

Boulder notes by Victor V. Albert

## I. CHARLIE KANE

The Gauss Bonnet theorem integrates the Gaussian curvature  $K = \frac{1}{R_1 R_2}$ , the inverse of the product of the curvature radii, to get an example of a topological invariant:

$$\frac{1}{4\pi} \int K dA = 1 - g.$$

Ground states of two systems with different values need a gap crossing, or critical point, and so cannot be adiabatically deformed into each other. Here, we focus on classifying **single-particle topological phases** — phases of matter that are equivalent to those which can be described using non-interacting fermions (when Hartree-Fock gets it right). We will also add translation symmetry, thereby focusing on crystalline insulators and leading to **topological band theory**.

To review band theory, Hamiltonian has discrete infinite translational symmetry  $T$  along some direction  $\vec{R}$ :

$$\left[ H, T(\vec{R}) \right] = 0.$$

Can simultaneously diagonalize both operators:

$$T(\vec{R})|\psi_k\rangle = e^{ikR}|\psi_k\rangle \quad H|\psi_k\rangle = E_k|\psi_k\rangle.$$

Can split up Hamiltonian and state as

$$H(k) \equiv e^{-ikr} H e^{ikr} \quad |\psi_k\rangle = e^{-ikr}|u(k)\rangle.$$

Since our Hamiltonian is now a continuous function of parameters, we can apply the formalism of the Berry phase (which reminds us of an electromagnetic gauge transformation):

$$|u(k)\rangle \longrightarrow e^{i\phi(k)}|u(k)\rangle.$$

We can define a Berry connection

$$\vec{A} = -i\langle u_k | \vec{\nabla}_k | u_k \rangle,$$

which transforms as an electromagnetic gauge potential:  $\vec{A} \longrightarrow \vec{A} + \vec{\nabla}_k \phi$ . The Berry phase for  $> 1D$  is then

$$\oint_C \vec{A} \cdot d\vec{k} = \int_S F d^2k,$$

where  $C$  is a contractible loop.

- Important example: TLS with Hamiltonian

$$H(k) = d_0(k)I + \vec{d}(k) \cdot \vec{\sigma} \quad \text{with eigenstates} \quad H(k)|u_{\pm}(k)\rangle = \left(d_0 + |\vec{d}|\right)|u_{\pm}(k)\rangle.$$

The Berry phase is one half the solid angle swept by path.

### I.1. Electric polarization in 1D

Classically,

$$P = \frac{\text{dipole moment}}{\text{length}}.$$

You can have a bound charge density

$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

and boundary charges

$$Q_{end} = \vec{P} \cdot \hat{n}.$$

We claim that the polarization is a Berry phase:

$$P = \frac{e}{2\pi} \int_{BZ} A dk \quad \text{with} \quad A = -i \langle u_k | \partial_k u_k \rangle.$$

One qualitative reason for this is that both share the same ambiguity:

1. On one hand, since one can add electrons  $e$  to the end charge of a 1D line,

$$Q_{end} = P \pmod{e}.$$

2. On the other hand, since in 1D the BZ is not a boundary, the integral is over a non-contractible loop. Therefore, we can have a winding (large gauge transformation) in which the phase difference changes by  $2\pi n$  with  $n \in \mathbb{Z}$ :

$$\gamma \longrightarrow \gamma + \frac{e}{2\pi} \int_{BZ} \partial_k \phi dk = \gamma + 2\pi n.$$

However, changes in polarization as a parameter  $\lambda$  is varied are not multivalued and are in fact gauge invariant:

$$\begin{aligned} \Delta P &= P(\lambda = 1) - P(\lambda = 0) & (1.1) \\ &= \frac{e}{2\pi} \left( \int_{BZ} A_{\lambda=1} dk - \int_{BZ} A_{\lambda=0} dk \right) \\ &= \frac{e}{2\pi} \int_C A dk \\ &\quad \longrightarrow \text{Contour } C \text{ is boundary of a surface; use Stokes.} \\ &= \frac{e}{2\pi} \int_C A dk \\ &= \frac{e}{2\pi} \int_S F dk d\lambda. \end{aligned}$$

The second argument in favor of the polarization being a Berry phase is that the original definition,

$$\vec{P} = e \langle \vec{r} \rangle,$$

requires a localized state. Therefore, let us introduce **Wannier states**

$$|\phi(R)\rangle = \int_{BZ} \frac{dk}{2\pi} e^{-ik(R-r)} |u(k)\rangle,$$

which can be defined if  $u(k)$  is a smooth function of  $k$ . These states are gauge dependent, but localized. Then :

$$P = e \langle \phi(R) | (r - R) | \phi(R) \rangle = -i \frac{e}{2\pi} \int_{BZ} \langle u_k | \partial_k u_k \rangle dk.$$

The gauge-dependence of Wannier states is equivalent to the gauge dependence of Berry phase.

## I.2. Su-Schrieffer-Heeger model

$$H = \sum_i t_1 c_{iA}^\dagger c_{iB} + t_2 c_{iB}^\dagger c_{i+1,A}$$

When  $t_1 \neq t_2$ , there is a gap. The regions  $t_1 > t_2$  and  $t_1 < t_2$  are two distinct phases, which we can see by calculating the polarization. Band Hamiltonian is

$$\begin{aligned} H &= \sum_k c_{kA}^\dagger c_{kB} (t_1 + t_2 e^{ika}) + h.c. \\ &= \sum_k (c_{kA}^\dagger \ c_{kB}^\dagger) \begin{pmatrix} h_{aa} & h_{ab} \\ h_{ba} & h_{bb} \end{pmatrix} \begin{pmatrix} c_{kA} \\ c_{kB} \end{pmatrix} \\ &= \sum_k c_a h_{ab} c_b \end{aligned}$$

where the Bloch vector of the Hamiltonian is

$$\vec{d} = \langle t_1 + t_2 \cos ka, -t_2 \sin ka, 0 \rangle .$$

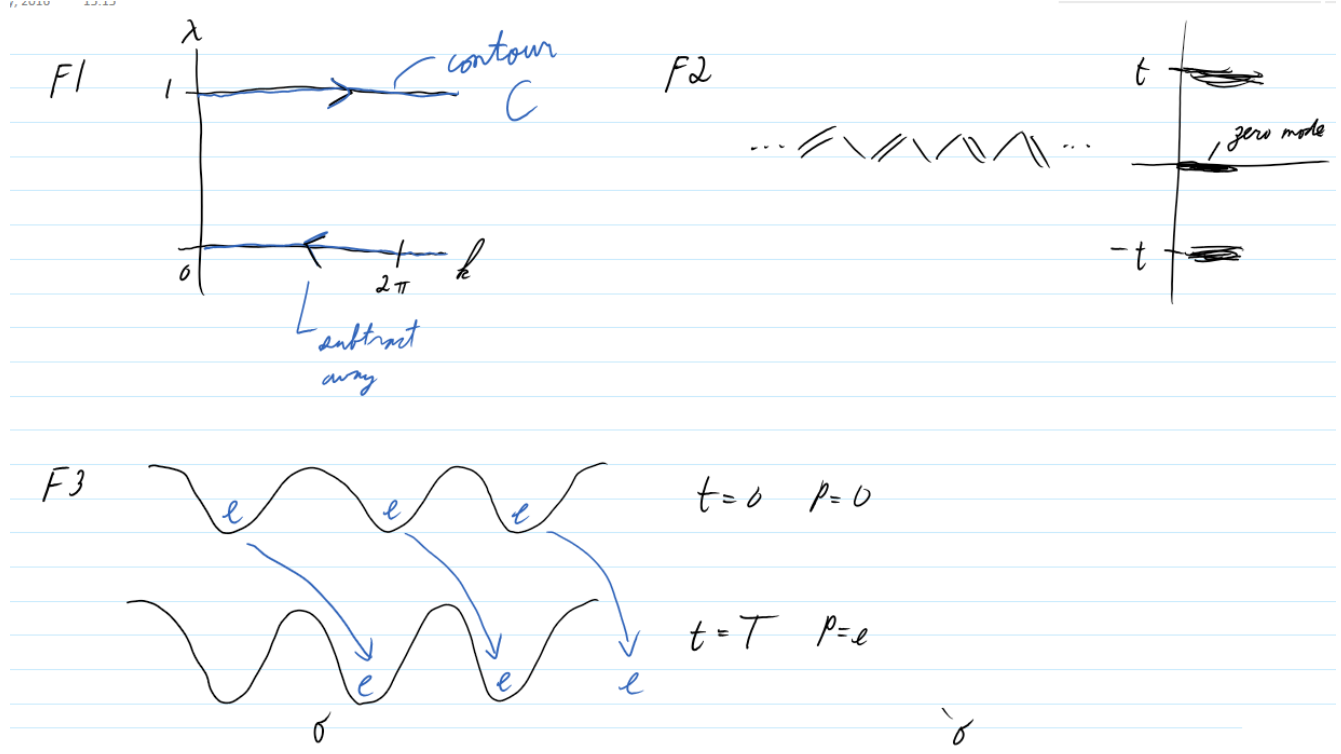
We can project the 3D Bloch sphere into a 2D plane and look at what happens as we change  $k$  in a loop.

- When  $t_1 > t_2$ , the origin (which in the original 3D Bloch sphere contains the location of the monopole) is not enclosed and the Berry phase is trivial.
- When  $t_1 < t_2$ , the origin is enclosed and the Berry phase is  $\pi$ .

Despite the two models looking very similar, if one picks one unit cell and calculates Berry phases for that cell, one gets phases that can be compared. Picking a different unit cell results in

$$H'(k) = \sum_k c_{kA}^\dagger c_{kB} (t_2 + t_1 e^{ika}) + h.c. .$$

gives you different Berry phases.



## II. CHARLIE KANE 2

For the SSH model  $H(k) = \vec{d}(k) \cdot \vec{\sigma}$ , the band Hamiltonian has several symmetries.

1. Chiral symmetry:

$$\{H(k), \sigma_z\} = 0.$$

This puts constraints on  $\vec{d}$ , namely

$$d_z = 0.$$

Given this symmetry and 1D, band structures can be characterized by an integer winding number. The SSH model has regions with two different windings. The symmetry leads to a particle-hole symmetric spectrum:

$$H|E\rangle = E|E\rangle \longrightarrow H(\sigma_z|\psi_E\rangle) = -E(\sigma_z|E\rangle).$$

2. Reflection symmetry of lattice:

$$H(-k) = \sigma_x H(k) \sigma_x$$

This puts constraints on  $\vec{d}$ , namely

$$d_x(-k) = d_x(k) \quad d_{y,z}(-k) = -d_{y,z}(k).$$

These symmetries turn out to give you two classes of possible loops. A heuristic argument for this is using the aforementioned polarization  $P \bmod e$ . Upon reflection,  $P \rightarrow -P$ , and since the Hamiltonian is invariant,  $P$  is either 0 or  $e/2$  modulo  $e$ .

Let's continue with SSH. On domain walls between two phases, as you turn up the coupling from  $t_{min} \geq 0$  to  $t_{max}$ , there exist zero modes which stay at zero and are topologically protected as long as they are gapped out from the rest of the  $\pm E$  spectrum. Thinking in terms of polarization, the reflection symmetry makes the polarization  $e/2$  and 0 on the two sides, respectively, meaning that the charge of the zero mode is fractional.

If we now consider an infinitesimally small gap ( $t_1 - t_2 \ll t_{1,2}$ ), we can construct a low-energy effective theory by expanding the Hamiltonian around the small gap at  $k = \pi/a$ . Expanding for small deviations  $q$  around  $\pi/a$ , we obtain  $\vec{d}(\frac{\pi}{a} + q) \approx \langle t_1 - t_2, t_2 a q, 0 \rangle$  and Hamiltonian (for  $v_F = t_2 a$  and  $m = t_1 - t_2$ )

$$H(q) \approx v_F q \sigma_y + m \sigma_x \text{ with energies } E_{\pm}(q) = \pm \sqrt{v_F^2 q^2 + m^2}.$$

Now consider a domain wall (Jackiw/Reby 1970) by making the mass smoothly vary with distance  $x$  such that  $m(x) \rightarrow \pm 1$  as  $x \rightarrow \pm \infty$ . To allow for spatial variation, let  $q \rightarrow -i\partial_x$  ( $k \cdot p$  **approximation**):

$$H \approx -i v_F \sigma_y \partial_x + m(x) \sigma_x. \quad (2.1)$$

The wavefunction of this  $H$  modulates the spatial envelope of the two states near the gap at  $k = \pi/a$ . The zero mode will satisfy

$$-i v_F \sigma_y \partial_x \psi + m(x) \sigma_x \psi = 0 \longrightarrow \partial_x \psi = \frac{m(x)}{v_F} \sigma_z \psi,$$

obtaining eigenstates  $\psi_{\pm} = e^{\mp \int_0^x \frac{m(x')}{v_F} dx'} |\pm z\rangle$ . Depending on the sign of  $m(x)$ , the properly normalized eigenstate will be different.

### II.1. Thouless charge pump

Consider an adiabatic cycle for a 1D insulator:  $H(k, t+T) = H(k, t)$ . For a concrete model, consider the nearly free electron gas under a periodic potential

$$V_G = V_0 e^{i\phi(t)} c_{k+G}^{\dagger} c_k + h.c.$$

where the phase  $\phi(t) = 2\pi \frac{t}{T}$  is advanced by  $2\pi$  every cycle. Advancing by  $2\pi n$  moves all electrons over by one site and changes the polarization by  $ne$  (F3 for  $n = 1$ ). But since the polarization is defined modulo  $e$ , the adiabatic path does not change it (as expected). Calculating this change (F4, but remembering F1) using eq. (1.1) yields

$$\Delta P = \frac{e}{2\pi} \int_S F dk dt. \quad (2.2)$$

The extra wrinkle is that now  $t = 0$  and  $t = T$  are also identified, making the surface a torus. Integrating the Berry curvature over a torus produces an integer. Letting

$$|u(k, T)\rangle = e^{-\phi(k)} |u(k, 0)\rangle,$$

we obtain the Chern number  $n$ :

$$\Delta P = \oint \left[ A(k, T) - \oint A(k, 0) \right] dk = \oint \partial_k \phi dk = 2\pi n.$$

This number is an obstruction to finding a continuous gauge throughout the torus. The same thing occurs on a sphere with a monopole in it, but not on a ring (since one can use large gauge transformations to remove any  $2\pi n$  phase). For a generic two level system, the Chern number tells how many times  $\vec{d}(k) : T^2 \rightarrow S^2$  has wrapped around the sphere (the **degree** of  $\vec{d}$  on  $S^2$ ).

## II.2. Integer quantum Hall effect & Laughlin's argument

This system is a 2D “insulator” with conductivities  $\sigma_{xx} = 0$  and  $\sigma_{xy} \neq 0$  (due to disorder) and current  $\vec{J} = \sigma_{xy} \hat{z} \times \vec{E}$ . Laughlin determined which values  $\sigma_{xy}$  can take using his argument (F5). Consider threading one flux quantum adiabatically through a cylinder:

$$\Phi(t=0) = 0 \quad \Phi(t=T) = \phi_0 = \frac{h}{e}.$$

An “integer” flux  $\Phi = \phi_0$  can be eliminated by a large gauge electromagnetic transformation

$$\begin{aligned} \psi(\vec{r}) &\longrightarrow e^{i\theta(\vec{r})} \psi(\vec{r}) \\ \vec{A}_{e\& m} &\longrightarrow \vec{A}_{e\& m} + \frac{\hbar}{e} \vec{\nabla} \theta(\vec{r}) \\ \Phi = \int \vec{A} \cdot d\vec{r} &\longrightarrow \Phi + 2\pi \frac{\hbar}{e} = \Phi + \frac{h}{e}. \end{aligned}$$

The Hamiltonian at  $t = 0$  and  $t = T$  are then related by a gauge transformation. Similar to the Thouless pump, charges will move after one cycle. Due to Faraday's law,

$$E_y = \frac{1}{2\pi R} \frac{d\Phi}{dt},$$

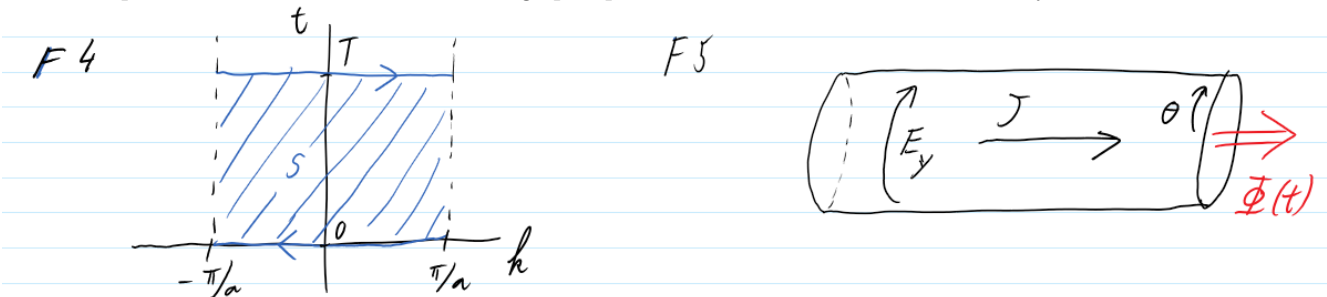
and there will be a current density  $J = \sigma_y E_y$  and current  $I = 2\pi R J = \frac{d\Phi}{dt}$ . The change in polarization is then

$$\Delta P = \int dt I(t) = \int dt \sigma_{xy} \frac{d\Phi}{dt} = \sigma_{xy} \frac{h}{e}.$$

Since we made a cycle, only an integer number of electrons is transferred, so we set

$$\sigma_{xy} \frac{h}{e} = ne \longrightarrow \sigma_{xy} = n \frac{e^2}{h}.$$

This is a real-space realization of Thouless's charge pump and remains valid in a disordered system as well.<sup>1</sup>



<sup>1</sup> However, the adiabatic limit is more difficult to make in a disordered system because the inter-LL gap will be closed.

### III. CHARLIE KANE 3

Last time we considered a 1+1D system with one band. Let's now consider a 2D Hamiltonian  $H(k_x, k_y)$  with PBC in the  $y$ -direction and thread flux in the  $x$ -direction. Due to periodic boundary conditions,  $k_y = \frac{1}{R} \left( m + \frac{\Phi}{\Phi_0} \right)$  is quantized by  $m$  and moved over by  $\Phi/\Phi_0$ , and the change in charge is just like the Thouless pump, but added over the bands:

$$\begin{aligned} \Delta Q &= \sum_m \frac{e}{2\pi} \int dk_x \int_0^{\Phi_0} d\Phi F(k_x, \Phi) \\ &\rightarrow \text{Obtains Thouless pump} \\ &= \sum_m \frac{e}{2\pi} \int \int dk_x dt F(k_x, t) \\ &= \sum_m \frac{e}{2\pi} \int_{BZ} d^2 k F(k_x, k_y) \\ &= \sum_m n(m) \\ &\equiv \sigma_{xy}. \end{aligned}$$

This shows that the Hall conductivity in a 2D system is quantized in a way that is based off of a 1D system.

#### III.1. Graphene

Consider

$$H = -t \sum_{\langle ij \rangle} c_{iA}^\dagger c_{jB} = \sum_k c_{ka}^\dagger c_{kb} h_{ab}(k) = \sum_k (c_{ka}^\dagger \ c_{kb}^\dagger) \begin{pmatrix} 0 & \sum_j e^{ik \cdot r'_j} \\ \sum_j e^{-ik \cdot r'_j} & 0 \end{pmatrix} \begin{pmatrix} c_{ka} \\ c_{kb} \end{pmatrix}$$

where the couplings and vector  $\vec{r}'$  (representing the three directions from one site to the three neighboring sites) are due to the geometry of the honeycomb lattice. In other words,

$$\vec{d} = -t \sum_j \langle \cos(k \cdot r'_j), \sin(k \cdot r'_j) \rangle.$$

This has the following symmetries:

1. Chiral symmetry:  $\{H, \sigma_z\} = 0$ .
2. Parity  $P$ :  $H(-k) = \sigma_x H(k) \sigma_x$ .
3. Time:  $H(-k) = H(k)^*$ .

Together,  $PT$  symmetry constrains  $d_z(k) = 0$ , allowing **Dirac points** — point defects in  $k$ -space around which  $\langle d_x, d_y \rangle$  wraps by  $2\pi$ . These points occur at two points in the BZ  $\pm K$  where

$$\vec{d}(\pm K) = 0,$$

which turn out to be on the edges of the BZ due to the structure of  $\vec{d}$ . We can expand near both of the Dirac points (just like we did for one point for the SSH model) to obtain two two-band Hamiltonians with energies:

$$H(\pm K + q) \approx v_F (\pm \sigma_x q_x + \sigma_y q_y).$$

We want to now open up the Dirac points to obtain a gapped phase. There are several ways to do this:

1. Break  $P$  (like in Boron Nitride) by letting

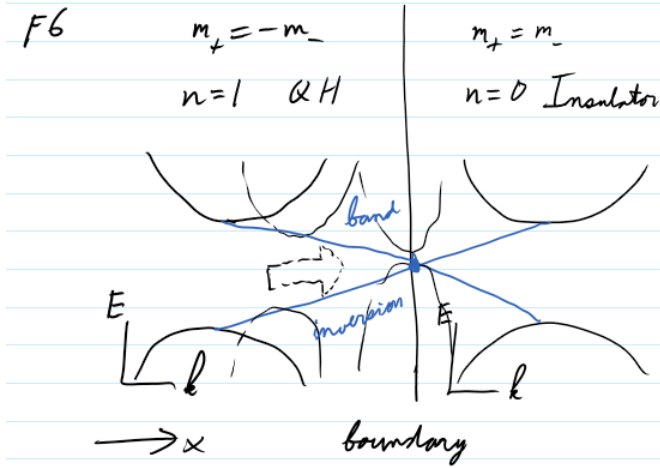
$$H(\pm K) \rightarrow H(\pm K) + m\sigma_z$$

The energies would then be  $E_{p,\pm}(q) = \pm \sqrt{v_F^2 q^2 + m^2}$ . However, the degree of this map (the number of times  $\vec{d}$  winds around the sphere) is zero, so this is a trivial insulator.

2. Break  $T$  (Haldane 87) by letting

$$H(\pm K) \longrightarrow H(\pm K) \pm m\sigma_z$$

Now  $\vec{d}$  wraps around the sphere once (with each  $\pm K$  part taking care of a hemisphere) and the degree  $n = 1$ . This is equal to the TKNN invariant and also the Chern number.



### III.2. Edge states

We can now look at a domain wall between the  $\pm m\sigma_z$  and  $m\sigma_z$  perturbations, making  $m$  slowly change along  $x$ -direction:

$$H(\pm K + q) = v_F(\pm i\sigma_x\partial_x + \sigma_yq_y) + m_{\pm}(x)\sigma_z$$

As you cross the domain wall, the bands invert (F6). This is equivalent to the Jackiw-Reby problem from eq. (2.1) and the states for a given  $k_y$  are

$$|\psi(k_x, k_y)\rangle = e^{iq_y y} e^{-\int_0^x \frac{m_{\pm}(x)}{v} dx} |\sigma_y, +\rangle.$$

If we plot the band structure vs.  $k_x$ , we get one edge state band crossing from the lower to the upper parabolic bulk bands. While this is possible in 2D, it is not possible in 1D due to the Nielsen-Ninomiya or Fermion Doubling theorem (“what goes up must come down”).

### III.3. Generalizations

- 3D QHE (Halperin 83): a trivial generalization which is adiabatically equivalent to stacking layers of 2D systems. Each of the three directions define a Chern number,  $\vec{n} = \langle n_x, n_y, n_z \rangle$ , which can be associated with a reciprocal lattice vector  $\vec{G} = \frac{2\pi}{a}\vec{n}$  and in turn with the 3D Hall conductivity

$$\sigma_{ij} = \frac{e^2}{h} \epsilon_{ijk} G_k.$$

- 4D QHE (Zhang/Hu 2001): Needs a non-Abelian Berry connection 1-form  $A_{ij}(k) = \langle u_i | \vec{\nabla}_k | u_j \rangle$  and curvature 2-form  $F = dA + A \wedge A$ . We can define the 2nd Chern number as the integral of the four-form  $F \wedge F$  over the 4D BZ  $T^4$ :

$$n = \frac{1}{8\pi^2} \int_{T^4} \text{Tr} \{F \wedge F\}.$$

The boundary states are then 3+1D chiral Dirac fermion, which are now commonly called **Weyl points**. It turns out  $F \wedge F$  is the total derivative of the Chern-Simons three-form

$$\int_{S_4} \text{Tr} \{F \wedge F\} = \int_{S_4} dQ_3 = \int_{\partial S_4} Q_3.$$

This mimicks the 1D case:

$$\int_{S_2} F = \int_S dA = \int_{\partial S} A.$$

3. Topological defects (Teo/Kane 2013): here the band structure varies slowly in space and is characterized by the 2nd Chern number. For example, consider vertically layered QHE states with a dislocation. Then, there will be an edge state living on that dislocation. The number of edge states can be quantified by Burgers vector of the dislocation and the 3D Chern number from example 1 above:

$$n = \frac{1}{2\pi} \vec{G} \cdot \vec{B}.$$

## IV. CHARLIE KANE 4

### IV.1. 2D Quantum Spin Hall Insulator

The graphene Hamiltonian (after introducing  $\tau_z$  as the  $\pm K$ -point degree of freedom) is

$$H(\pm K + q) \approx v_F (\pm \sigma_x q_x + \sigma_y q_y) \rightarrow v_F (\sigma_x \tau_z q_x + \sigma_y q_y)$$

This is two copies of the Haldane model, and spin  $s_z$  is preserved. Let's once again consider opening energy gaps in graphene:

1. Break  $P$ :  $V = m_I \sigma_z$ , trivial insulator.
2. Break  $T$ :  $V = m_{Haldane} \sigma_z \tau_z$ , get IQHE since no longer TRS.
3. Doesn't break anything:  $V = m_{SO} \sigma_z \tau_z s_z$ . This comes from microscopic spin-orbit coupling  $V_{SO} = (\vec{s} \times \vec{p}) \cdot \vec{E}$ . This coupling is very small in real graphene, but this model coupling leads to a new phase.

When written as a Block matrix in the  $s_z$  (spin) degree of freedom (where  $H_{Haldane}$  is a  $4 \times 4$  matrix), it is clear the model is  $\sigma_z$ -conserving:

$$H = \begin{pmatrix} H_{Haldane} & 0 \\ 0 & H_{Haldane}^* \end{pmatrix}. \quad (4.1)$$

The model has edge states (for both types of edges), shown in F7 (copied from arXiv:cond-mat/0411737), and there is a degeneracy between the edge modes at  $k = 0$ . This degeneracy is not broken by terms which do not commute with  $s_z$ . It turns out that this degeneracy is protected by time reversal symmetry  $\Theta = e^{i\pi s_y} K$  (with  $K$  complex conjugation). It acts on spin as

$$\Theta \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \begin{pmatrix} \psi_\uparrow^* \\ -\psi_\downarrow^* \end{pmatrix} \rightarrow \Theta^2 = -1.$$

Such an operator means there is a two-fold degeneracy for all states (**Kramers' theorem**).

- Proof: Assume (by contradiction) that  $|\chi\rangle$  is not degenerate. Act with  $\Theta$ :

$$\Theta|\chi\rangle = c|\chi\rangle \rightarrow \Theta^2|\chi\rangle = c^*c|\chi\rangle \rightarrow \Theta^2 = |c|^2 \neq -1 \rightarrow \text{contradiction.}$$

The two edge states are Kramers' partners, and any operator which couples them will split the degeneracy, so elastic backscattering between edge states is forbidden. More strongly, all states are extended even for strong disorder (F8). The problem boils down to a transmission/reflection problem, and the reflection coefficient  $r \rightarrow -r$  under  $\Theta$ . Therefore, under  $\Theta$ -preservation,  $r = 0$  and  $|t| = 1$ . There are, however, inelastic processes (i.e., when kinetic energy not conserved) which can occur under  $\Theta$ .

For TRS Hamiltonians (i.e.,  $H(-k) = \Theta H(k) \Theta$ ), the property in the bulk which characterizes the edge protection is a  $\mathbb{Z}_2$  topological invariant  $\nu$ . The reason there are only two is because edge states either connect back with each other at  $k = \pm \frac{\pi}{a}$  or zig-zag (F9).

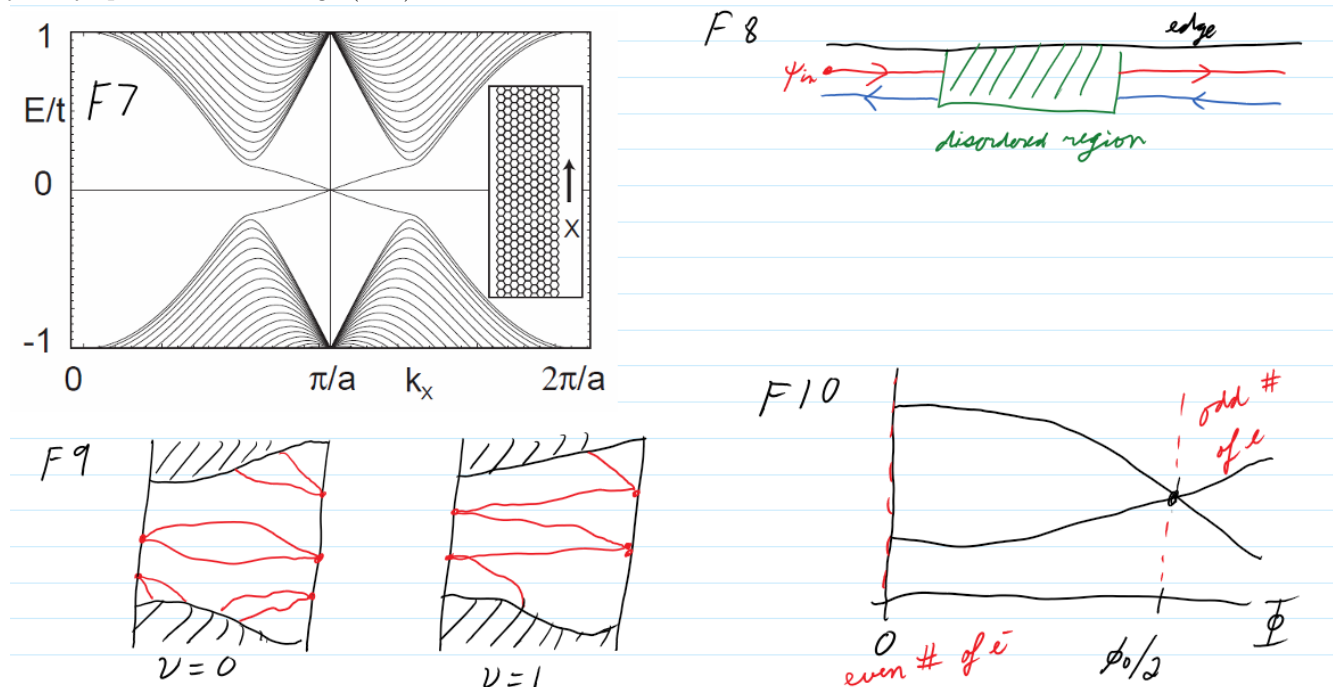
We can also explain this effect via a variant on Laughlin's argument. Consider the original cylindrical system (F5), but now thread half of a flux quantum ( $\Delta\Phi = \frac{1}{2}\Phi_0$ ). You cannot gauge away the resulting flux, but the point



here is to show that threading creates a degeneracy and the presence of a spin current. In the model (4.1) where  $s_z$  is preserved, half of an up spin goes right and half of a down spin goes right. The total spin at the end is then changed by a  $\frac{1}{2}$ :

$$S_z \rightarrow S_z + \frac{1}{2}.$$

Therefore, the number of electrons at the edge goes up by one. If we assume the number is initially even, then the number is odd after threading. Thus, the net spin is half integer and there is a Kramers' degeneracy in the many-body spectrum of the edge (F10).<sup>2</sup>



#### IV.1.1. Formula for the $\mathbb{Z}_2$ invariant

Consider  $N$  filled bands (each with a spin degeneracy) with two-component wavefunctions  $|u_n(k)\rangle$  and look at matrix elements of  $\Theta$ :

$$W_{mn}(k) = \langle u_m(k) | \Theta | u_n(-k) \rangle \in U(N).$$

Since  $\Theta^2 = -1$ , we can see that  $W(k) = -W^T(-k)$ . In particular, at the four  $k$ -points  $\Lambda_1 = (0, 0)$ ,  $\Lambda_2 = (\pi, 0)$ ,  $\Lambda_3 = (\pi, \pi)$ , and  $\Lambda_4 = (0, \pi)$  in the 2D BZ which are invariant under TRS,

$$W(\Lambda_a) = -W^T(\Lambda_a)$$

and so  $W$  is antisymmetric. Such matrices have the property that

$$\det \{W(\Lambda_a)\} = \text{Pf}\{W(\Lambda_a)\}^2.$$

Picking one branch for the square root, one introduces

$$\delta(\Lambda_a) = \frac{\text{Pf}\{W(\Lambda_a)\}}{\sqrt{\det\{W(\Lambda_a)\}}} = \pm 1$$

<sup>2</sup> However, since Laughlin's argument only shows what happens after a full flux threading, it is not clear whether exactly half of a charge gets transported after half-flux threading.

and multiplies these to get the gauge-invariant

$$\nu = \delta(\Lambda_1) \delta(\Lambda_2) \delta(\Lambda_3) \delta(\Lambda_4) = \pm 1. \quad (4.2)$$

This is only meaningful if you define a continuous gauge. With additional parity symmetry  $P$ , each  $\delta$  is a product of the parity eigenvalues (taking one for each Kramers' pair, since product of all of them is one):

$$\delta(\Lambda_a) = \prod_m \xi_m.$$

#### IV.1.2. Realizations

The QSH effect exists in HgCdTe quantum wells (Bernevig/Hughes/Zhang 2006; Molenkamp et al. 2007). An example is the BHZ model

$$H = (m + k^2) \tau_z + v \tau_x \vec{\sigma} \cdot \vec{k}.$$

If  $m > 0$  ( $< 0$ ), this is called the uninverted (inverted) case. You can repeat the analysis of the quantum hall effect to see that there are edge states.

## V. CHARLIE KANE 4.5

### V.1. 3D Topological insulators

Just like the Dirac point in the hexagonal graphene example, the four time-reversal invariant points  $\Lambda_a$  on the 2D BZ surface of a 3D TI are also Dirac points. Along any line connecting two of them, one can see how they connect (either directly or in a zig-zag; see F9). Types of insulators include:

1. Trivial insulator.
2. Weak TI: layers of weakly coupled 2D layers. The Fermi surface divides the BZ into two regions, and each region encloses two Dirac points (F11). One could have Dirac lines, but the only protected regions are the four points. We think of the zero-coupling limit between the layers, but this includes cases which are adiabatically connected to the zero-coupling case. There are then three  $\mathbb{Z}_2$  invariants:

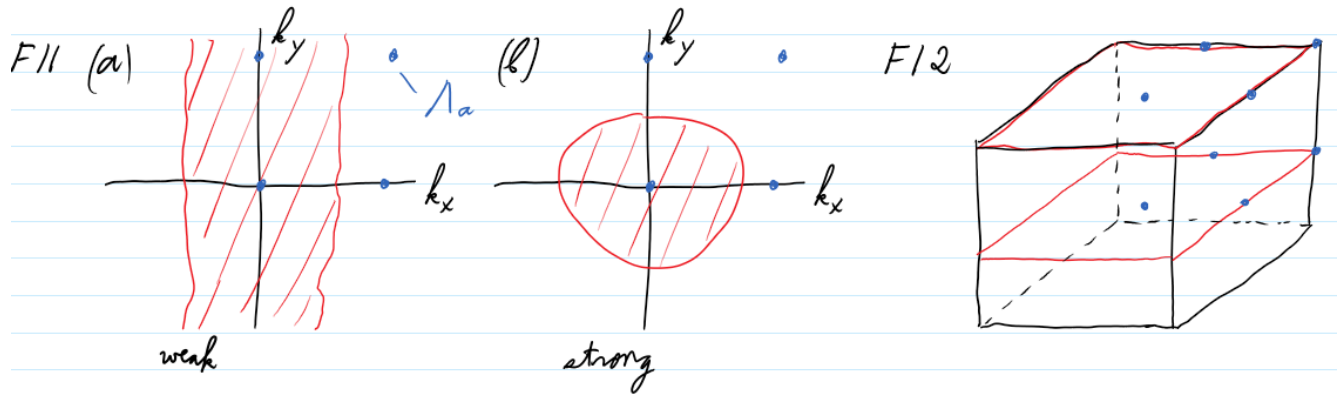
$$\vec{G} = \frac{2\pi}{a} \langle \nu_x, \nu_y, \nu_z \rangle$$

For example,  $\nu_z$  is just the invariant of the type (4.2) at  $k_z = 0$  and (identically)  $k_z = \frac{\pi}{a}$ . Any weak TI will be continuously connected to a stacking of layers in the  $\vec{G}$  direction. This is called a weak TI because it relies on translation symmetry of the lattice.

3. Strong TI: Now, the values of the  $\nu$ 's at  $k = 0$  and  $\frac{\pi}{a}$  are different, so now the invariant

$$\nu_0 = \nu(k_z = 0) \nu\left(k_z = \frac{\pi}{a}\right) = \prod_{a \in \text{TRS pt.}} \delta(\Lambda_a) = -1.$$

This invariant remains the same if instead  $k_{x,y}$  are considered. Now, the Fermi surface isn't connected at the top and bottom and is instead a circle (F11b). Therefore, it encloses either one of three Dirac points. Due to the bulk-boundary correspondence, the boundary invariant related to the bulk invariant  $\nu_0$  is then the  $e^{i\pi}$  Berry phase obtained by encircling the boundary of the Fermi surface. Recall that a Fermi surface with one Dirac point cannot exist in 2D due to the Nielsen/Ninomiya doubling theorem. However, in 3D, one Dirac point can exist on each of the two surfaces.



## V.2. Breaking $T$ symmetry

1. Orbital field. This leads to a surface QHE. Putting a Dirac point in a magnetic field leads to Landau levels (which can split at the surface; see related talk by Pablo Jarillo-Herrero). The conductivity is then

$$\sigma_{xy} = \frac{e^2}{h} \left( n + \frac{1}{2} \right).$$

This is weird because no electron edge state can carry half  $e^2/h$ . However, in a thin slab geometry, the top and bottom each contribute half, and together this leads to  $e^2/h$ . Moreover, there is a single chiral edge mode along the sides of the slab.

2. Zeeman field

$$H = v_F \vec{\sigma} \cdot \vec{p} + m(x) \sigma_z.$$

At points where  $m$  crosses zero, there is a domain wall and a chiral edge mode. Folding the two regions of positive/negative  $m$  on top of each other obtains case (1).

3. Gauge symmetry, leading to superconductivity.

$$H_{BdG} - \mu N = \tau_z (v_F \vec{\sigma} \cdot \vec{p} - \mu) + \Delta_1 \tau_x + \Delta_2 \tau_y \quad \Delta = \Delta_1 + i\Delta_2 = |\Delta| e^{i\phi}.$$

This is similar to the  $p + ip$  superconductor discussed by Nick Read in Sec. ??, whose vortex excitations are Majorana zero modes ( $\mu = 0$  case handled by Jackiw/Rossi).

One can then look at 1D and 2D domain walls between two gapped systems, one of which has a “magnetic” gap (2) while the other has a “superconducting” gap (3). These host chiral Majorana modes.