are equivalent via basis changes. The coefficients of the basis change are given by the F-matrix, which now has to be added to our set of data about the algebraic structure.

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III.1. Multi Chern Simons theories & their related Abelian topological phases

Returning to multi Chern-Simons theory and letting $I = 1, 2, ..., N$ (Read, Wen/Zee 1990s), we have

$$S = \int \frac{1}{4\pi} K_{IJ} a_\mu^I \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda^J + j_I^\mu a_\mu^I\,.$$

We will characterize such theories, which then can be related to a classification of Abelian topological phases. The matrix K is non-degenerate, symmetric, and integer.

- The $N = 2$ case, $K = X$, is related to the toric code.
- The $N = 1$ case, $K = 3$, is related to the $\nu = 1/3$ Laughlin state.

The matrix K has $N = N_+ + N_-$ eigenvalues, where $N_\pm$ is the number of positive/negative eigenvalues (these tell you how many left/right movers there are on the edge of a disk). The ath eigenvector equation is

$$K_{IJ} e_{Ja} = \lambda_a e_{Ia} \text{ (no sum over } a)\,.$$

We make the vectors orthogonal, but normalize by eigenvalues:

$$e_{Ia} e_{Ib} = |\lambda_a|\, \delta_{ab} \longrightarrow K_{IJ} = \eta^{ab} e_{Ia} e_{Jb} = \operatorname{sgn}(\lambda_a)\, e_{Ia} e_{Ja}\,.$$

Now we suppress indices a, b and define $\vec{e}_I = e_{Ia}\hat{x}_a$ with $\hat{x}_a \in \mathbb{R}^{N_+, N_-}$. We can then define a lattice

$$\Lambda = \left\{ m^I \vec{e}_I \,|\, m^I \in \mathbb{Z} \right\}$$

and K (with $K_{IJ} = \vec{e}_I \cdot \vec{e}_J$) becomes the Gram matrix of the lattice Λ. The dual lattice is

$$\Lambda^\star = \left\{ n_I \vec{f}^I \,|\, n_I \in \mathbb{Z} \right\}\,,$$

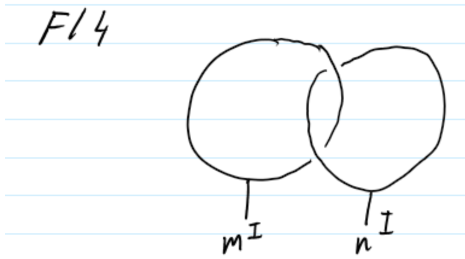
meaning that $\vec{e}_I \cdot \vec{f}_J = \delta_{IJ}$. Then

$$\left(K^{-1}\right)^{IJ} = \vec{f}^I \cdot \vec{f}^J\,.$$

Transformations $K \to W^T K W$ where $W \in GL(N, \mathbb{Z})$ are changes of basis of Λ. So two Chern-Simons theories are equivalent up to the W's.

Consider braiding two particles. The EOM says

$$\epsilon^{ij} \partial_i a_j^I = \left(K^{-1}\right)^{IJ} j_J^0 2\pi\,.$$



The K^{-1} tells you what integer charges the particles have under which gauge fields. The phase upon braiding them (F14) is then conveniently given by

$$\theta = 2\pi \left(K^{-1}\right)^{IJ} m_I n_J \,,$$

where m_I and n_J are integer vectors specifying the charges created for the two loops that are being braded:

$$j_I^0 = m_I \delta\left[\vec{x} - \vec{x}\left(\tau\right)\right]$$

(with τ the time parameterizing the braiding). Alternatively, letting $\vec{v} = m_I \vec{f}^I$ and $\vec{v}' = n_I \vec{f}^I$ with $\vec{v}, \vec{v}' \in \Lambda^\star$, one can write

$$\theta = 2\pi \vec{v} \cdot \vec{v}' \,.$$

You can think of particles in the theory as vectors in the dual lattice! Moreover, shifting $\vec{v}$ by a lattice transformation in Λ, $\vec{v} \to \vec{v} + \vec{\lambda}$ for $\vec{\lambda} \in \Lambda$, by the property of dual lattices that $\vec{v} \cdot \vec{\lambda}$ is an integer,

$$e^{i\theta} = e^{2\pi i \vec{v}\cdot\vec{v}'} \longrightarrow e^{2\pi i \left(\vec{v}+\vec{\lambda}\right)\cdot\vec{v}'} = e^{2\pi i \vec{v}\cdot\vec{v}'} \,.$$

The topologically distinct particle types are given by $\Lambda^\star$ modded out by $\Lambda \subset \Lambda^\star$:

$$D = \frac{\Lambda^\star}{\Lambda} \,.$$

Both $\Lambda, \Lambda^\star$ are infinite Abelian groups, so D is a finite Abelian group — **the discriminant group**. We can obtain its size by looking at how many vectors of $\Lambda^\star$ are sitting in the unit cell of Λ, i.e., the ratio of volumes:[2]

$$|D| = \frac{\mathrm{vol}\Lambda^\star}{\mathrm{vol}\Lambda} = \frac{\sqrt{\det K}}{\sqrt{\det K^{-1}}} = \det K \,.$$

Elements of D are equivalence classes $[\vec{v}]$ with $\vec{v} \in \Lambda^\star$ which, for this Abelian case, is the total quantum dimension of the theory. The equivalence relation is the S-matrix

$$S_{[\vec{v}],[\vec{v}']} = \frac{1}{\sqrt{|D|}} e^{2\pi i \vec{v}\cdot\vec{v}'} \approx \left(R^{[\vec{v}][\vec{v}']}_{[\vec{v}+\vec{v}']}\right)^2 \,.$$

The twist or T-matrix (F10) is then

$$T_{[\vec{v}],[\vec{v}']} = \delta_{[\vec{v}],[\vec{v}']} \underbrace{e^{i\pi\vec{v}^2}}_{\theta_{[\vec{v}]}} e^{-i\frac{2\pi}{24}(N_+ - N_-)} \,.$$

- Mathematical tool — quadratic Gauss sum: for a prime $p \neq 2$,

$$\sum_{n=0}^{p-1} e^{i\frac{2\pi}{p}n^2} = \begin{cases} \sqrt{p} & p \equiv 1 \mod 4 \\ i\sqrt{p} & p \equiv 3 \mod 4 \end{cases} \,.$$

[2] The volume of the unit cell is the square root because it is something like a Jacobian, changing variables from a hypercubic lattice to Λ. It is similar to writing the metric in an integral.

The above formula is associated with the lattice $D = \mathbb{Z}_p$ and generalizes to the Gauss-Milgram sum

$$\frac{1}{\sqrt{|D|}} \sum_{[\vec{v}] \in D} e^{2\pi i \left(\frac{[\vec{v}]^2}{2}\right)} = e^{i\frac{2\pi}{8}(N_+ - N_-)} \tag{3.1}$$

for general Abelian groups D, given a quadratic form $q : D \to \mathbb{Q}/\mathbb{Z}$ with $q(nv) = n^2 q(v)$ modulo integers. However, the matrix D alone does not encode all of the topological information. **You need D, the above quadratic form q** (determined from the inner product on the dual lattice, which in turn is determined by K)**, and some integer r modulo 8** such that

$$\frac{1}{\sqrt{|D|}} \sum_{a \in D} e^{2\pi i q(a)} = e^{i\frac{2\pi}{8} r} \qquad\qquad r = N_+ - N_-$$

Then you can lift to a lattice, and this information uniquely determines the bulk topological phase. The K matrix does not uniquely do so, so there are distinct K matrices for the same bulk topological phase. The edge properties, however, will be different.

The above sums can be used to prove that

$$S^2 = I \qquad\qquad (ST)^3 = I\,,$$

which is a representation of the modular group of the torus: Dein twists T and rotations S. These equations would not work if the phase $e^{-i\frac{2\pi}{24}(N_+ - N_-)}$ is not present in T.

- Gauss-Smith normal form: $K = PMQ$ with integer diagonal M and integer P, Q with $|\det P| = |\det Q| = 1$. Then can construct $D = \prod_J \mathbb{Z}_{M_{IJ}}$ and use to see if different K's are distinct.

III.2. Examples

1. Bosonic Laughlin state $\nu = \frac{1}{2}$: $K = 2$, so lattice is 1D: $\Lambda = \sqrt{2}\mathbb{Z}$. The dual is $\Lambda^\star = \frac{1}{\sqrt{2}}\mathbb{Z}$ and

$$\begin{aligned} D &= \mathbb{Z}_2 = \{0, 1\} \\ q(0) &= 0 \\ q(1) &= \frac{1}{4} \longleftarrow \text{due to half from Gauss-Milgram sum.} \end{aligned}$$

Then we just need $N_+ - N_-$ and Gauss-Milgram sum (3.1) is

$$\frac{1}{\sqrt{2}}(1 + i) = e^{i\frac{2\pi}{8}}\,.$$

2. Fermionic Laughlin state $\nu = \frac{1}{3}$: $K = 3$, so lattice is 1D: $\Lambda = \sqrt{3}\mathbb{Z}$. The dual is $\Lambda^\star = \frac{1}{\sqrt{3}}\mathbb{Z}$ and

$$\begin{aligned} D &= \mathbb{Z}_3 = \{0, 1, 2\} \\ q(0) &= 0 \\ q(1) &= \frac{1}{6} \\ q(2) &= \frac{2}{3}\,. \end{aligned}$$

However, Gauss-Milgram sum (3.1) has gone wrong:

$$\frac{1}{\sqrt{3}}\left(1 + e^{i\frac{\pi}{3}} + e^{i\frac{4\pi}{3}}\right) \neq e^{i\frac{2\pi}{8} \times \text{integer}}\,.$$

This is because of the fermionic nature of the state, and the equations above as written don't quite hold for arbitrary K matrices. Those only hold for even K-matrices and even integral lattices so that $\vec{v}^2 \in 2\mathbb{Z}$ and K_{II} are even. This covers all bosons, but not all fermions, since adding another fermion changes twist factors. This makes S not well-defined. However, you can modify this to make it work (Belov/Moore 2005).

3. Hierarchy states (Haldane/Halperin): Here $\nu = \frac{1}{m+\frac{1}{2p_1+\frac{1}{2p_1+\ldots}}}$ and

$$K = \begin{pmatrix} m & 1 & & \\ 1 & -2p_1 & 1 & \\ & 1 & -2p_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix} .$$

4. Jain sequence: $\nu = \frac{n}{2pn+1}$ and

$$K = \begin{pmatrix} 2p+1 & 2p & 2p & \cdots \\ 2p & 2p+1 & & \\ 2p & & 2p+1 & \\ \vdots & & & \ddots \end{pmatrix}$$

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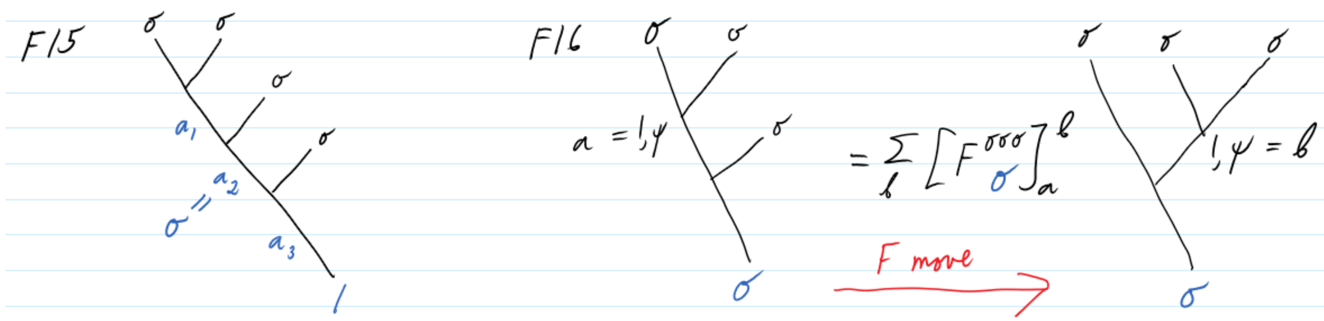


IV.1. Ising Anyons: consistency conditions

Let's come back to the Ising anyons $1, \sigma, \psi$ with fusion rules

$$\begin{aligned} \sigma \times \sigma &= 1 + \psi \\ \psi \times \psi &= 1 \\ \sigma \times \psi &= \sigma \,. \end{aligned}$$

Recall that σ's can be thought of as forming a degenerate ground-state subspace on a punctured sphere or a subspace of excitations above a (unique) ground state in the plane. To see that this leads to non-Abelian statistics, we can consider fusing several σ's (F15), given the condition that the last fusion must be the trivial particle, and count the ground state degeneracy on the punctured sphere (i.e., dimension of logical qu$n$it)

$$n_\sigma \equiv \sum_{a_1,\ldots,a_{n-1}} N^{a_1}_{\sigma\sigma} N^{a_2}_{a_1\sigma} N^{a_3}_{a_2\sigma} \ldots N^{1}_{a_{n-1}\sigma} \,.$$

Each can be thought of as a matrix:

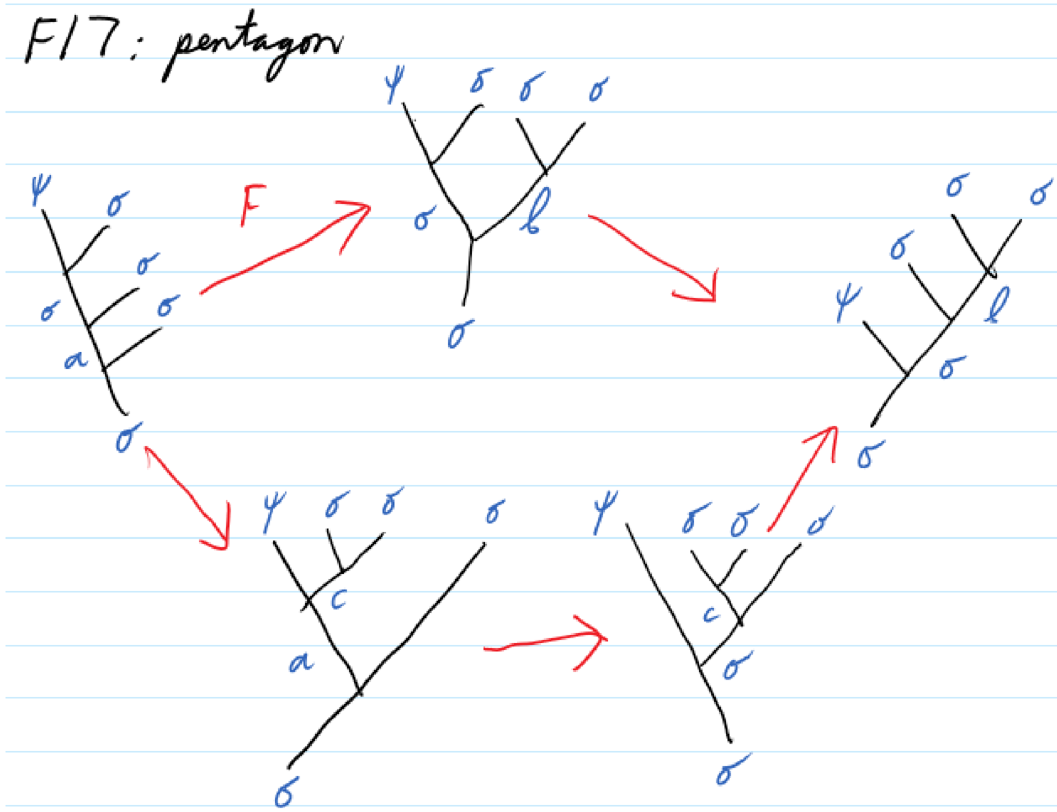
$$N^b_{a\sigma} \to (N_\sigma)^b_a = \begin{array}{cc} \begin{array}{ccc} 1 & \sigma & \psi \end{array} & \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ \sigma \\ \psi \end{array} \end{array} .$$

We can asymptotically ($N \to \infty$) approximate the degeneracy by the largest eigenvalue, which for the above is $+\sqrt{2}$ (others are $0, -\sqrt{2}$):

$$n_\sigma \sim d_\sigma^N = 2^{N/2} \,.$$

The largest eigenvalue of $N_{a\sigma}^{b}$ is then called the **quantum dimension** d_σ **of particle** σ. Asymptotically, the logical subspace of a large number σ's equals the quantum dimension.

We can fuse particles in different order, and fusion patterns are related to each other via basis transformations with F matrices. These depend on all particles that are being fused, so the F matrix for four σ's (F16) is actually $[F_{\sigma}^{\sigma\sigma\sigma}]_a^b$. In general, one way of fusing $a, b, ..., z$ into d corresponds to one matrix $F_{\sigma}^{ab...z}$ or one F-**move**.



For consistency, different F-moves need to get us to the same diagram. This leads to the **pentagon equation** — the only consistency condition which involves exclusively changes of basis. Let us write the consistency equation for F-moves for the two different paths in F17:

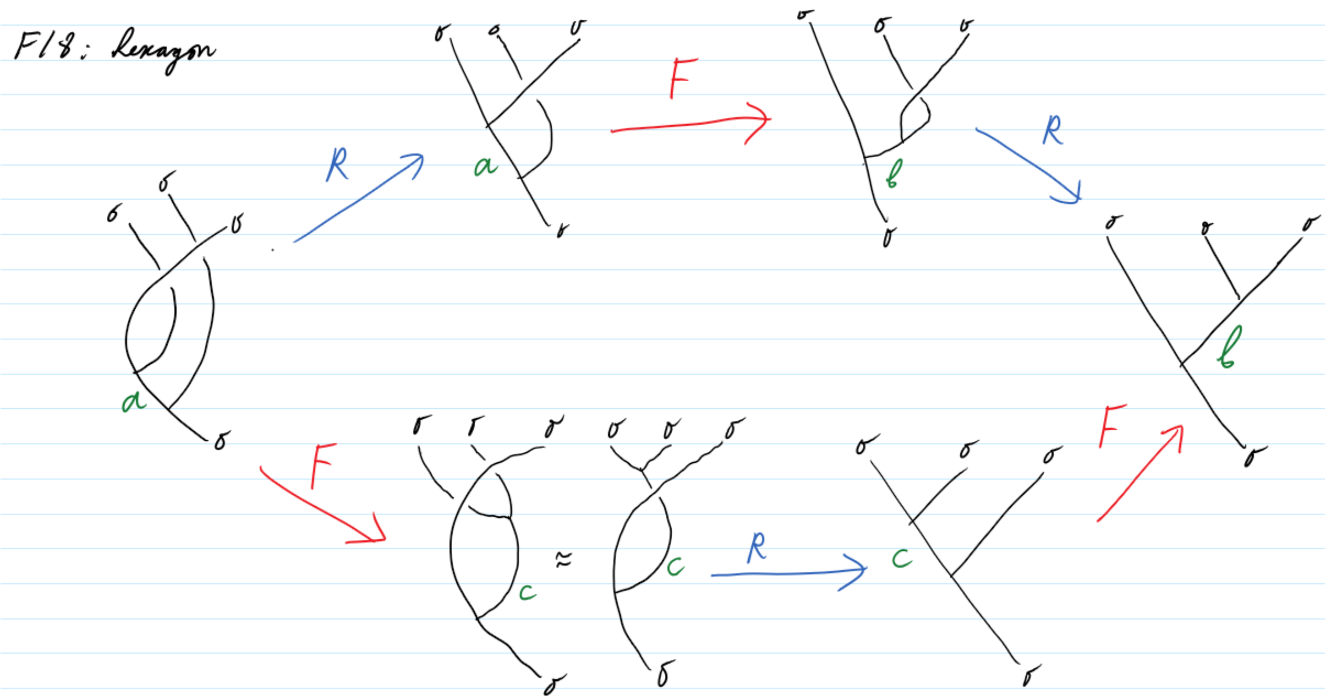
$$[F_{\sigma}^{\sigma\sigma\sigma}]_a^b \left[F_{\sigma}^{\psi\sigma b}\right]_{\sigma}^{\sigma} = \left[F_{a}^{\psi\sigma\sigma}\right]_{\sigma}^{c} \left[F_{\sigma}^{\psi c\sigma}\right]_{a}^{\sigma} [F_{\sigma}^{\sigma\sigma\sigma}]_c^b \; .$$

We now make guesses and try to solve for F. Taking $\left[F_{a}^{\psi\sigma\sigma}\right]_{\sigma}^{c} \left[F_{\sigma}^{\psi c\sigma}\right]_{a}^{\sigma} = X$ and simultaneously working with two possible solutions for $\left[F_{\sigma}^{\psi\sigma b}\right]_{\sigma}^{\sigma} = Z$, we obtain

$$[F_{\sigma}^{\sigma\sigma\sigma}]_a^b \, Z = X \, [F_{\sigma}^{\sigma\sigma\sigma}]_c^b \longrightarrow [F_{\sigma}^{\sigma\sigma\sigma}]_a^b = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \; . \tag{4.1}$$

This yields $\left[F_{\sigma}^{\psi\sigma\psi}\right] = -1$ and all other F's are $+1$.

- For the case of Majorana fermions $\gamma_{1,2,3,4}$, the diagram change between bases of eigenstates of $i\gamma_1\gamma_2$ and $i\gamma_2\gamma_3 = i\gamma_1\gamma_4$ (assuming total fermion parity is $+1$, which makes a qubit). In that case, $Z = i\gamma_1\gamma_2$ and $X = i\gamma_1\gamma_4$.



Recall that we also have braiding matrices R^{ab}_c, or R-**moves** (F9). Different combinations of F- and R-moves have to take us to the same diagram for consistency, leading to the **hexagon equation**. This equation allows one to solve for the R-matrices. Writing F18 out yields

$$R^{\sigma\sigma}_a \left[F^{\sigma\sigma\sigma}_\sigma\right]^b_a R^{\sigma\sigma}_b = \left[F^{\sigma\sigma\sigma}_\sigma\right]^c_a R^{c\sigma}_\sigma \left[F^{\sigma\sigma\sigma}_\sigma\right]^b_c .$$

Multiplying out both sides using eq. (4.1) yields

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \left(R^{\sigma\sigma}_1\right)^2 & R^{\sigma\sigma}_1 R^{\sigma\sigma}_\psi \\ R^{\sigma\sigma}_1 R^{\sigma\sigma}_\psi & -\left(R^{\sigma\sigma}_\psi\right)^2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} R^{\sigma 1}_\sigma + R^{\psi\sigma}_\sigma & R^{\sigma 1}_\sigma - R^{\psi\sigma}_\sigma \\ R^{\sigma 1}_\sigma - R^{\psi\sigma}_\sigma & R^{\sigma 1}_\sigma + R^{\psi\sigma}_\sigma \end{pmatrix} .$$

Braiding σ with 1 shouldn't give anything (in a reasonable gauge), so $R^{\sigma 1}_\sigma = 1$. We guess that braiding σ around ψ gives $R^{\psi\sigma}_\sigma = -i$. Solving both sides yields

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \left(R^{\sigma\sigma}_1\right)^2 & R^{\sigma\sigma}_1 R^{\sigma\sigma}_\psi \\ R^{\sigma\sigma}_1 R^{\sigma\sigma}_\psi & -\left(R^{\sigma\sigma}_\psi\right)^2 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} \end{pmatrix} \longrightarrow \begin{cases} R^{\sigma\sigma}_1 = e^{-i\frac{\pi}{8}} \\ R^{\sigma\sigma}_\psi = e^{i\frac{3\pi}{8}} \end{cases} .$$

IV.2. Fibonacci Anyons

Here the particles are $1, \epsilon$ and $\epsilon \times \epsilon = 1 + \epsilon$. The matrix

$$N_\epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow \lambda_{max} = \text{Golden ratio} = \frac{1+\sqrt{5}}{2} .$$

The quantum dimension of ϵ is the same as the total quantum dimension and it scales as

$$d_\epsilon \sim \phi^K$$

for a large number K of Fibonacci anyons. The exact number of states n^1_K given K particles and total charge 1 is

$$n^1_K = n^\epsilon_{K-1}$$
$$n^\epsilon_K = n^1_{K-1} + n^\epsilon_{K-1} = n^\epsilon_{K-1} + n^\epsilon_{K-1} ,$$

and follows the recursion relation for the Fibonacci sequence. Going through the same consistency conditions and solving yields

$$\left[F^{\epsilon\epsilon\epsilon}_\epsilon\right] = \begin{pmatrix} \phi^{-1} & \phi^{-\frac{1}{2}} \\ \phi^{-\frac{1}{2}} & \phi^{-1} \end{pmatrix} \qquad R^{\epsilon\epsilon}_1 = e^{i\frac{4\pi}{5}} \qquad R^{\epsilon\epsilon}_\epsilon = e^{-i\frac{3\pi}{5}} .$$

IV.3. Quantum computing connection

A **quantum error-correcting code** is a subspace $\mathcal{C} = \text{span}\{|a\rangle, |b\rangle\}$ of a Hilbert space $\mathcal{H}$. Denote its projection as $P_{\mathcal{C}}$. It protects some errors $\left\{E_i^\dagger\right\}$ if the quantum error-correction conditions are satisfied:

$$P_{\mathcal{C}} E_i^\dagger E_j P_{\mathcal{C}} = c_{ij} P_{\mathcal{C}} \,.$$

This equation is followed by topological phases for all local errors, up to exponential corrections; see eq. (1.4).

An example of encoding information using Ising anyons, we can imagine two encodings: a **dense encoding** with two σ's of total charge 1 or ψ and a **sparse encoding** with four σ's of total charge 1 (which uses a 2D subspace). The group of braids does not provide universal gates, but measurements can provide the remaining non-topological gates.

For Fibonacci anyons, the dense encoding has three ϵ's with total charge ϵ (if charge is 1, then space is 1D) and the sparse encoding with four ϵ's and total charge 1. Because the F-matrices contain the irrational number ϕ, the group of braids provides a way to do universal quantum computation.