Low dimensional quantum gases

T. Giamarchi

http://dqmp.unige.ch/giamarchi/
Plan of the lectures

- Equilibrium, basic notions of Tomonaga-Luttinger liquid
- More on TLL (factionalization, topology, etc.)
  Effect of perturbations (Lattice and disorder)
- Notions of transport in 1D
  From 1D to 3D; Ladders; Dimensional crossover
Why one dimension?
Three urban legends about 1D

- It is a toy model to understand higher dimensional systems.

- It does not exist in nature! This is only for theorists!

- Everything is understood there anyway!
One dimension is specially interesting

- No individual excitation can exist (only collective ones)
- Strong quantum fluctuations

\[ \psi = |\psi| e^{i\theta} \quad \text{Difficult to order} \]
Many CM or cold atoms Systems

Quantum Spin Hall Effect
Drastic evolution of the 1d world

- **New methods** (DMRG, correlations from BA, etc.)
- **New systems** (cold atoms, magnetic insulators, etc.)
- **New questions** (strong SOC, out of equilibrium, etc.)
How to treat?
- "Standard" many body theory

- Exact Solutions (Bethe ansatz)

- Field theories (bosonization, CFT)

- Numerics (DMRG, MC, etc.)
References

TG, Quantum physics in one dimension, Oxford (2004)


M. Cazalilla et al., Rev. Mod. Phys. 83 1405 (2011)


And now we start....
Typical problem (e.g. Bosons)

- **Continuum:**

\[ H = \int dx \frac{(\nabla \psi)\dagger(\nabla \psi)}{2M} + \frac{1}{2} \int dx \, dx' \, V(x - x') \rho(x) \rho(x') - \mu \int dx \, \rho(x) \]

- **Lattice:**

\[ H = -J \sum_{\langle i,j \rangle} b_i^\dagger b_j + U \sum_i n_i (n_i - 1) - \mu \sum_i n_i \]
Luttinger liquid physics
Labelling the particles

\[ \rho(x) = \sum_i \delta(x - x_i) = \sum_n |\nabla \phi_l(x)| \delta(\phi_l(x) - 2\pi n) \]

1D: unique way of labelling
\[ \phi_l(x) = 2\pi \rho_0 x - 2\phi(x) \]

\[ \rho(x) = \left[ \rho_0 - \frac{1}{\pi} \nabla \phi(x) \right] \sum_p e^{i2p(\pi \rho_0 x - \phi(x))} \]

\( \phi(x) \) varies slowly

\( q \sim 0 \quad q \sim 2\pi \rho_0 \)
\[ \psi^\dagger(x) = [\rho(x)]^{1/2} e^{-i\theta(x)} \]

\[ \left[ \frac{1}{\pi} \nabla \phi(x), \theta(x') \right] = -i\delta(x - x') \]

\[ H = \frac{\hbar}{2\pi} \int dx \left[ \frac{uK}{\hbar^2} (\pi \Pi(x))^2 + \frac{u}{K} (\nabla \phi(x))^2 \right] \]
Correlations

\[ \langle \psi(r) \psi^\dagger(0) \rangle = A_1 \left( \frac{\alpha}{r} \right)^{\frac{1}{2K}} + \cdots \]

\[ \langle \rho(r) \rho(0) \rangle = \rho_0^2 + \frac{K}{2\pi^2} \frac{y_\alpha^2 - x^2}{(y_\alpha^2 + x^2)^2} + A_3 \cos(2\pi \rho_0 x) \left( \frac{1}{r} \right)^{2K} + \cdots \]
\( S(q, \gamma) \) J.S. Caux et al PRA 74 031605 (2006)
Finite temperature

Conformal theory

\( e^{-(2K)\frac{\pi\chi}{\beta}} = e^{-x/\xi_{\beta}} \)

\( \chi \)

\( \left( \frac{1}{r} \right)^{2K} \)
Other 1D systems
Spins

Use boson or fermions mapping

\[ S^+ = (-1)^i e^{i\theta} \cos(2\phi) + e^{i\theta} \cos(2\phi) \]

\[ S^z = \frac{-1}{\pi} \nabla \phi + (-1)^i \cos(2\phi) \]

Powerlaw correlation functions

\[ \langle S^z(x, 0) S^z(0, 0) \rangle = C_1 \frac{1}{x^2} + C_2 (-1)^x \left( \frac{1}{x} \right)^{2K} \]

\[ \langle S^+(x, 0) S^-(0, 0) \rangle = C_3 \left( \frac{1}{x} \right)^{2K + \frac{1}{2K}} + C_4 (-1)^x \left( \frac{1}{x} \right)^{\frac{1}{2K}} \]

Non universal exponents \( K(h, J) \)
Fermions

\[ \psi_{F}^{\dagger}(x) = \psi_{B}^{\dagger}(x) e^{i \frac{1}{2} \phi(x)} \]

\[ \psi_{F}^{\dagger}(x) = \left[ \rho_0 - \frac{1}{\pi} \nabla \phi(x) \right]^{1/2} \sum_{p} e^{i(2p+1)(\pi \rho_0 x - \phi(x))} e^{-i\theta(x)} \]

Right (+k_F) and left (-k_F) particles

\[ \langle \rho(x, \tau) \rho(0) \rangle = \rho_0^2 + \frac{K}{2\pi^2} \frac{y_\alpha^2 - x^2}{(x^2 + y_\alpha^2)^2} + \rho_0^2 A_2 \cos(2\pi \rho_0 x) \left( \frac{\alpha}{r} \right)^{2K} \]

\[ + \rho_0^2 A_4 \cos(4\pi \rho_0 x) \left( \frac{\alpha}{r} \right)^{8K} \]
Luttinger liquid concept

• How much is perturbative?

• Nothing (Haldane): provided the correct $u, K$ are used

• Low energy properties: Luttinger liquid (fermions, bosons, spins…)
Luttinger parameters

General Relation of Correlation Exponents and Spectral Properties of One-Dimensional Fermi Systems: Application to the Anisotropic $S = \frac{1}{2}$ Heisenberg Chain

F. D. M. Haldane

Attractive Hubbard model

TG + B. S. Shastry PRB 51 10915 (1995)
``Quantitative” theory possible

- Trick: use thermodynamics and BA or numerics
- Compressibility: $u/K$
- Response to a twist in boundary: $u K$
- Specific heat: $T/u$
- Etc.
Tonks limit

U = 1 : spinless fermions

Not for \( n(k) : \psi_F \neq \psi_B \)

Free fermions: \[ \langle \rho(x)\rho(0) \rangle \propto \cos(2k_F x) \left( \frac{1}{x} \right)^2 \]

K=1

Note: \[ \langle \psi_B(x)\psi_B(0)^\dagger \rangle \propto \left( \frac{1}{x} \right)^{1/2} \]
Tests of Luttinger liquids
Organic conductors

\( \sigma(\omega) \sim \omega^{-\nu} \)

TG PRB (91) :
A. Schwartz et al. PRB 58 1261 (1998)

First observation of LL/powerlaw !!
Cold gases
Observation of a One-Dimensional Tonks-Girardeau Gas

Toshiya Kinoshita, Trevor Wenger, David S. Weiss*
Optical lattices (dilute)


\[ n(k) = \int dx e^{ikx} \langle \psi^\dagger(x) \psi(0) \rangle \]
Atom chips

\[ \int_0^L dr \langle \psi(r) \psi^\dagger(0) \rangle \]

K large (42)

ULTRACOLD ATOMS
CONFIRM 55-YEAR-OLD PHYSICS THEORY

https://www.futurity.org/one-dimensional-electrons-physics-1858622/
Other important 1D properties
Fractionalization of excitations
Fractionalization of excitations

\[ H = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1} \]

\[ \Delta S = -1 \quad E = \epsilon(q) \quad \text{Magnon} \]

\[ \Delta S = -1/2 \quad \text{Spinons} \quad \Delta S = -1/2 \]
Magnons and spinons: $1 = \frac{1}{2} + \frac{1}{2}$

- Hidden (topological) order parameters

- Continuum of excitations

\[ E(k) = \cos(k_1) + \cos(k_2) \]

\[ k = k_1 + k_2 \]
Deconstruction of the electron: spin-charge separation
Spin and Charge Resolved Quantum Gas Microscopy of Antiferromagnetic Order in Hubbard Chains


The inset shows the decay of the rectified spin correlations \((-1)^d C(d)\) in a logarithmic plot together with an exponential fit \(C(d) \propto \exp(-d/\xi)\), which reveals an average correlation length of \(\xi = 0.9(1)\) sites and \(\xi = 1.4(4)\) sites for the lowest entropy tube. All error bars represent one s.e.m.
Doped Hubbard model

Spin-Charge Separation higher D?

Spin

Charge

Energy increases with spin-charge separation

Confinement of spin-charge: « quasiparticle »
Confinement of spinons BaCoVO

Take home message

- Good theoretical methods to deal with the case of a "simple" equilibrium 1d systems (analytics and numerics)

- Stepping stone to go beyond: many exciting questions and problems (out of equilibrium, disorder, many chains, etc.)

- Controlled experimental realizations in condensed matter and cold atoms
Effect of lattices: 
Mott transition
Mott transition

- Mott insulator \((n=1)\)
- \(T < T_N\) : antiferromagnetic phase
Periodic lattice

\[ H = \int dx V_0 \cos(Qx) \rho(x) \]

- Incommensurate: \( Q \neq 2 \pi \rho_0 \)

\[ H = \int dx V_0 \cos(Qx) \rho_0 e^{i(2\pi \rho_0 x - 2\phi(x))} \]

- Commensurate: \( Q = 2 \pi \rho_0 \)

\[ H = \int dx \cos(2\phi(x) + \delta x) \]

\[ H = \int dx \cos(2\phi(x)) \]
Competition

\[ S_0 = \int \frac{dxd\tau}{2\pi K} \left[ \frac{1}{u} (\partial_\tau \phi(x, \tau))^2 + u (\partial_x \phi(x, \tau))^2 \right] \]

\[ S_L = -V_0 \rho_0 \int dxd\tau \cos(2\phi(x)) \]

Beresinskii-Kosterlitz-Thouless transition at \( K=2 \)

String order parameter
BKT transition

\[ H = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) \]

- BKT: remarkable transition going outside the paradigm of Landau’s phase transitions
- A transition without an order parameter
- Topological Vortex excitations
**Vortex operator**

\[ e^{iaP} \left| x \rightangle = \left| x + a \rightangle \]

\[ \phi(x, \tau) = \pi \int_{-\infty}^{x} dx' \Pi_\theta(x', \tau) \]

\[ \cos(2\phi(x_1, \tau_1)) \]

- Vortex operator for \( \theta \)
- \( K \) : inverse temperature
- \( g \) : vortex fugacity

\[ S = \frac{K}{2\pi} \int dx d\tau \left[ \frac{1}{u} (\partial_x \theta) + u(\partial_x \theta)^2 \right] - g \int dx \cos(2\phi) \]
Mott insulator: 
ϕ is locked

Density is fixed

TG, Physica B
230 975(97):
arXiv/0605472
(Salerno lectures);
Oxford (2004);

M. Cazalilla et al.,
Rev. Mod. Phys.83
1405 (2011)

Gap in the excitation spectrum

\[ G(r) \propto e^{-r/\xi} \]

G. Boeris et al. PRA 93 93, 011601(R) (2016)
Non local (topological) order

\[ \rho(x) \sim \nabla \phi(x) \]

\[ \mathcal{O}_p^2 = \lim_{l \to \infty} \mathcal{O}_p^2(l) = \lim_{l \to \infty} \left( \prod_{k \leq j \leq k+l} e^{i \pi \delta n_j} \right) \]

E. Berg, E. Dalla Torre, T. Giamarchi, E. Altman,

Observation of Correlated Particle-Hole Pairs and String Order in Low-Dimensional Mott Insulators


Science (2011)
Topological excitations is the norm in 1D
Disorder (equilibrium)
Example: localization of 1D interacting bosons

\[ H = \sum_{i=1}^{N} \frac{P_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} V(R_i - R_j) + \sum_{i=1}^{N} D(R_i) \]


Competition Disorder vs Interactions

Existence of a Many-Body localized phase: Bose glass
Bose glass phase

\[ \langle \psi(x) \psi^\dagger(0) \rangle \sim e^{-x/\xi} \]
Other potentials: Biperiodics

\[ V(x) = V_0 \cos(Q_0 x) + V_1 \cos(Q_1 x) \]

- \( U = 0 \)
  - Aubry-André model
- Localization transition

Effect of interactions?
Same as "true" disorder?

J. Vidal, D. Mouhanna, TG PRL 83 3908 (1999); PRB 65 014201 (2001)

G. Roux et al. PRA 78 023628 (2008);
X. Deng et al PRA 78, 013625 (2008);
Quasi-periodicics and interactions

C. D’Errico, E. Lucioni et al. PRL (2014);
L. Gori et al PRA 93 033650 (2016)
\[ V(x) = V_0 \cos(Q_0 x) + V_0 \cos(Q_1 x) \]
d.c. transport at finite T

- In 1D all states are exponentially localized
- Finite T: sweeps energy $E$ with probability $f(E)$
- No interactions $\sigma(T)=0$ for all $T$
- Needs coupling to a thermal bath (phonons, etc.)
- Or with interactions can the system be its own thermal bath?
With a bath
d.c. transport (high $T$)

Transport with a Thermostat

- Mott variable range hopping

$$e^{-\beta E} e^{-L/\xi_{loc}}$$
$$N_0 L^d E$$

- 1D with interactions


arXiv:cond-mat/0403487

$$\sigma(T) \propto e^{- \left( \frac{S^*}{h} \right)} = \exp\left[ -\frac{4\pi}{K^*} \sqrt{2\beta \Delta} \right].$$
Without a bath

• With interactions can other particles be a ``bath'' for one particle?

• Can the system reach the thermodynamic equilibrium? And explore ergodically the phase space?
Many body localization

Basko, Aleiner, Altshuler; Gornyi, Mirlin Polyakov; Huse, ……


No thermostat $\sigma(T) = 0$ if $T<T^*$ even with interactions

The system is not ergodic even at finite temperatures/energies
Take home message

- Remarkable interplay between localization and interactions
- Consequences and challenges both in equilibrium (LIP) and out of equilibrium (MBL).
- Experimental possibilities to explore these phenomena
- Need better tools!
Transport
Transport

- Condensed matter: a «routine» probe for materials

- Theoretically: complicated!
  Out of equilibrium

- Typical situation

- Often (but not always !) : linear response $I = G V$
Questions

- $I = f(V)$ Reflects the properties of the system (interferences – metal, insulator, etc.)

- Methods?
  
  Kubo; Memory function; Landauer; Keldysh;
  
  …..

- Expectations for small $V$:
  
  $$V = RI$$
  
  $$R = R_{\text{contact}} + R_{\text{system}}$$

- Other transport(s): spin, temperature, etc.
Cold atoms

Conduction of Ultracold Fermions Through a Mesoscopic Channel

Jean-Philippe Brantut, Jakob Meineke, David Stadler, Sebastian Krinner, Tilman Esslinger

SCIENCE VOL 337 31 AUGUST 2012
1D ballistic: quantized conductance

Quantized Conductance of Point Contacts in a Two-Dimensional Electron Gas

B. J. van Wees
Department of Applied Physics, Delft University of Technology, 2628 CJ Delft, The Netherlands

H. van Houten, C. W. J. Beenakker, and J. G. Williamson,
Philips Research Laboratories, 5600 JA Eindhoven, The Netherlands

L. P. Kouwenhoven and D. van der Marel
Department of Applied Physics, Delft University of Technology, 2628 CJ Delft, The Netherlands

and

C. T. Foxon
Philips Research Laboratories, Redhill, Surrey RH1 5HA, United Kingdom
(Received 31 December 1987)

Observation of quantized conductance in neutral matter

Sebastian Känneri, David Stadler, Dominik Husmann, Jean-Philippe Brantut & Tilman Esslinger

64 | NATURE | VOL 517 | 1 JANUARY 2015
Atomtronic

M. Lebrat, P. Grisins et al., PRX 8 011053 (18)
No interactions: band insulator
What happens with interactions?

\[ H = H_{GY} + H_{lattice}, \]

\[ H_{GY} = -\frac{\hbar^2}{2m} \sum_i \frac{\partial^2}{\partial y_i^2} + g_1 \sum_{i<j} \delta(y_i - y_j), \]

\[ H_{lattice} = \int dy \ V(y) \rho(y), \]
Luther-Emery liquid

- **Gap in the spin sector** (singlet pairing)

\[
\rho(y) = \rho_0 - \frac{\sqrt{2}}{\pi} \nabla \phi_c(y) \\
+ 2\rho_0 f_s \cos \left( 2k_F y - \sqrt{2}\phi_c(y) \right) \\
+ 2C\rho_0 \cos \left( 4k_F y - 2\sqrt{2}\phi_c(y) \right),
\]

- **Conductance determined by the charge sector**

\[
\mathcal{H}_\mu(y) = -\mu(y)\rho(y) = \mu(y) \frac{\sqrt{2}}{\pi} \nabla \phi_c,
\]

\[
I_{\uparrow\downarrow}(y) = \frac{\sqrt{2}}{\pi} \partial_t \phi_c(y, t)
\]
Many-body insulator "pinned" L.E. liquid
Experimental evidence for L.E. liquid
Spin transport

Spin transport and spin drag

\[ J_\sigma = \langle j_{\uparrow} - j_{\downarrow} \rangle = G_\sigma (\mu_{\uparrow} - \mu_{\downarrow}) \]

**Attractive**

**Repulsive**

Spin Drag: \( \Delta \mu_\uparrow \) \( I_\downarrow \) \( I_\downarrow = \Gamma \Delta \mu_\uparrow \)
Solution (using Sine-Gordon)

\[ H^0 = \frac{1}{2\pi} \int dx \left\{ v_v K_v [\partial_x \theta_v(x, t)]^2 + \frac{v_v}{K_v} [\partial_x \phi_v(x, t)]^2 \right\}. \]

\[ \frac{2g_{1\perp}}{(2\pi \alpha)^2} \int_{-L/2}^{L/2} dx \cos[2\sqrt{2}\phi_\sigma(x, t)], \]

\[ J \propto \partial_t \phi_\sigma \]

<table>
<thead>
<tr>
<th>weak coupling</th>
<th>strong coupling</th>
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<tr>
<td>( L \ll L_\Delta )</td>
<td>( L_\Delta \ll L )</td>
</tr>
<tr>
<td>( \Delta_\sigma \ll T_L )</td>
<td>( T_L \ll \Delta_\sigma )</td>
</tr>
<tr>
<td>memory function</td>
<td></td>
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<tr>
<td>( G_\sigma - 1 \propto -y_{1\perp}^2 L(T/\Lambda)^{4K_\sigma - 3} )</td>
<td>( e^{-\frac{\Delta_\sigma}{T}} )</td>
</tr>
</tbody>
</table>

- high temperature \( \Delta_\sigma, T_L \ll T \)
- intermediate temperature \( T_L \ll T \ll \Delta_\sigma, \Delta_\sigma \ll T \ll T_L \)
- low temperature \( T \ll T_f \ll T_L, \Delta_\sigma \)
- low temperature \( T \ll T_f \ll T_L, \Delta_\sigma \)

(a) \( G_\sigma \)

(b) \( G_\sigma \)

2-step RG

3-step RG
Comparison with experiments

![Graph showing comparison with experiments]
Lindblad reservoirs and losses

T. Jin, M. Filippone, TG PRB 102 205131 (2021)

\[ J_L = \frac{1}{2} \left\{ (\alpha_L - \beta_L - \alpha_R + \beta_R) + \frac{i}{2} \int \frac{d\epsilon}{2\pi} \text{Tr} \left[ (\alpha_L + \beta_L)\gamma_L - (\alpha_R + \beta_R)\gamma_R \right] G^\xi \right\}, \]

\[ J_{D,L} = \frac{i}{2} \int \frac{d\epsilon}{2\pi} \text{Tr} \left[ (\alpha_L + \beta_L)\gamma_L + (\alpha_R + \beta_R)\gamma_R \right] G^\xi. \]

T. Jin, J. Ferreira, M. Filippone, TG arxiv/2106.14765

\[ \int dxdt \xi(x,t) \rho(x,t) \]

Ballistic ?
Diffusive ?
Artificial gauge fields
Meissner effect in bosonic ladders

E. Orignac, TG, PRB 64 144515 (2001)
\[ H = -t_\parallel \sum_{i,p=1,2} (b_{i,p+1} e^{ie^* a A_{\parallel,p}(i)} b_{i,p} + b_{i,p} e^{-ie^* a A_{\parallel,p}(i)} b_{i+1,p}) \]

\[ -t_\perp \sum_i (b_{i,2} e^{ie^* A_{\perp,i}(i)} b_{i,1} + b_{i,1} e^{-ie^* A_{\perp,i}(i)} b_{i,2}) \]

\[ + U \sum_{i,p} n_{i,p} (n_{i,p} - 1) + V n_{i,1} n_{i,2}, \]

(1)

\[ \int \vec{A} \cdot d\vec{l} = \Phi \]

\[ H = H_s^0 + H_a^0 - \frac{t_\perp}{\pi a} \int dx \cos[\sqrt{2} \theta_a + e^* A_{\perp}(x)] \]

\[ + \frac{2 V a}{(2 \pi a)^2} \int dx \cos \sqrt{8} \phi_a, \]
Orbital currents (``Meissner'' effect)

Field ``Hc1'': appearance of vortices
Artificial gauge field (cold atoms)

M Atala et al. Nat Phys, 10 588 (2014)
Hall effect

Non interacting "simple"

\[ R_h = \frac{V_\perp}{I \parallel B} \propto \frac{1}{n} \]

No interactions: curvature of fermi surface
- topological formula (Thouless-Kohmoto)

With interactions: open question
Hall effect: measurements
Hall effect on ladders

S. Greshner, M. Filippone, TG,
PRL 122, 083402 (19)

Analytic: calculation on a ring
Experiments: out of equilibrium
Vanishing Hall response

M. Filippone, C.E. Bardyn, S. Greshner, TG, PRL 123, 086803 (19)

Exact zero of the Hall effect for LB geometry
Hall voltage vs polarization

M. Buser, S. Greshner, U. Schollwöck, TG, PRL 126, 030501 (21)

Practical protocol to compute Hall voltage $V_h$ from Hall polarization $P_h$
Dimensional crossover
• Largely open physics
• Strong difficulties to treat (analytics, numeric)
• Allows to incorporate SOC and magnetic field effects
• Many studies limited to non-interacting case
• Relevant for many experimental systems
Deconfinement

\((\text{TMTTF})_2\text{PF}_6\)

\[
\begin{array}{c}
T(K) \\
\hline
100
\end{array}
\]

\[
\begin{array}{c}
P(\text{GPa}) \\
\hline
0 \quad 2 \quad 4 \quad 6 \quad 8
\end{array}
\]


Deconfinement

Activation energy, $T^*$ (K)

Pressure (kbar)

Spin-Peierls

C SDW

IC SDW


Chain (Cluster) - DMFT

S. Biermann, A. Georges, A. Lichtenstein, TG, PRL 87 276405 (2001)
C. Berthod et al. PRL 97, 136401 (2006)

Bath must be self consistently determined
Transverse transport

$T > t'$: tunneling, not usual transport

\[ \sigma(\omega, T) \propto (\omega, T)^{2\alpha - 1} \]

\[ \alpha = \frac{1}{4} (K + K^{-1}) - \frac{1}{2} \]
Understanding repulsively mediated superconductivity of correlated electrons via massively parallel density matrix renormalization group

A. Kantian, M. Dolfi, M. Troyer, and T. Giamarchi

<table>
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<th>$N_{ch}$</th>
<th>$V/t = 0.5$</th>
<th>$V/t = 1$</th>
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<td>8</td>
<td>9.45</td>
<td></td>
<td>7.6(1.3)$^c$</td>
<td></td>
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Conclusions

- Tour of one dimensional physics

- Luttinger liquid theory provides a framework to study this physics, and to go beyond

- Beautiful open problems: out of equilibrium, disorder, coupled chains, impurities, etc. ……

- Requires interplay of analytical and numerical techniques (and new ideas!) to make progress

- Remarkable and complementary realization in condensed matter and cold atomic gases