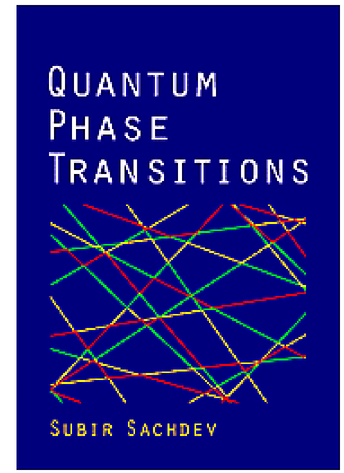
Quantum phase transitions

- http://onsager.physics.yale.edu/c41.pdf

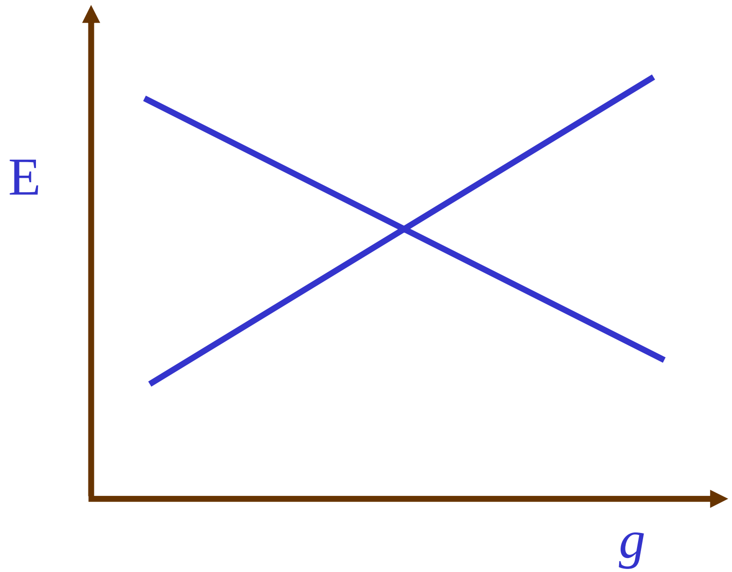
- cond-mat/0109419

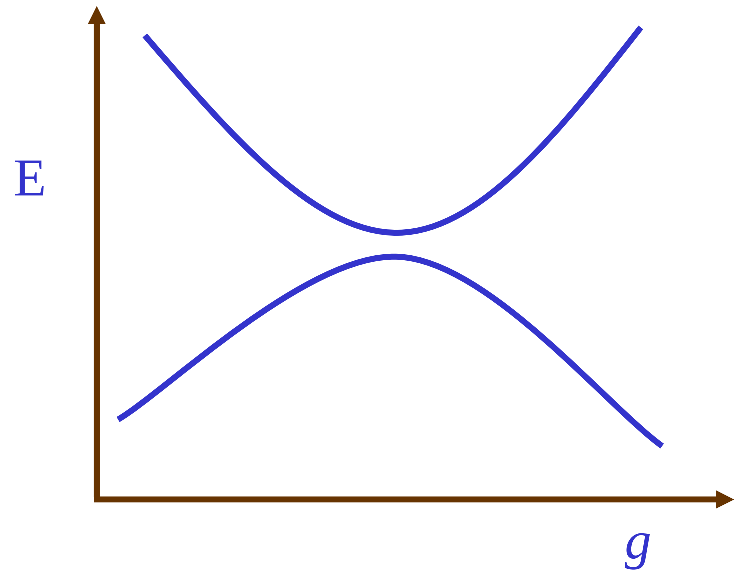


Quantum Phase Transitions
Cambridge University Press

What is a quantum phase transition ?

Non-analyticity in ground state properties as a function of some control parameter *g*



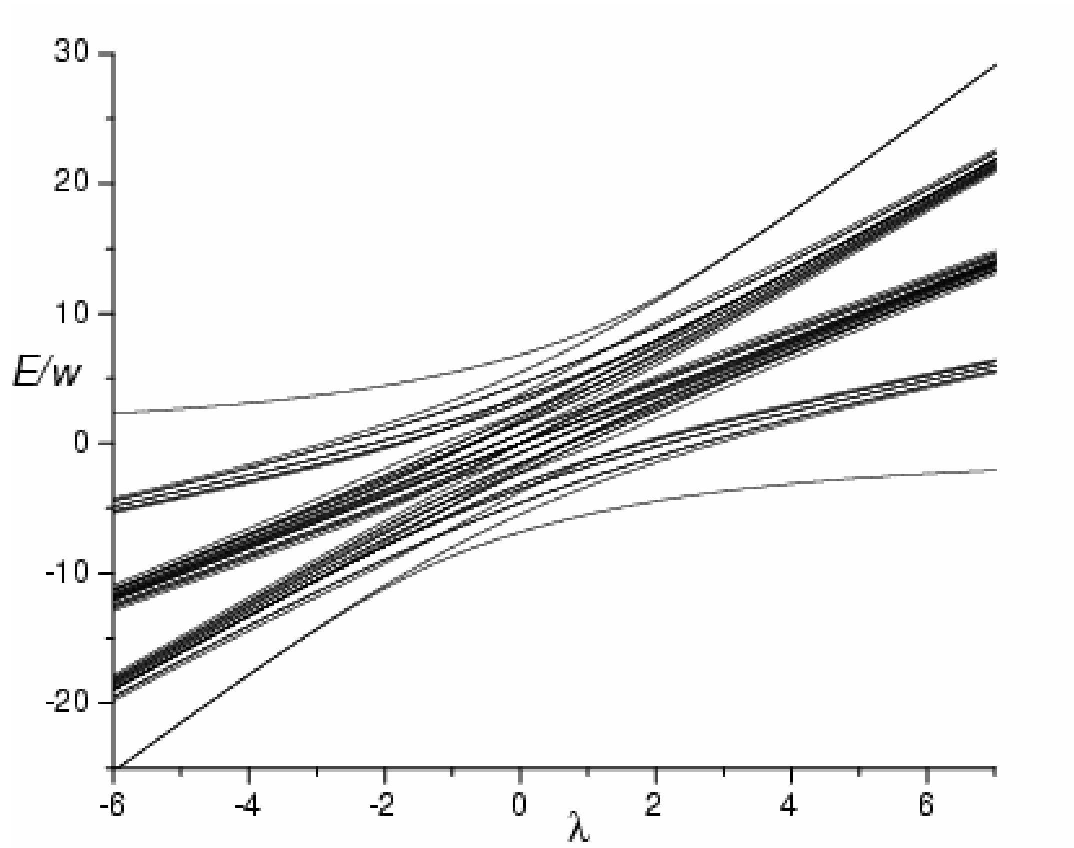


True level crossing:

Usually a *first*-order transition

Avoided level crossing which becomes sharp in the infinite volume limit:

second-order transition



30
20
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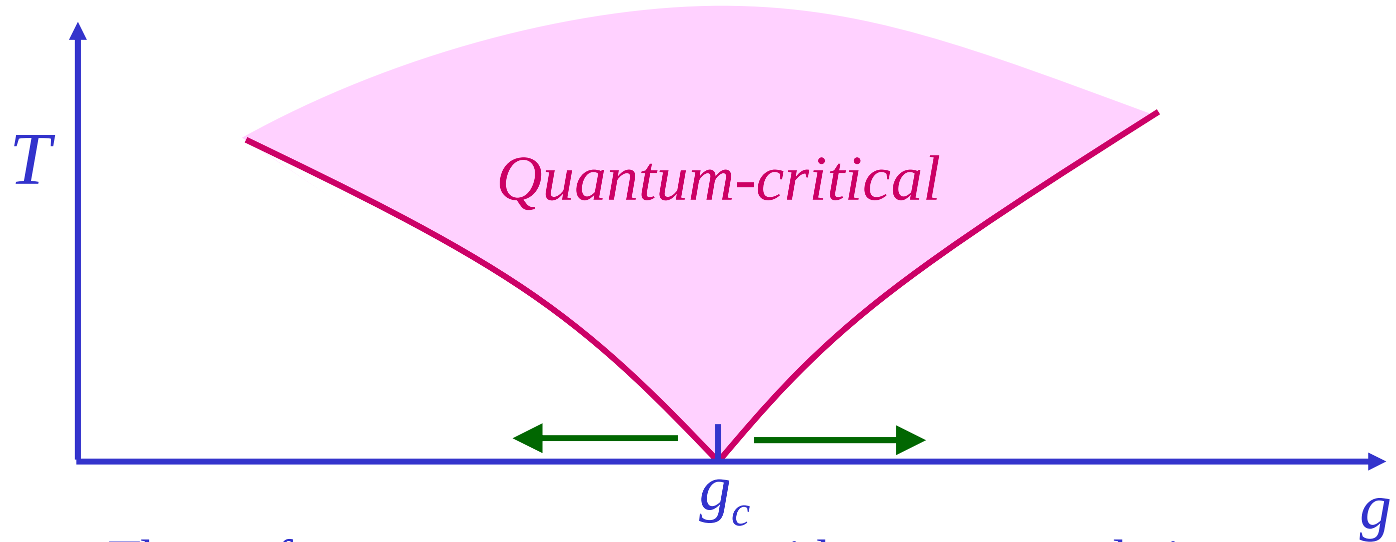
Why study quantum phase transitions ?



• Theory for a quantum system with strong correlations: describe phases on either side of g_c by expanding in deviation from the quantum critical point.

• Critical point is a novel state of matter without quasiparticle excitations

• Critical excitations control dynamics in the wide *quantum-critical* region at non-zero temperatures.

Important property of ground state at $g=g_c$:
temporal and spatial scale invariance;
characteristic energy scale at other values of g: $\Delta \sim |g - g_c|^{z\nu}$

Outline

I. Quantum Ising Chain

II. Landau-Ginzburg-Wilson theory
  Mean field theory and the evolution of the excitation spectrum.

III. Superfluid-insulator transition
  Boson Hubbard model at integer filling.

IV. Tilting the Mott insulator
  Density wave order at an Ising transition.

V. Bosons at fractional filling
  Beyond the Landau-Ginzburg-Wilson paradigm.

I. Quantum Ising Chain

I. Quantum Ising Chain

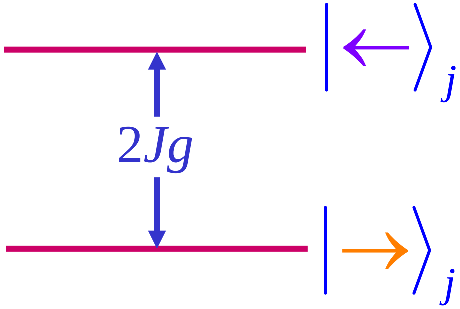
Degrees of freedom: $j = 1 \ldots N$ qubits, N "large"

$$\left|\uparrow\right\rangle_j , \left|\downarrow\right\rangle_j$$

or $$\left|\rightarrow\right\rangle_j = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_j + \left|\downarrow\right\rangle_j\right) , \quad \left|\leftarrow\right\rangle_j = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_j - \left|\downarrow\right\rangle_j\right)$$

Hamiltonian of decoupled qubits:

$$H_0 = -Jg\sum_j \sigma_j^x$$



Coupling between qubits:

$$H_1 = -J\sum_j \sigma_j^z \sigma_{j+1}^z$$

$$\Big(|\rightarrow\rangle_j\langle\leftarrow| + |\leftarrow\rangle_j\langle\rightarrow|\Big)\Big(|\rightarrow\rangle_{j+1}\langle\leftarrow| + |\leftarrow\rangle_{j+1}\langle\rightarrow|\Big)$$

Prefers neighboring qubits

are *either* $|\uparrow\rangle_j|\uparrow\rangle_{j+1}$ *or* $|\downarrow\rangle_j|\downarrow\rangle_{j+1}$

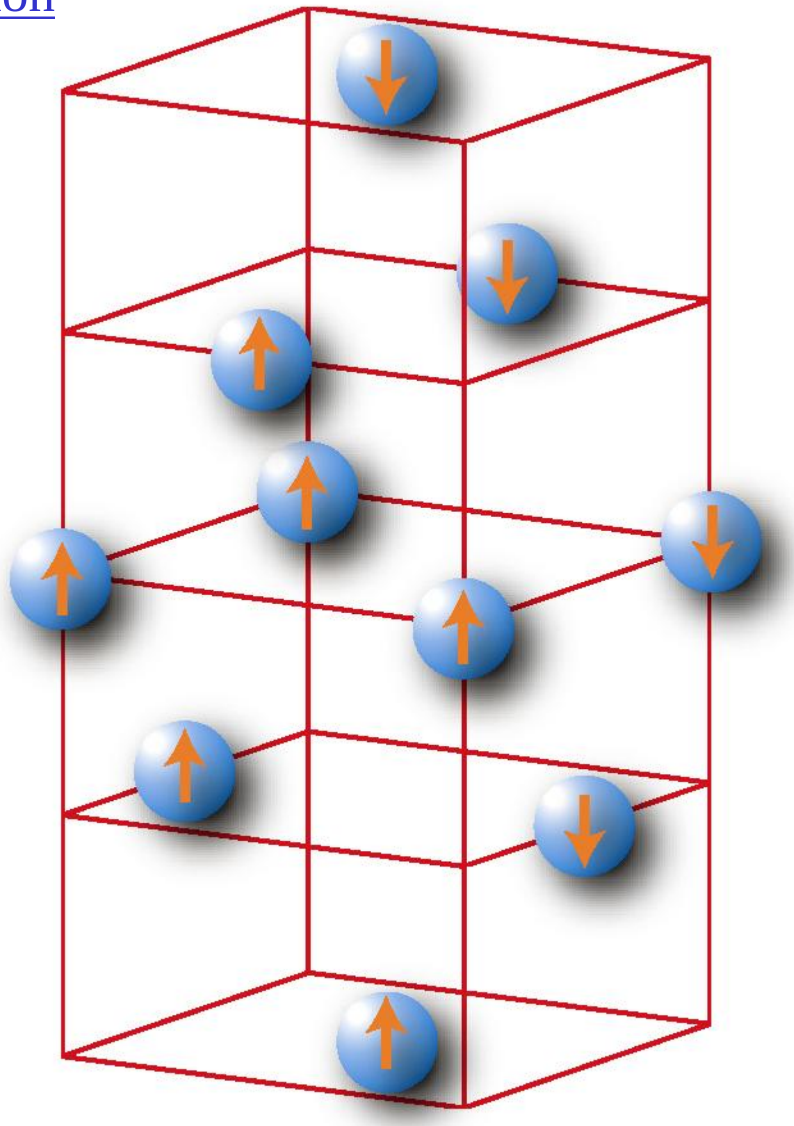
(not entangled)

Full Hamiltonian

$$H = H_0 + H_1 = -J\sum_j \left(g\sigma_j^x + \sigma_j^z \sigma_{j+1}^z \right)$$

leads to entangled states at *g* of order unity

Experimental realization



$LiHoF_4$

Weakly-coupled qubits $\left(g \gg 1\right)$

Ground state:

$$\left|G\right\rangle = \left|\cdots\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\cdots\right\rangle$$

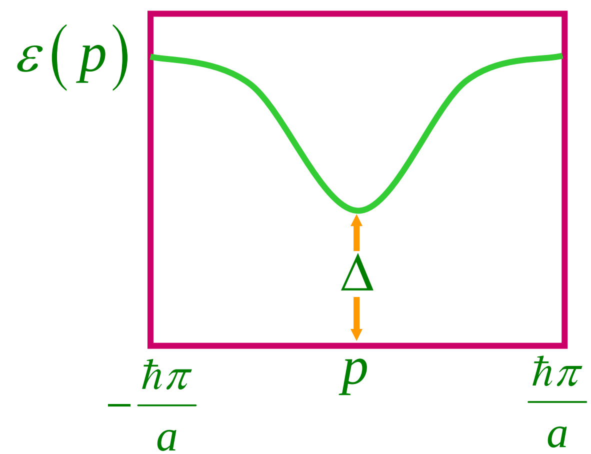
$$-\frac{1}{2g}\left|\cdots\rightarrow\rightarrow\rightarrow\rightarrow\leftarrow\leftarrow\rightarrow\rightarrow\rightarrow\rightarrow\cdots\right\rangle - \cdots$$

Lowest excited states:

$$\left|\ell_j\right\rangle = \left|\cdots\rightarrow\rightarrow\rightarrow\rightarrow\leftarrow_j\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\cdots\right\rangle + \cdots$$

Coupling between qubits creates "flipped-spin" *quasiparticle* states at momentum p

$$\left|p\right\rangle = \sum_j e^{ipx_j/\hbar}\left|\ell_j\right\rangle$$



Excitation energy $\varepsilon(p) = \Delta + 4J\sin^2\left(\frac{pa}{2\hbar}\right) + O\left(g^{-1}\right)$

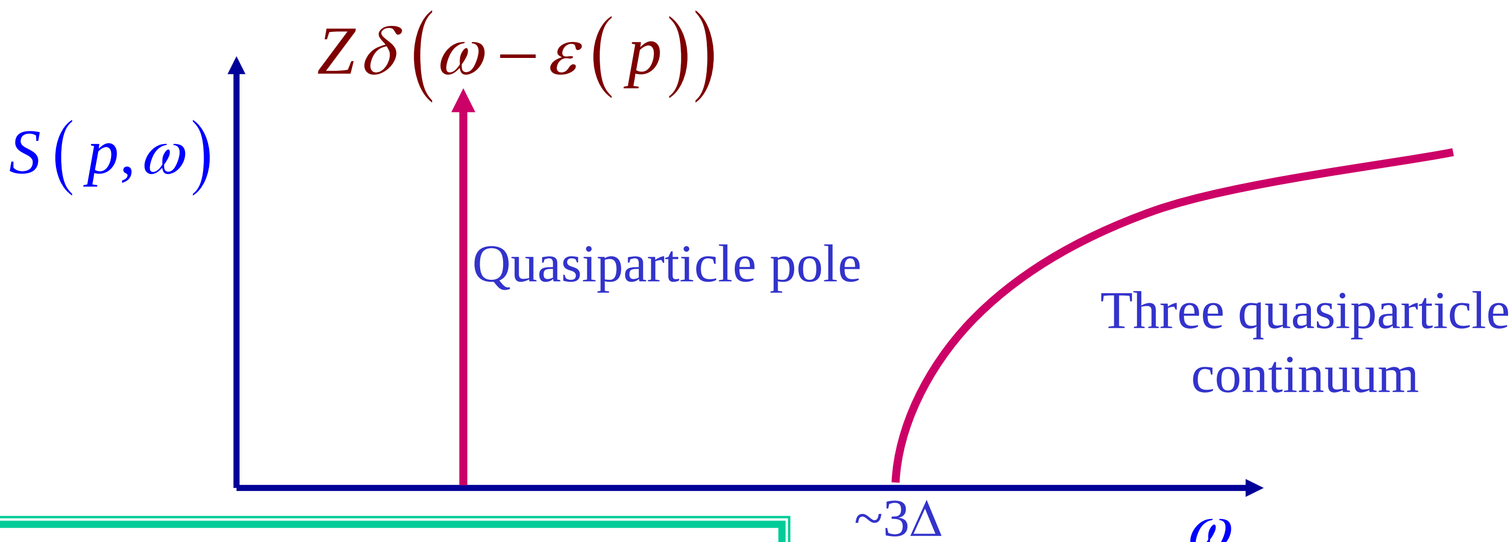
Excitation gap $\Delta = 2gJ - 2J + O\left(g^{-1}\right)$

Entire spectrum can be constructed out of multi-quasiparticle states

Dynamic Structure Factor $S(p,\omega)$:

Weakly-coupled qubits $(g \gg 1)$

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa) while transferring energy $\hbar\omega$ and momentum p



Structure holds to all orders in $1/g$

At $T > 0$, collisions between quasiparticles broaden pole to a Lorentzian of width $1/\tau_\varphi$ where the ***phase coherence time*** τ_φ is given by

$$\frac{1}{\tau_\varphi} = \frac{2k_BT}{\pi\hbar} e^{-\Delta/k_BT}$$

S. Sachdev and A.P. Young, *Phys. Rev. Lett.* **78**, 2220 (1997)

Strongly-coupled qubits $\left(g \ll 1\right)$

Ground states:

$$\left|G\uparrow\right\rangle = \left|\cdots\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\right\rangle$$

$$-\frac{g}{2}\left|\cdots\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\right\rangle - \cdots$$

Ferromagnetic moment

$$N_0 = \left\langle G\right|\sigma^z\left|G\right\rangle \neq 0$$

Second state $\left|G\downarrow\right\rangle$ obtained by $\uparrow \Leftrightarrow \downarrow$

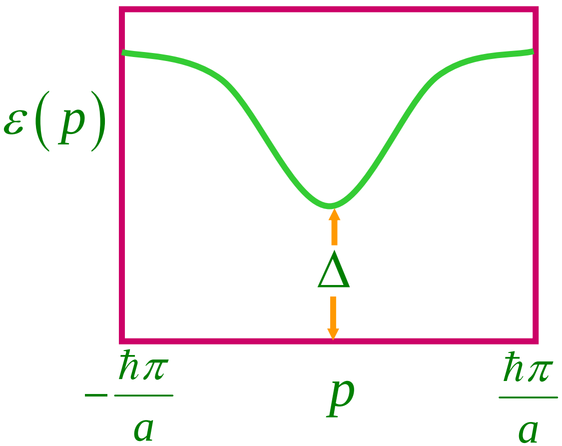
$\left|G\downarrow\right\rangle$ and $\left|G\uparrow\right\rangle$ mix only at order g^N

Lowest excited states: domain walls

$$\left|d_j\right\rangle = \left|\cdots\uparrow\uparrow\uparrow\uparrow\uparrow_j\downarrow\downarrow\downarrow\downarrow\downarrow\cdots\right\rangle + \cdots$$

Coupling between qubits creates new "domain-wall" *quasiparticle* states at momentum p

$$\left|p\right\rangle = \sum_j e^{ipx_j/\hbar}\left|d_j\right\rangle$$



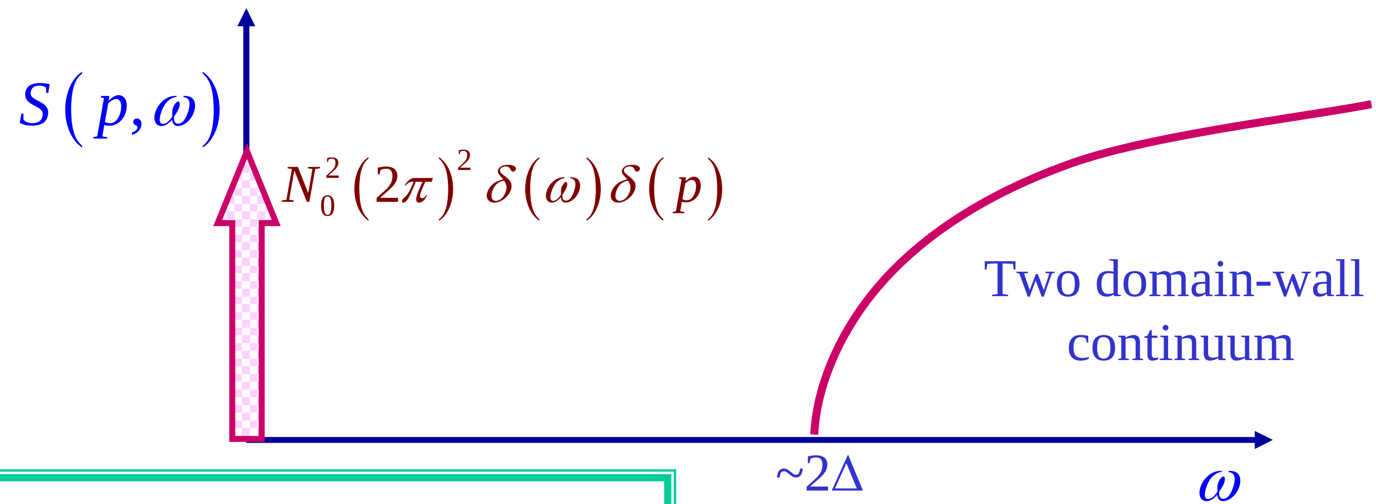
Excitation energy $\varepsilon(p) = \Delta + 4Jg\sin^2\left(\frac{pa}{2\hbar}\right) + O\left(g^2\right)$

Excitation gap $\Delta = 2J - 2gJ + O\left(g^2\right)$

Strongly-coupled qubits $(g \ll 1)$

Dynamic Structure Factor $S(p,\omega)$:

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa) while transferring energy $\hbar\omega$ and momentum p



Structure holds to all orders in g

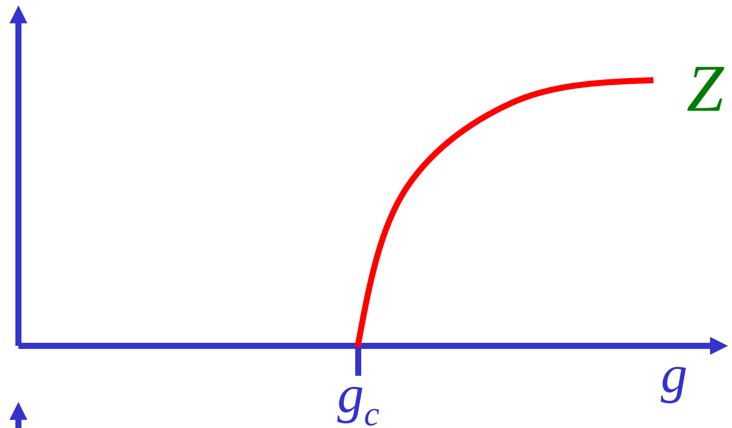
At $T > 0$, motion of domain walls leads to a finite ***phase coherence time*** τ_φ, and broadens coherent peak to a width $1/\tau_\varphi$ where

$$\frac{1}{\tau_\varphi} = \frac{2k_B T}{\pi\hbar} e^{-\Delta/k_B T}$$

S. Sachdev and A.P. Young, *Phys. Rev. Lett.* **78**, 2220 (1997)

Entangled states at g of order unity

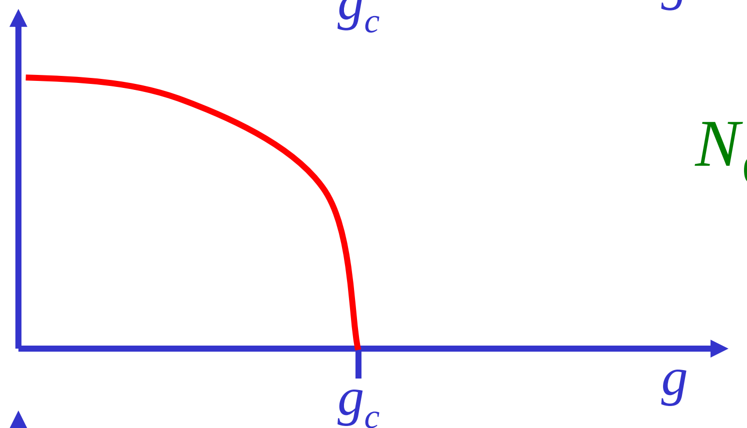
"Flipped-spin" Quasiparticle weight Z



$$Z \sim (g - g_c)^{1/4}$$

A.V. Chubukov, S. Sachdev, and J.Ye, *Phys. Rev.* B **49**, 11919 (1994)

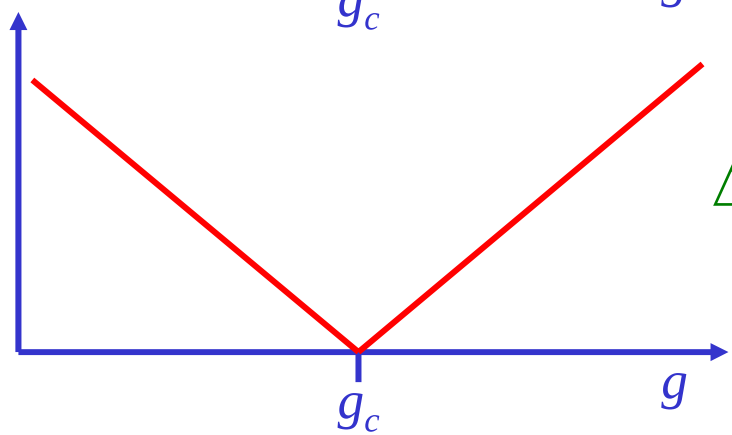
Ferromagnetic moment N_0



$$N_0 \sim (g_c - g)^{1/8}$$

P. Pfeuty *Annals of Physics*, **57**, 79 (1970)

Excitation energy gap Δ



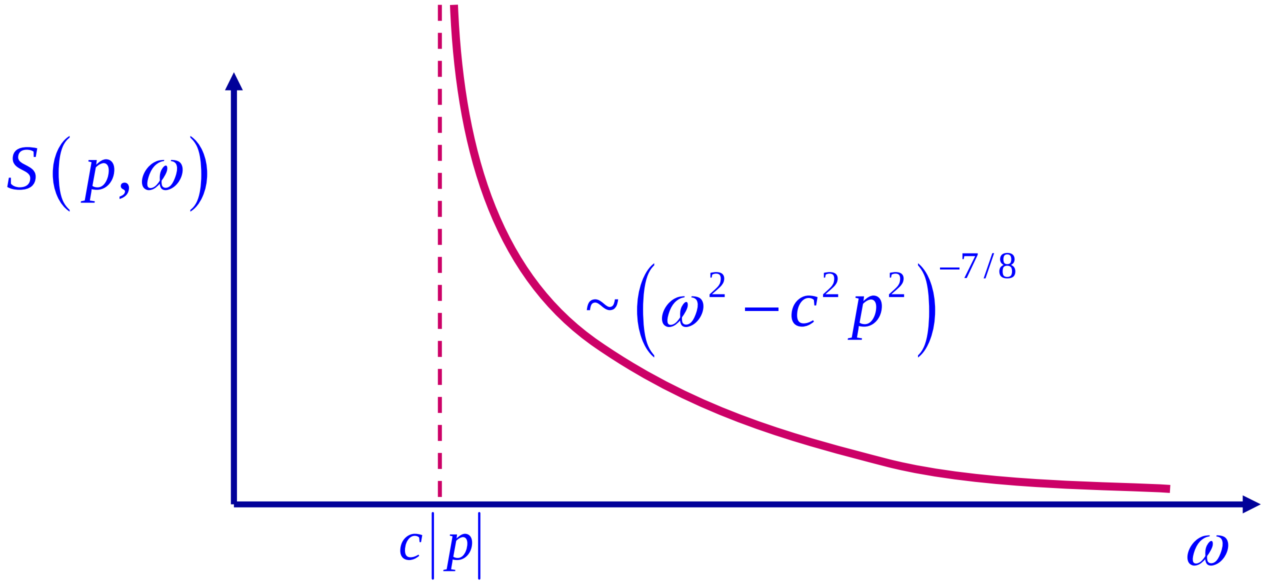
$$\Delta \sim |g - g_c|$$

Dynamic Structure Factor $S(p,\omega)$:

Critical coupling $(g = g_c)$

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa)

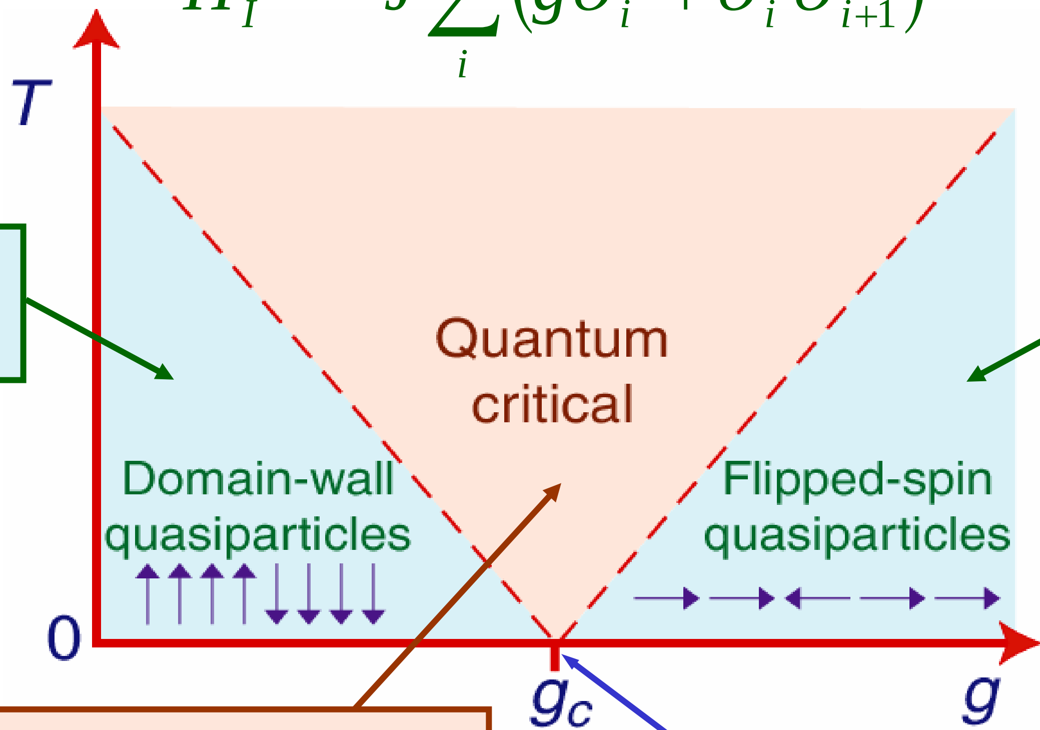
while transferring energy $\hbar\omega$ and momentum p



No quasiparticles --- dissipative critical continuum

$$H_I = -J\sum_i \left(g\sigma_i^x + \sigma_i^z \sigma_{i+1}^z \right)$$

Quasiclassical dynamics



Quasiclassical dynamics

$$\chi(\omega) = \frac{i}{\hbar}\sum_k \int_0^\infty dt \left\langle \left[\sigma_j^z(t), \sigma_k^z(0) \right] \right\rangle e^{i\omega t}$$

$$= \frac{A}{T^{7/4}\left(1 - i\omega/\Gamma_R + \ldots\right)}$$

$$\Gamma_R = \left(2\tan\frac{\pi}{16} \right) \frac{k_B T}{\hbar}$$

$$\left\langle \sigma_j^z \sigma_k^z \right\rangle \sim \frac{1}{\left| j-k \right|^{1/4}}$$

P. Pfeuty *Annals of Physics*, **57**, 79 (1970)

S. Sachdev and J. Ye, Phys. Rev. Lett. **69**, 2411 (1992).
S. Sachdev and A.P. Young, Phys. Rev. Lett. **78**, 2220 (1997).

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II. Landau-Ginzburg-Wilson theory

Mean field theory and the evolution of the excitation spectrum

- Identify order parameter $\phi(x,\tau) \sim \sigma_j^z$
- Symmetries:

$$\begin{aligned} \text{Spin inversion:} \quad & \phi \rightarrow -\phi \\ \text{Time reversal} \quad & \tau \rightarrow -\tau \\ \text{Spatial inversion} \quad & x \rightarrow -x \end{aligned}$$

- Write down most general Lagrangian consistent with symmetries

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\phi(x,\tau) \exp\left(-\int d^d x \int d\tau \mathcal{L}[\phi] \right) \\ \mathcal{L}[\phi] &= \frac{1}{2}(\partial_\tau \phi)^2 + \frac{c^2}{2}(\nabla_x \phi)^2 + \frac{r}{2}\phi^2 + \frac{u}{4}\phi^4 + \ldots \end{aligned}$$

- Identify phases at $r \gg 0$ and $r \ll 0$ with the paramagnet and the ferromagnet respectively.

Quantum field theory formally resembles the classical statistical mechanics of an Ising model in $d+1$ dimensions. Theory of second-order classical phase transitions implies that at the critical point the susceptibility depends on the $d+1$ dimensional momentum k as

$$\chi(k) \sim \frac{1}{k^{2-\eta}}$$

After analytic continuation, and using the "Lorentz invariance" of the critical theory, the quantum critical point therefore has the following dynamic susceptibility at $T = 0$.

$$\chi(p,\omega) \sim \frac{1}{(c^2p^2 - \omega^2)^{1-\eta/2}}$$

At $T > 0$, we have to consider a classical statistical mechanics problem in finite geometry with a 'temporal' direction of extent $L_\tau = \hbar/(k_B T)$. *Finite size scaling* now implies that the susceptibility at the critical point obeys

$$\chi(k) \sim L_\tau^{2-\eta} F\left(kL_\tau\right)$$

After analytic continuation, the quantum system has the dynamic response (note: can no longer use "Lorentz invariance")

$$\chi''(0,\omega) \sim \frac{1}{T^{2-\eta}} \Phi\left(\frac{\hbar\omega}{k_B T}\right)$$

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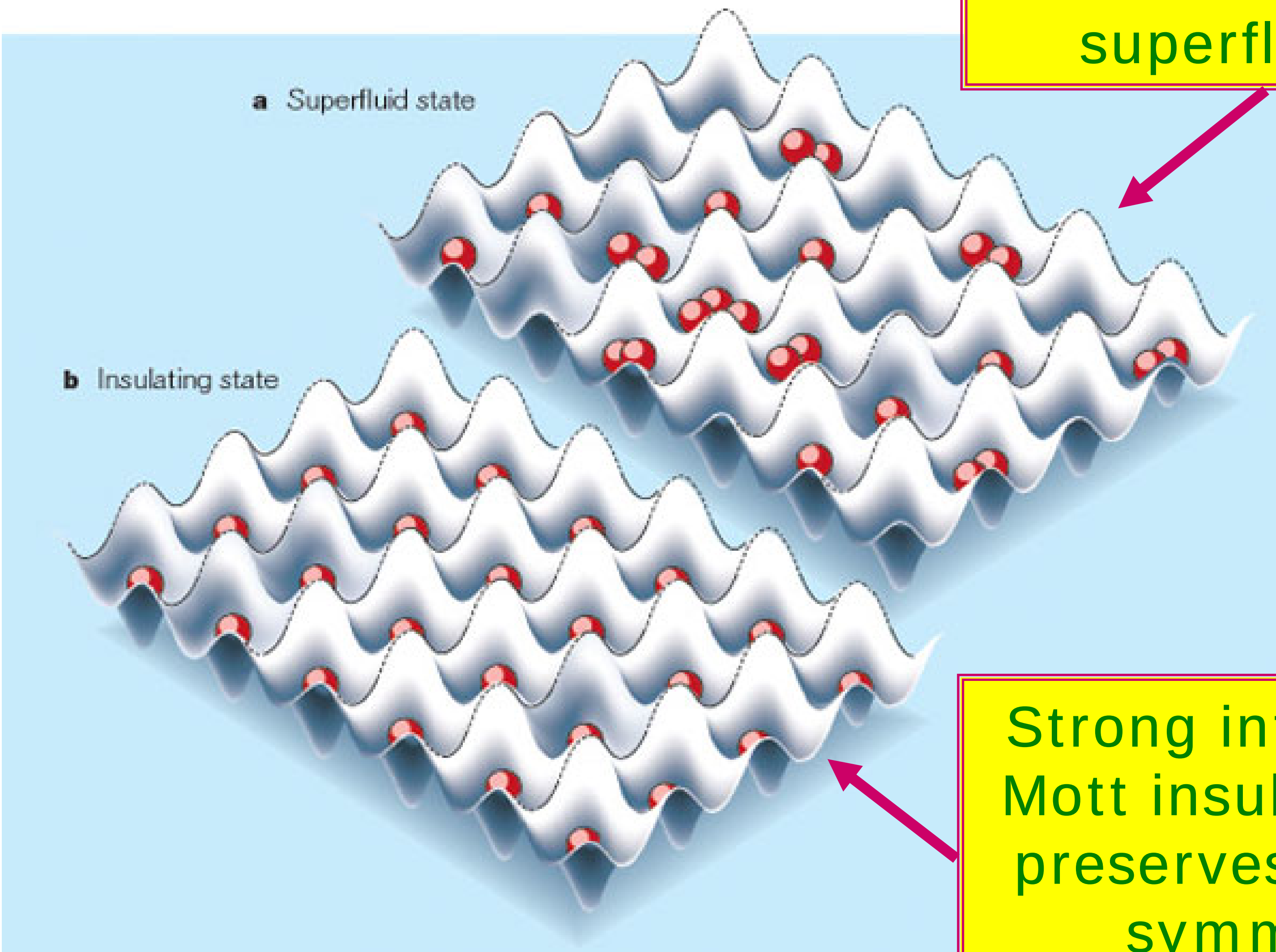
III. Superfluid-insulator transition

Boson Hubbard model at integer filling

Bosons at density $f = 1$



Weak interactions: superfluidity

Strong interactions: Mott insulator which preserves all lattice symmetries

LGW theory: continuous quantum transitions between these states

M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).

I. The Superfluid-Insulator transition

Boson Hubbard model

Degrees of freedom: Bosons, $b_j^\dagger$, hopping between the sites, j, of a lattice, with short-range repulsive interactions.

$$H = -t\sum_{\langle ij\rangle} b_i^\dagger b_j - \mu\sum_j n_j + \frac{U}{2}\sum_j n_j(n_j - 1) + \cdots$$

$$n_j \equiv b_j^\dagger b_j$$

M.P.A. Fisher, P.B. Weichmann, G. Grinstein, and D.S. Fisher *Phys. Rev. B* **40**, 546 (1989).

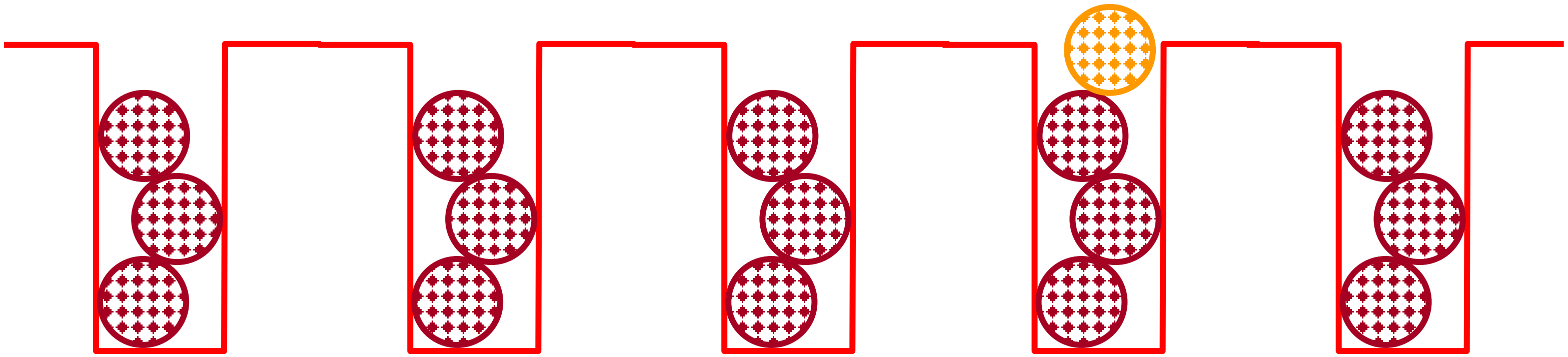
For small *U*/*t*, ground state is a superfluid BEC with

superfluid density $\approx$ density of bosons

What is the ground state for large *U/t* ?

Typically, the ground state remains a superfluid, but with

superfluid density $\ll$ density of bosons

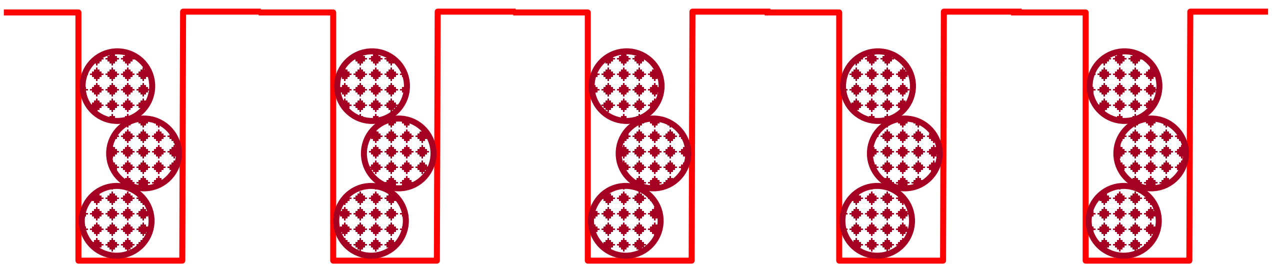


The superfluid density evolves smoothly from large values at small *U/t*, to small values at large *U/t*, and there is no quantum phase transition at any intermediate value of *U/t*.

(In systems with Galilean invariance and at zero temperature, superfluid density=density of bosons always, independent of the strength of the interactions)

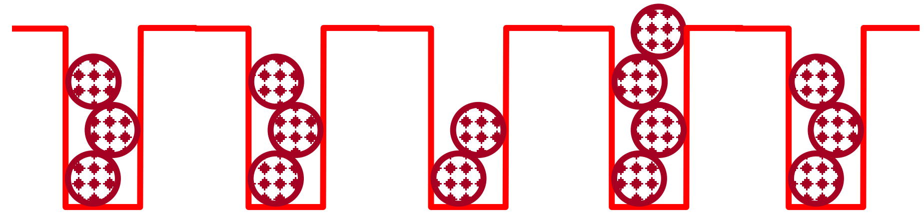
What is the ground state for large U/t ?

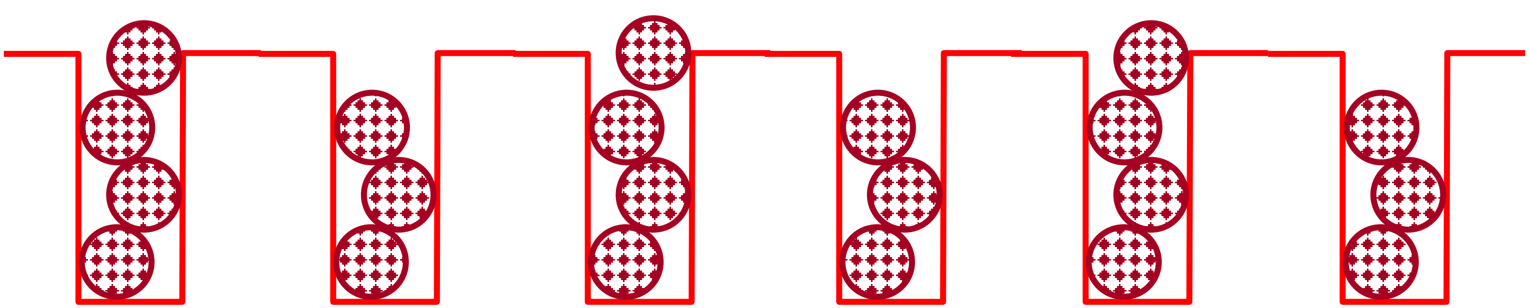
Incompressible, insulating ground states, with zero superfluid density, appear at special commensurate densities



$$\langle n_j \rangle = 3$$

$$-\frac{t}{U}$$

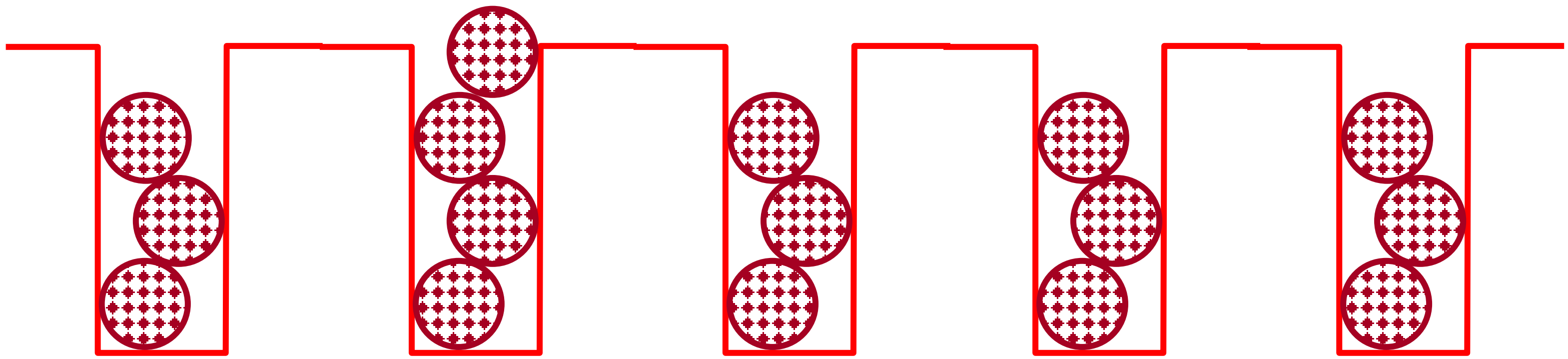




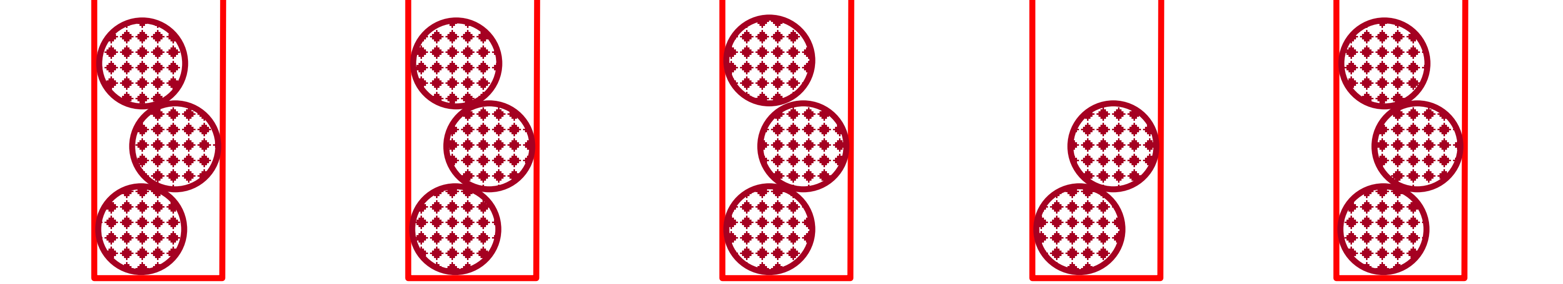
$$\langle n_j \rangle = 7/2$$

Ground state has "density wave" order, which spontaneously breaks lattice symmetries

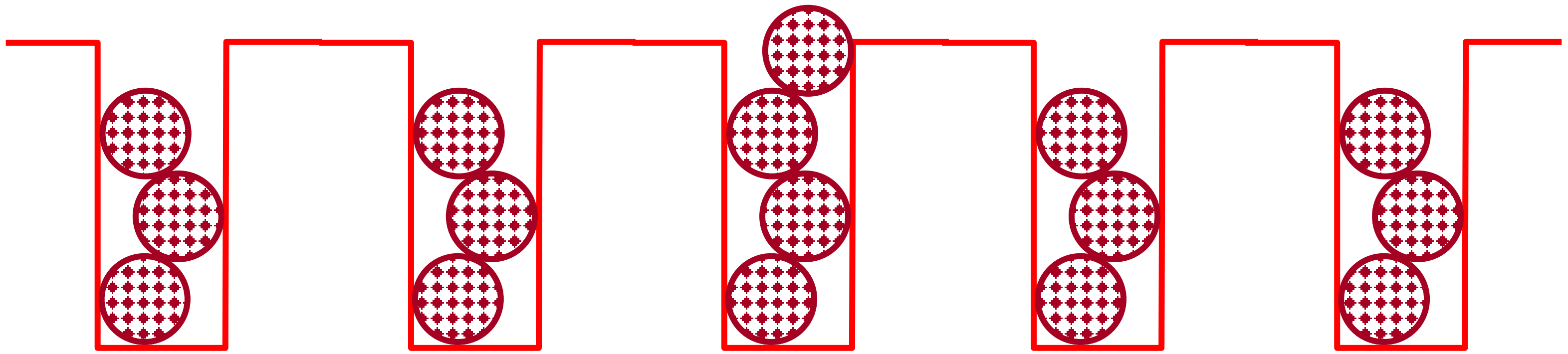
Excitations of the insulator: infinitely long-lived, finite energy *quasiparticles* and *quasiholes*



Energy of quasi-particles/holes: $\varepsilon_{p,h}(p) = \Delta_{p,h} + \frac{p^2}{2m^*_{p,h}}$



Excitations of the insulator: infinitely long-lived, finite energy *quasiparticles* and *quasiholes*



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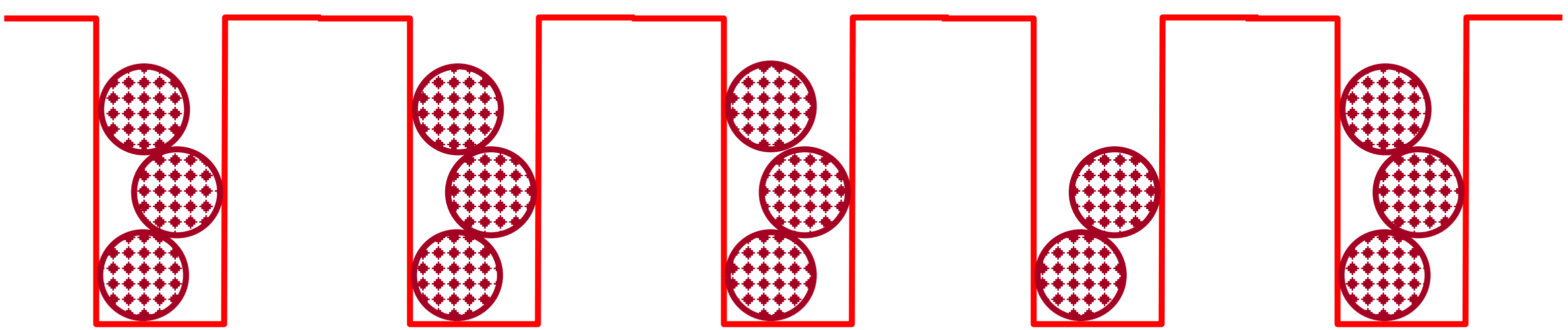
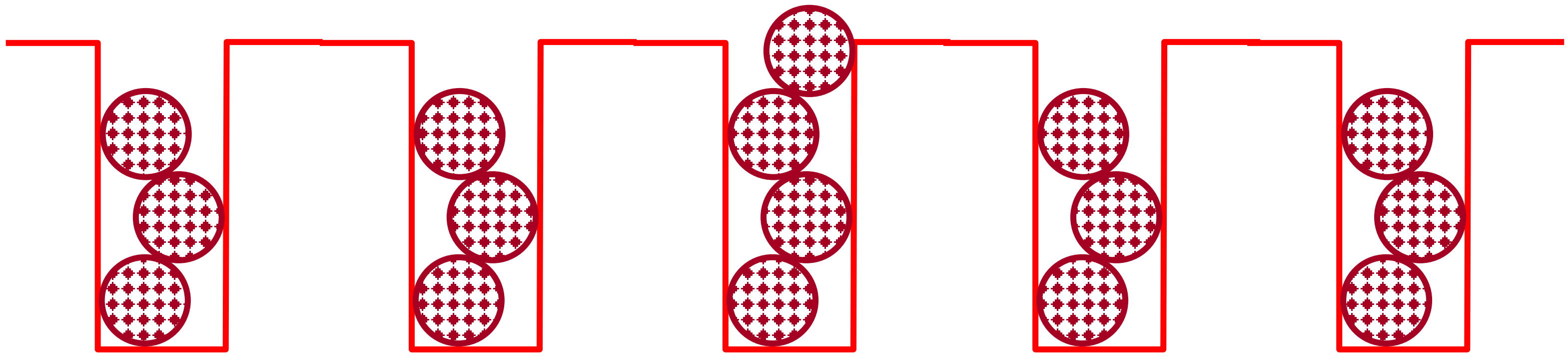
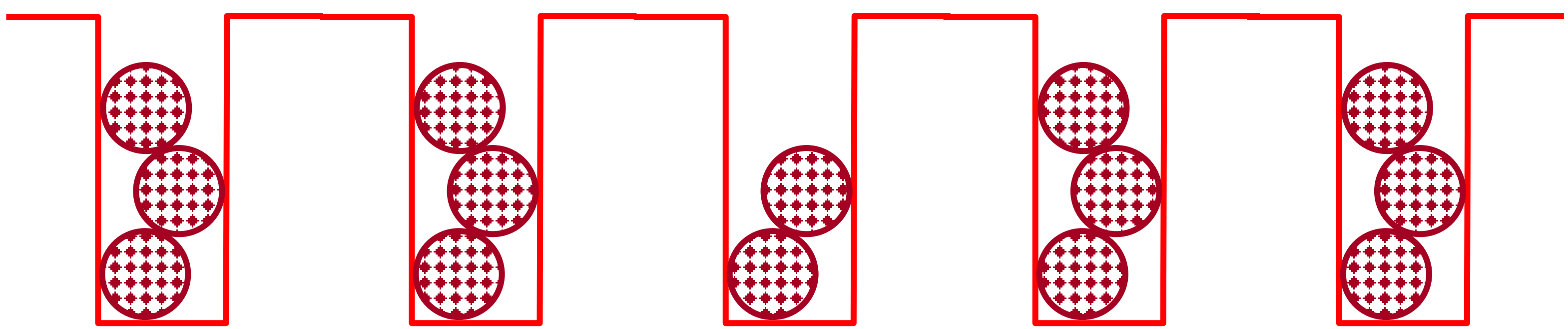


Excitations of the insulator: infinitely long-lived, finite energy *quasiparticles* and *quasiholes*



Energy of quasi-particles/holes: $\varepsilon_{p,h}(p) = \Delta_{p,h} + \frac{p^2}{2m^*_{p,h}}$



LGW theory of the superfluid insulator transition

- Identify order parameter $\Psi(x,\tau) \sim b_j^\dagger$
- Symmetries:

$$
\begin{aligned}
\text{Gauge invariance:} \quad & \Psi \to \Psi e^{i\theta} \\
\text{Time reversal} \quad & \tau \to -\tau \ ; \ \Psi \to \Psi^* \\
\text{Spatial inversion} \quad & x \to -x
\end{aligned}
$$

- Write down most general Lagrangian consistent with symmetries

$$
\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}\Psi(x,\tau) \exp\left(- \int d^d x \int d\tau \mathcal{L}[\Psi] \right) \\
\mathcal{L}[\Psi] &= K \Psi^* \frac{\partial \Psi}{\partial \tau} + |\partial_\tau \Psi|^2 + c^2 |\nabla_x \Psi|^2 + r |\Psi|^2 + \frac{u}{2} |\Psi|^4 + \dots
\end{aligned}
$$

- Identify phases at $r \gg 0$ and $r \ll 0$ with the insulator and the superfluid respectively.
- For $K \neq 0$, the particle and hole excitations have different energies.

- Gauge-invariance of the underlying boson Hamiltonian shows that

$$K = -\frac{\partial r}{\partial \mu}$$

- In mean-field theory, the ground state energy, E, across the superfluid-insulator transition has the non-analytic term

$$E = E_0 - \frac{r^2}{2u}\theta(-r)$$

(Beyond mean-field theory, the non-analytic term is $E \sim r^{(d+z)\nu}$).

- Because the density of bosons $= -\partial E/\partial \mu$, this implies a change in the boson density across the transition *unless* $\partial r/\partial \mu = 0$

- A superfluid-insulator transition at fixed boson density must have.

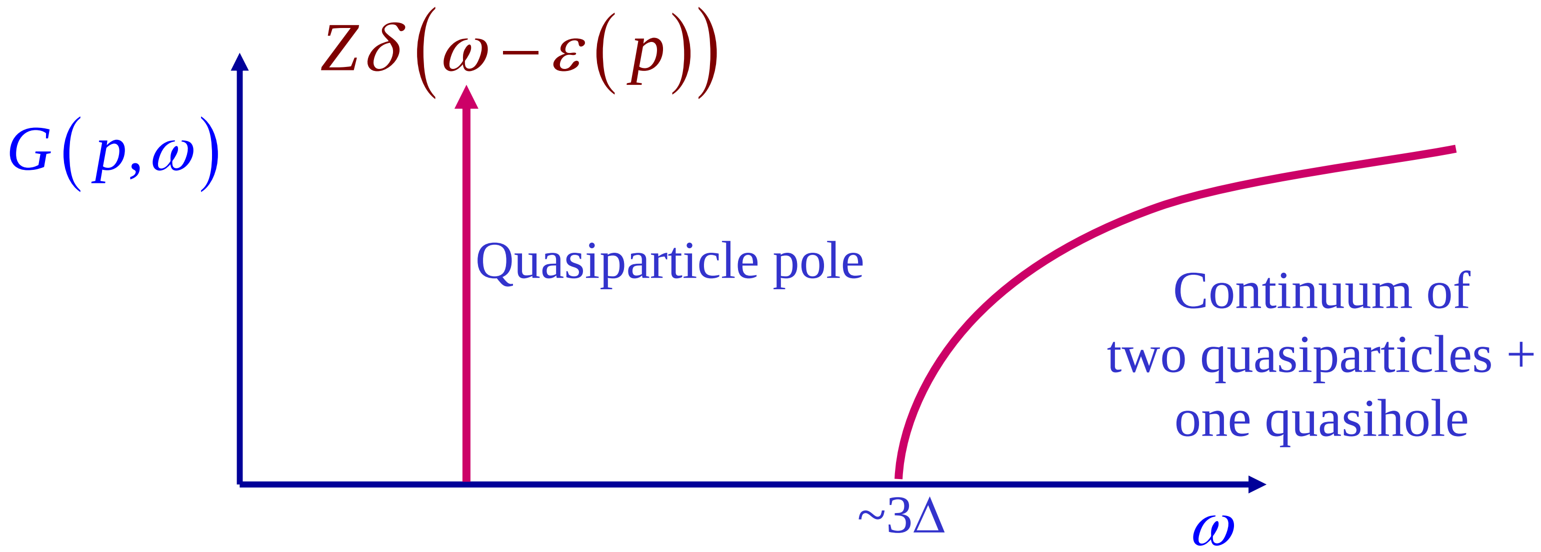
$$K = 0$$

Boson Green's function $G(p,\omega)$:

Insulating ground state

Cross-section to add a boson

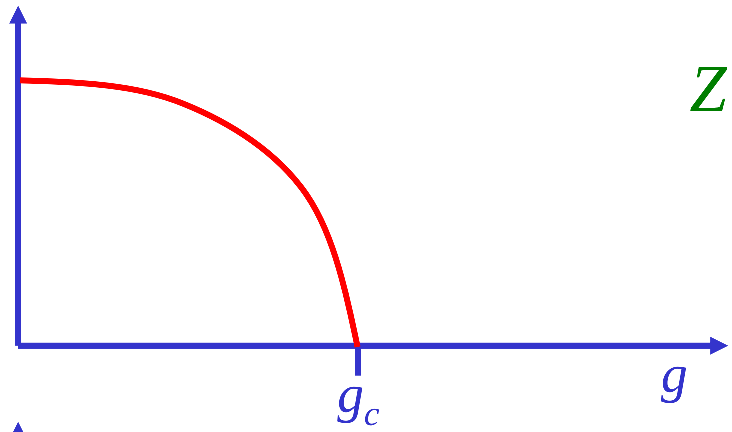
while transferring energy $\hbar\omega$ and momentum p



Similar result for quasi-hole excitations obtained by removing a boson

Entangled states at $g \equiv t/U$ of order unity

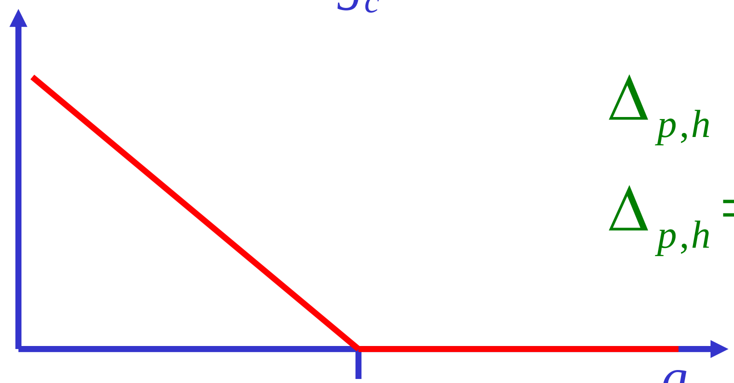
Quasiparticle weight Z



$$Z \sim \left(g_c - g\right)^{\eta\nu}$$

A.V. Chubukov, S. Sachdev, and J.Ye, *Phys. Rev.* B **49**, 11919 (1994)

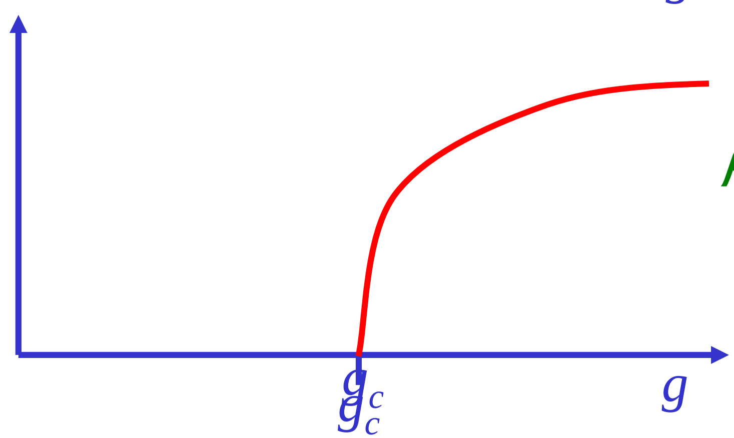
Excitation energy gap Δ



$$\Delta_{p,h} \sim \left(g_c - g\right)^{\nu} \text{ for } g < g_c$$

$$\Delta_{p,h} = 0 \text{ for } g > g_c$$

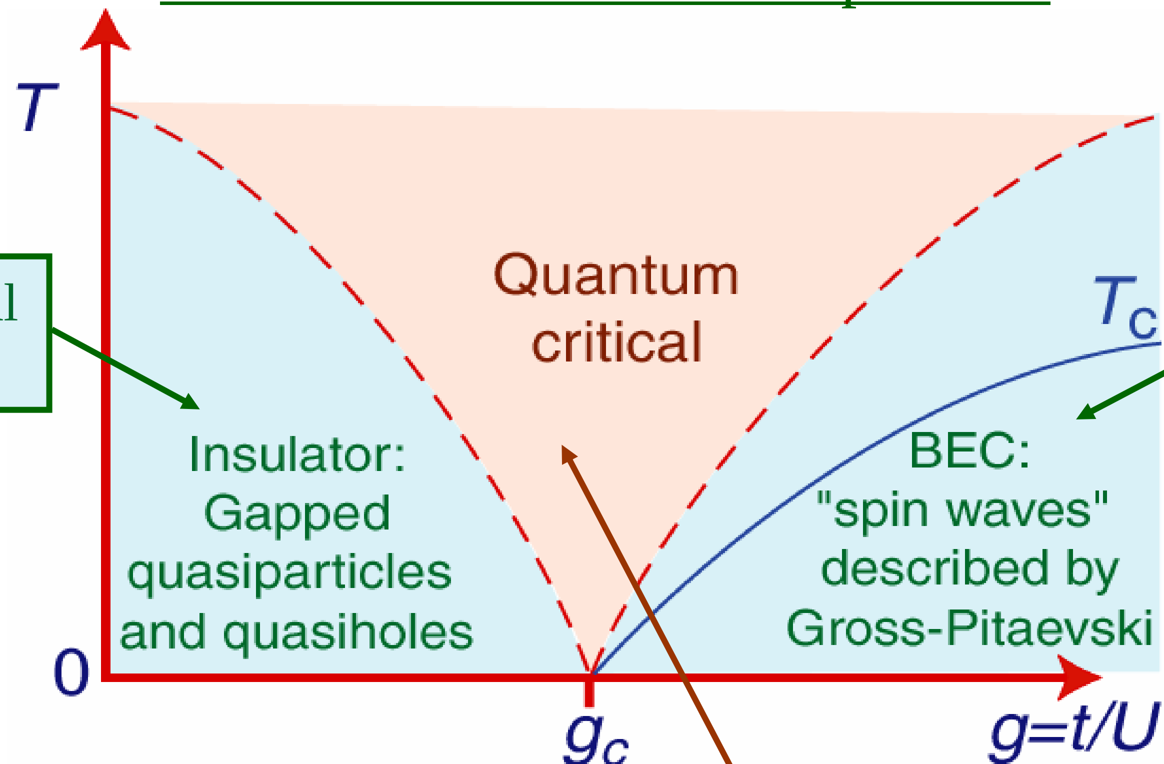
Superfluid density ρ_s



$$\rho_s \sim \left(g - g_c\right)^{(d+z-2)\nu}$$

Crossovers at nonzero temperature

Quasiclassical dynamics



Quasiclassical dynamics

Relaxational dynamics ("Bose molasses") with phase coherence/relaxation time τ_φ given by

$$\frac{1}{\tau_\varphi} = \left(\text{Universal number}\right)\frac{k_B T}{\hbar} \quad \left(1\mu\text{K} = 20.9\text{kHz}\right)$$

S. Sachdev and J. Ye, *Phys. Rev. Lett.* **69**, 2411 (1992).
K. Damle and S. Sachdev *Phys. Rev.* B **56**, 8714 (1997).

$$\text{Conductivity (in d=2)} = \frac{Q^2}{h}\Sigma\left(\frac{\hbar\omega}{k_B T}\right) \qquad \Sigma \rightarrow \text{universal function}$$

M.P.A. Fisher, G. Girvin, and G. Grinstein, *Phys. Rev. Lett.* **64**, 587 (1990).
K. Damle and S. Sachdev *Phys. Rev.* B **56**, 8714 (1997).

Outline

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    Boson Hubbard model at integer filling.

IV. Tilting the Mott insulator
    Density wave order at an Ising transition.

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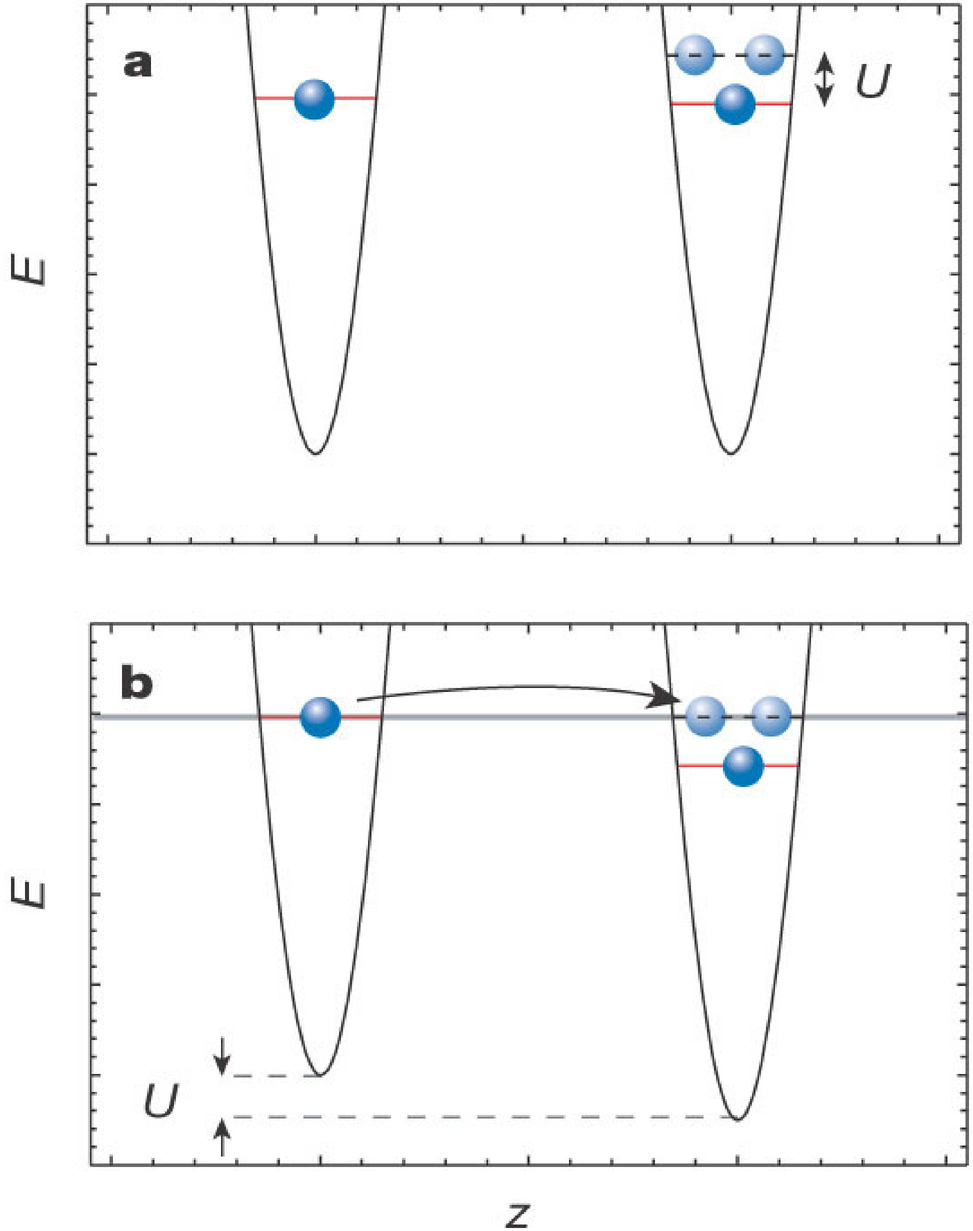
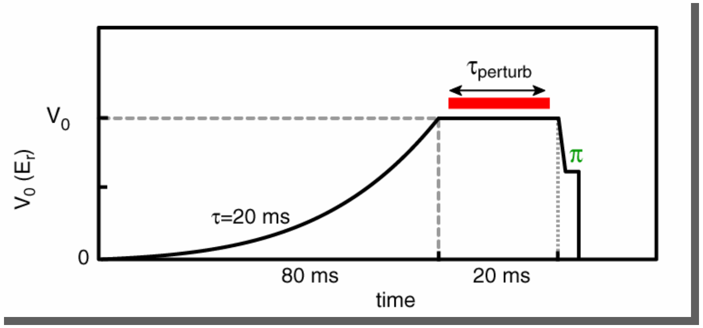
    Beyond the Landau-Ginzburg-Wilson paradigm.

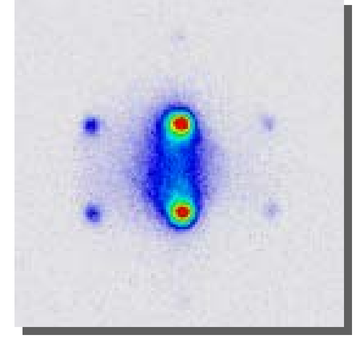
IV. Tilting the Mott insulator

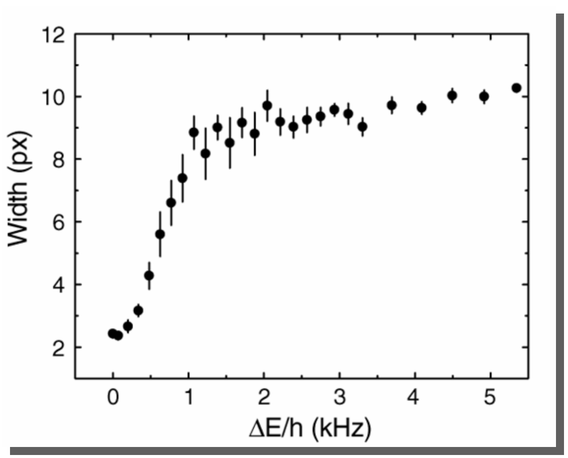
Density wave order at an Ising transition

Applying an "electric" field to the Mott insulator

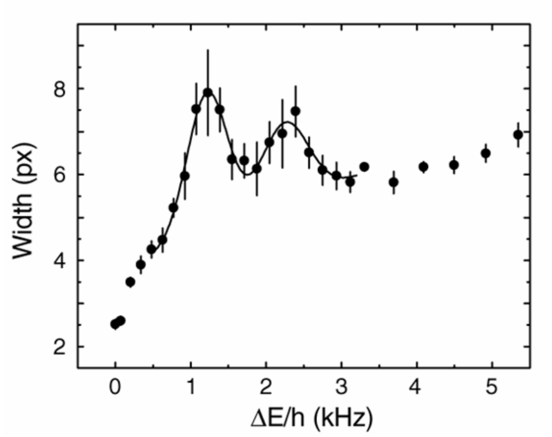






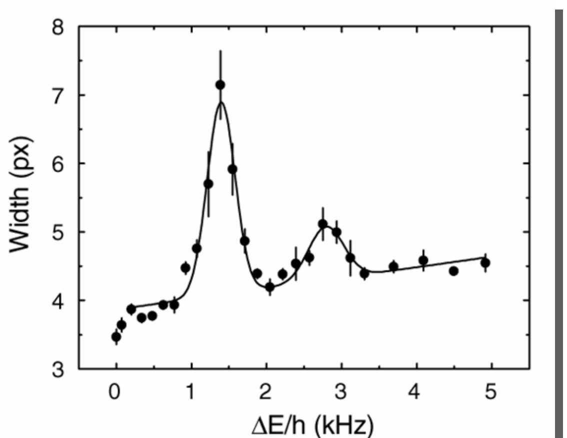


V_0=10 E_{recoil} **$\tau_{perturb}$ = 2 ms**

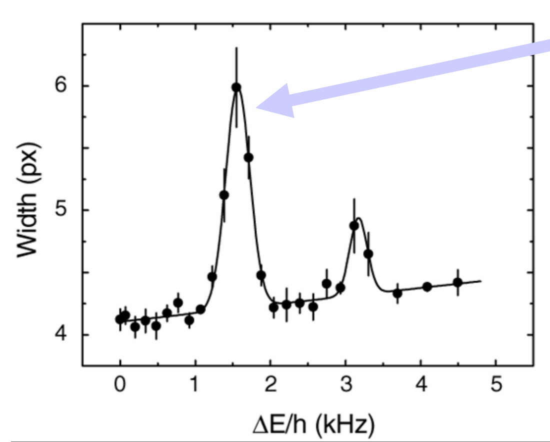


V_0= 13 E_{recoil} **$\tau_{perturb}$ = 4 ms**

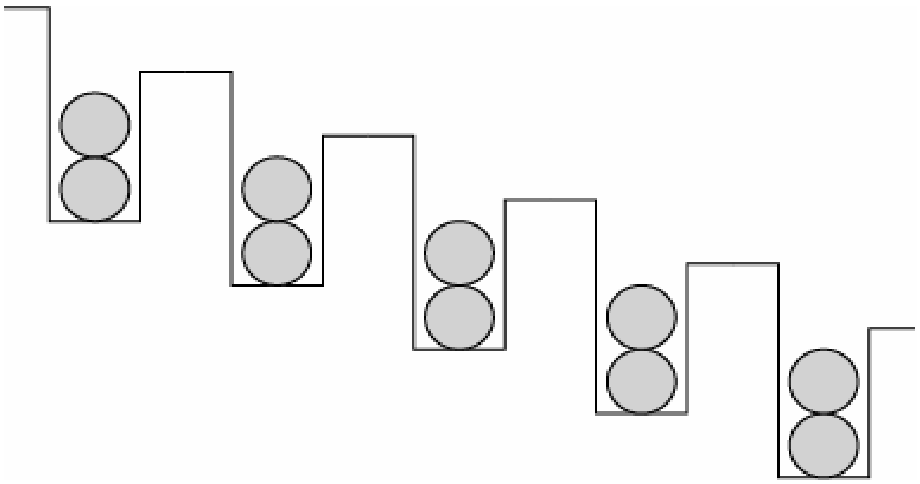
What is the quantum state here ?



V_0= 16 E_{recoil} **$\tau_{perturb}$ = 9 ms**



V_0= 20 E_{recoil} **$\tau_{perturb}$ = 20 ms**



$$H = -t\sum_{\langle ij\rangle}\left(b_i^\dagger b_j + b_j^\dagger b_i\right) + \frac{U}{2}\sum_i n_i\left(n_i - 1\right) - \sum_i \boldsymbol{E}\cdot\boldsymbol{r}_i n_i$$

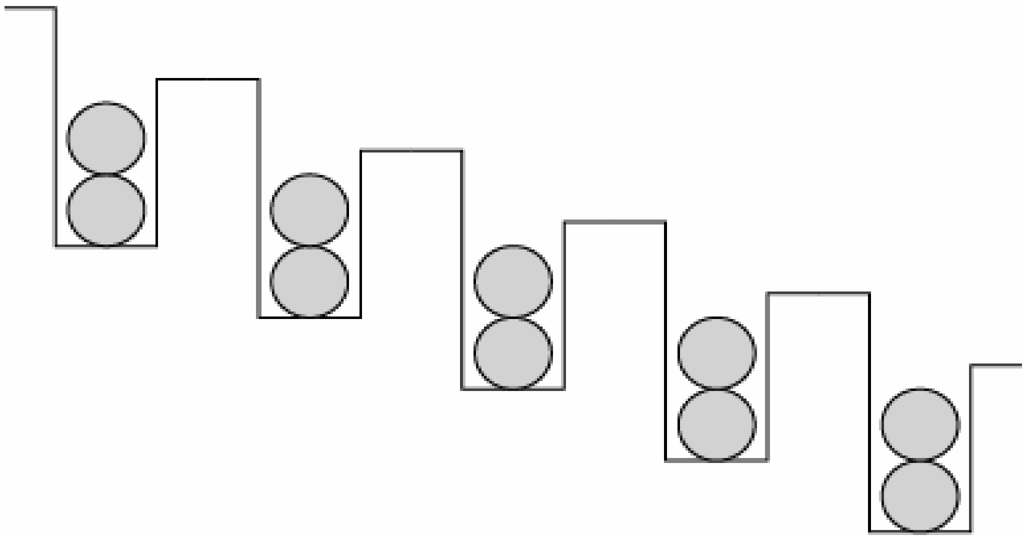
$$n_i = b_i^\dagger b_i$$

$$\left|U - E\right|, t \ll E, U$$

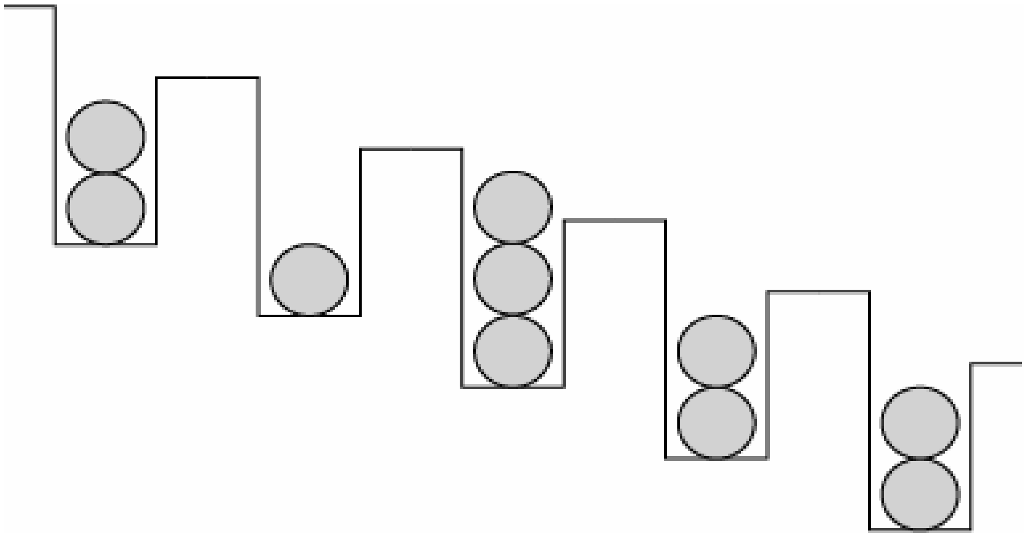
Describe spectrum in subspace of states resonantly coupled to the Mott insulator

S. Sachdev, K. Sengupta, and S.M. Girvin, *Physical Review* B **66**, 075128 (2002)

Important neutral excitations (in one dimension)

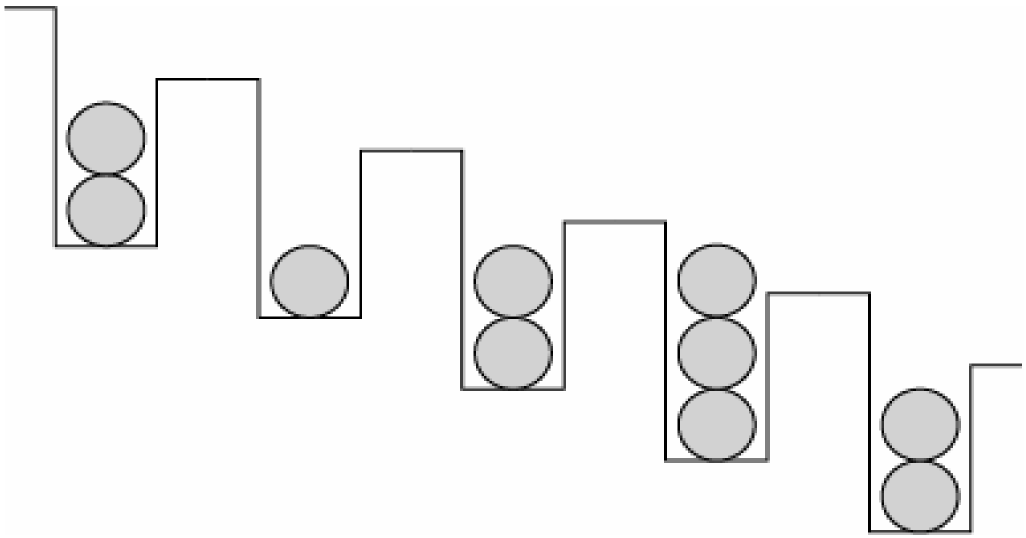


Important neutral excitations (in one dimension)



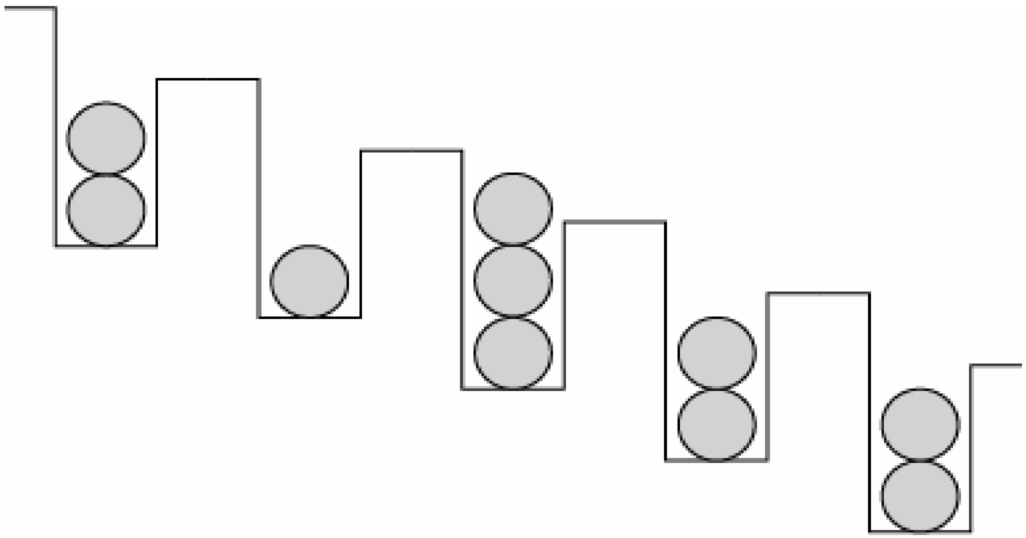
Nearest neighbor dipole

Important neutral excitations (in one dimension)



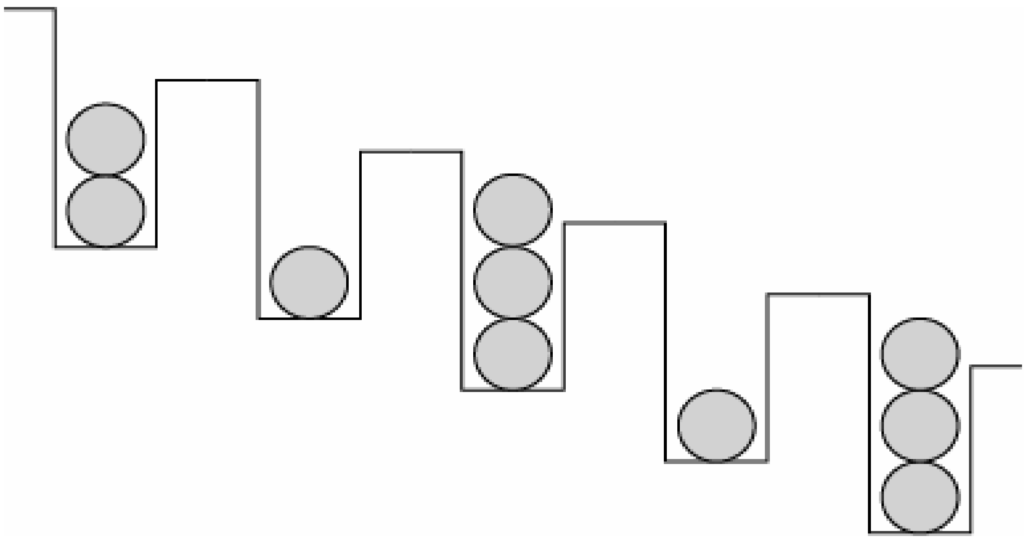
Creating dipoles on nearest neighbor links creates a state with relative energy *U-2E ;* such states are *not* part of the resonant manifold

Important neutral excitations (in one dimension)



Nearest neighbor dipole

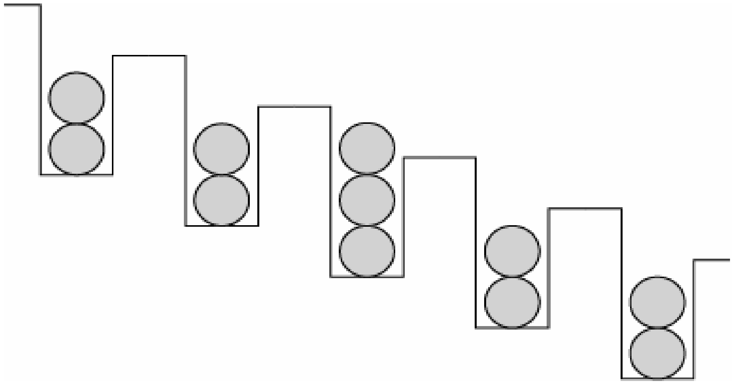
Important neutral excitations (in one dimension)

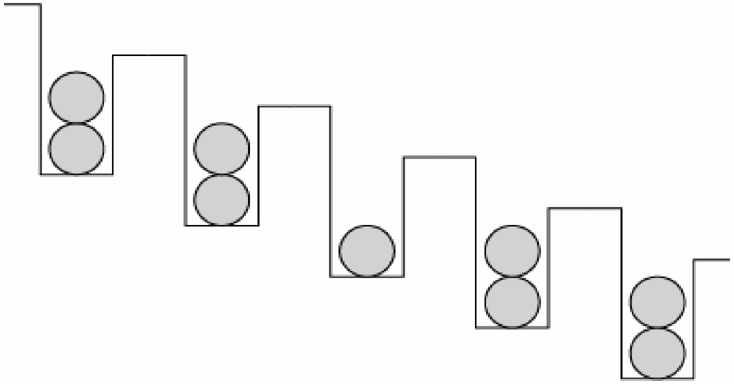


Nearest-neighbor dipoles

Dipoles can appear resonantly on non-nearest-neighbor links.
Within resonant manifold, dipoles have infinite on-link
and nearest-link repulsion

Charged excitations (in one dimension)





Effective Hamiltonian for a quasiparticle in one dimension (similar for a quasihole):

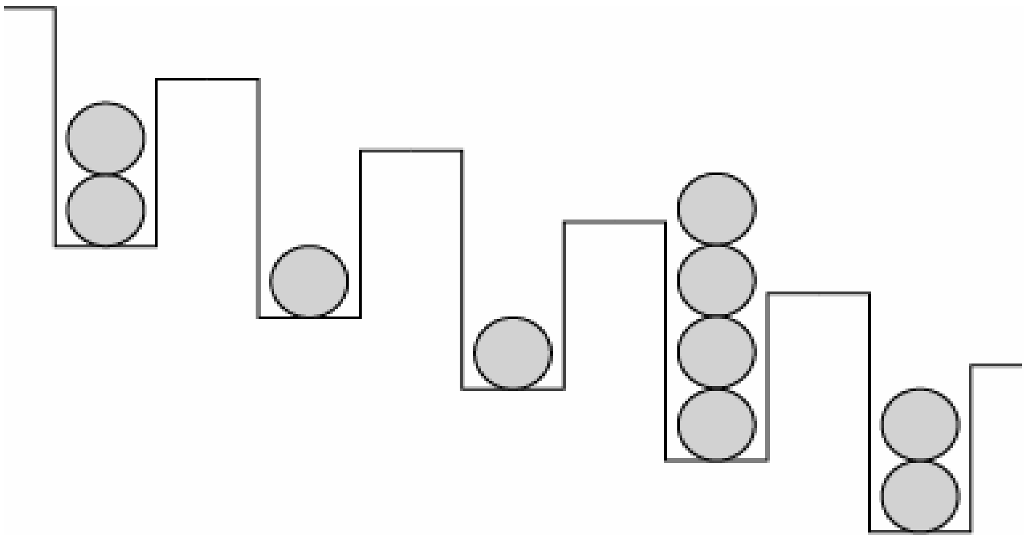
$$H_{\text{eff}} = -\sum_j \left[3t\left(b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j\right) + Ejb_j^\dagger b_j \right]$$

Exact eigenvalues $\varepsilon_m = Em \quad ; \quad m = -\infty \cdots \infty$

Exact eigenvectors $\psi_m(j) = J_{j-m}(6t/E)$

All charged excitations are strongly localized in the plane perpendicular electric field.
Wavefunction is periodic in time, with period *h/E* (Bloch oscillations)
Quasiparticles and quasiholes are not accelerated out to infinity

A non-dipole state



State has energy *3(U-E)* but is connected to resonant state by a matrix element smaller than t^2/U

State is not part of resonant manifold

Hamiltonian for resonant dipole states (in one dimension)

$$d_\ell^\dagger \Rightarrow \text{Creates dipole on link } \ell$$

$$H_d = -\sqrt{6}t\sum_\ell \left(d_\ell^\dagger + d_\ell \right) + (U-E)\sum_\ell d_\ell^\dagger d_\ell$$

$$\text{Constraints:} \quad d_\ell^\dagger d_\ell \leq 1 \quad ; \quad d_{\ell+1}^\dagger d_{\ell+1} d_\ell^\dagger d_\ell = 0$$

Determine phase diagram of H_d as a function of $(U-E)/t$

Note: there is <u>no explicit dipole hopping term</u>.

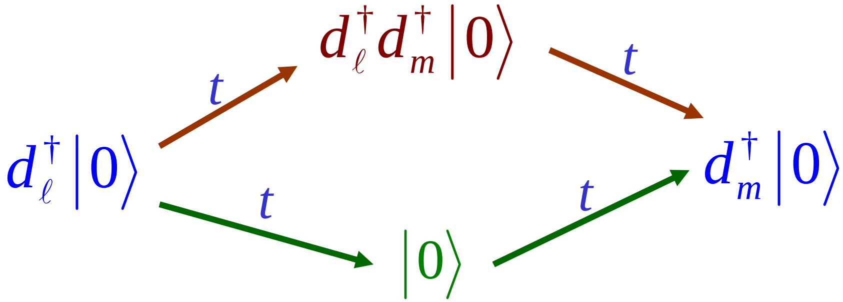
However, dipole hopping is generated by the interplay of terms in H_d and the constraints.

Weak electric fields: $(U\text{-}E) \gg t$

Ground state is dipole vacuum (Mott insulator) $|0\rangle$

First excited levels: single dipole states $d_\ell^\dagger |0\rangle$

Effective hopping between dipole states



If both processes are permitted, they exactly cancel each other.

The top processes is blocked when ℓ, m are nearest neighbors

$\Rightarrow$ A nearest-neighbor dipole hopping term $\sim \dfrac{t^2}{U-E}$ is generated

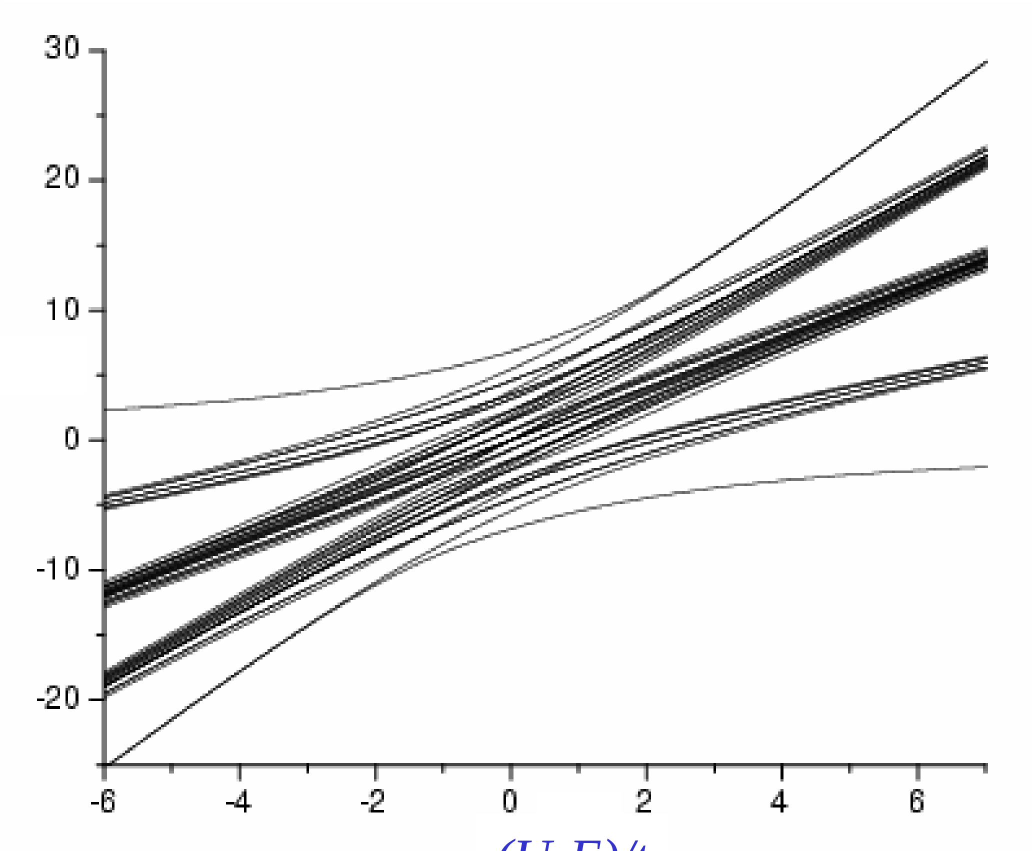
Strong electric fields: $(E\text{-}U) \gg t$

Ground state has maximal dipole number.

Two-fold degeneracy associated with Ising density wave order:

$$\cdots d_1^\dagger d_3^\dagger d_5^\dagger d_7^\dagger d_9^\dagger d_{11}^\dagger \cdots \left|0\right\rangle \quad or \quad \cdots d_2^\dagger d_4^\dagger d_6^\dagger d_8^\dagger d_{10}^\dagger d_{12}^\dagger \cdots \left|0\right\rangle$$

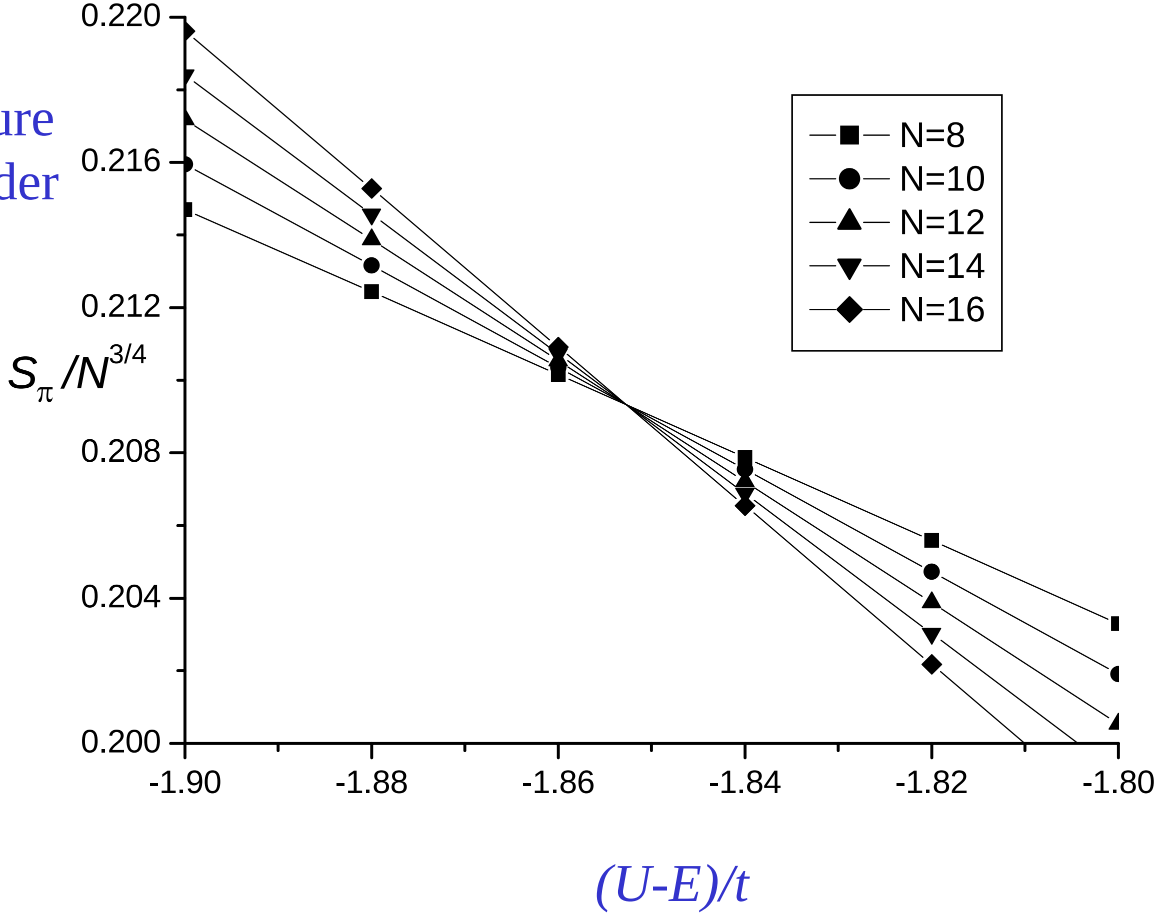
Eigenvalues



(U-E)/t

Ising quantum critical point at *E-U=1.08 t*

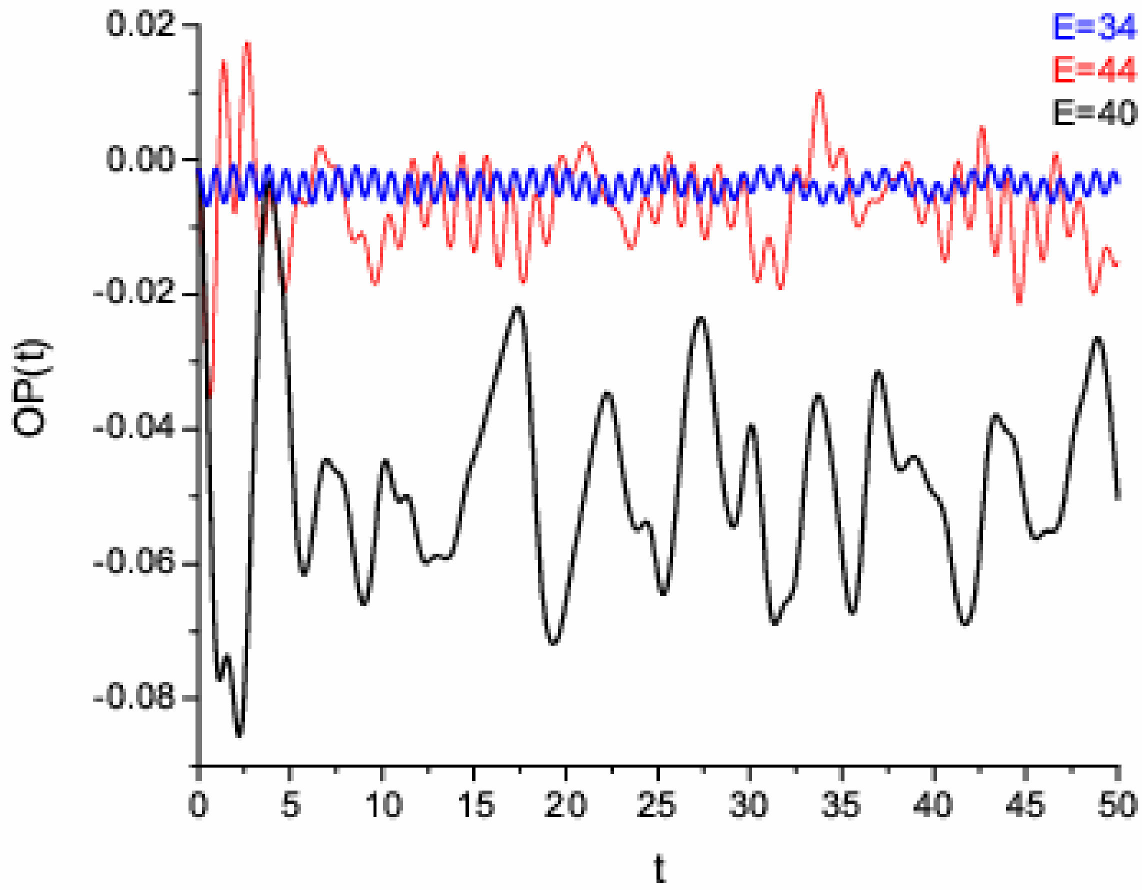
Equal-time structure factor for Ising order parameter



S. Sachdev, K. Sengupta, and S.M. Girvin, *Physical Review* B **66**, 075128 (2002)

Non-equilibrium dynamics in one dimension

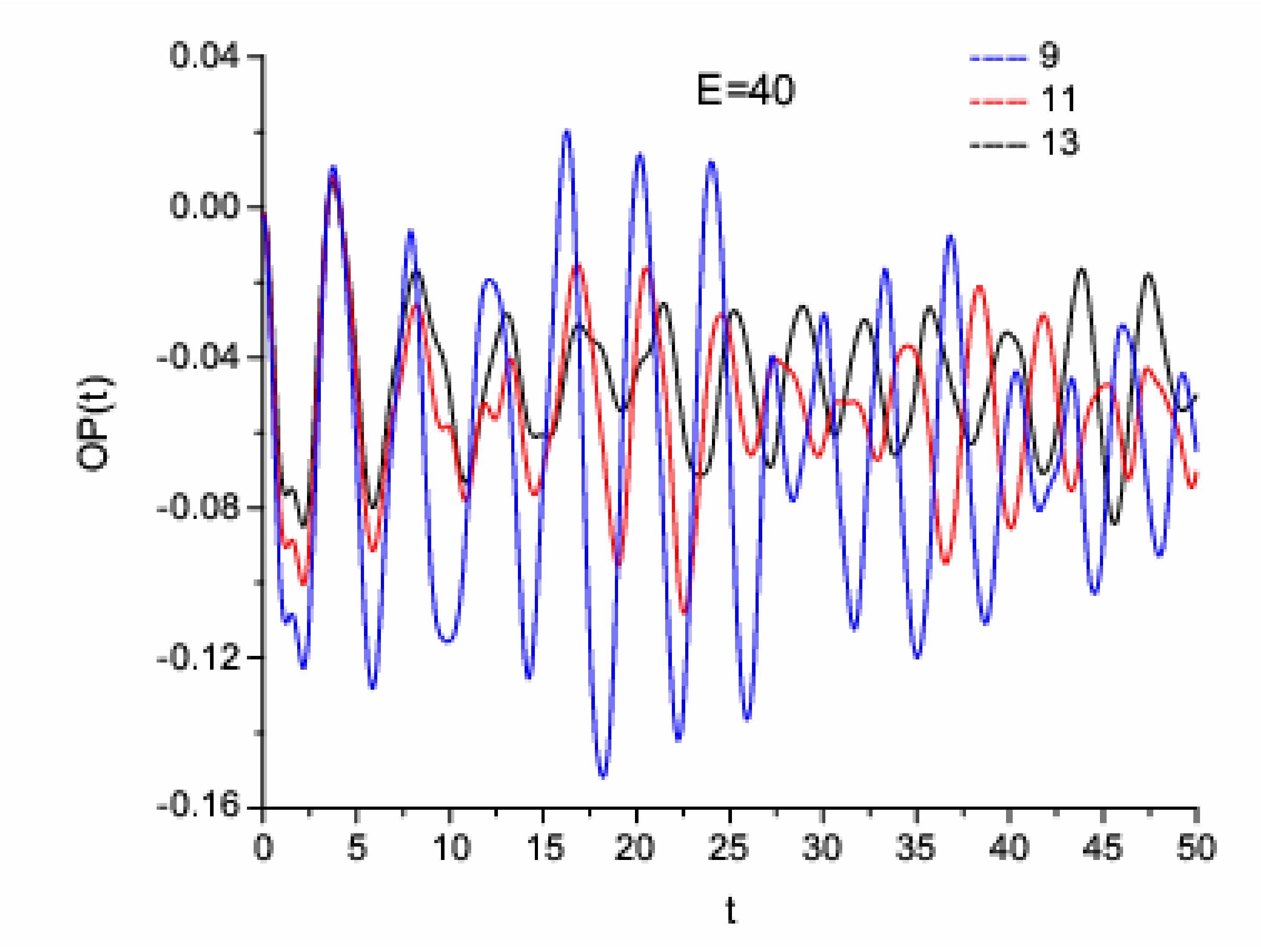
Start with the ground state at *E=32* on a chain with open boundaries. Suddenly change the value of *E* and follow the evolution of the wavefunction



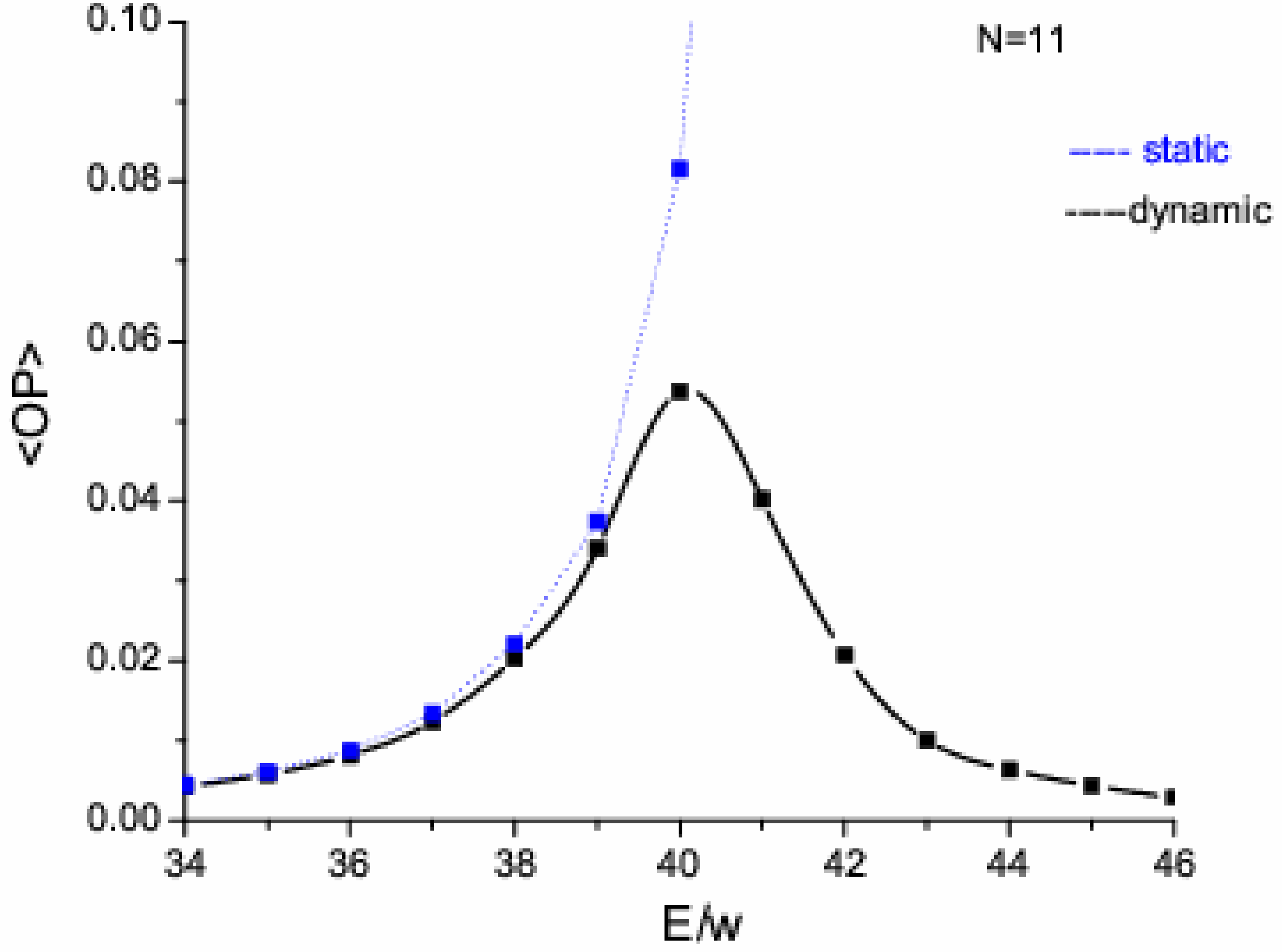
Critical point at *E*=41.85

Non-equilibrium dynamics in one dimension

Dependence on chain length



Non-equilibrium dynamics in one dimension



Non-equilibrium response is maximal near the Ising critical point

K. Sengupta, S. Powell, and S. Sachdev, *Physical Review* A **69**, 053616 (2004)

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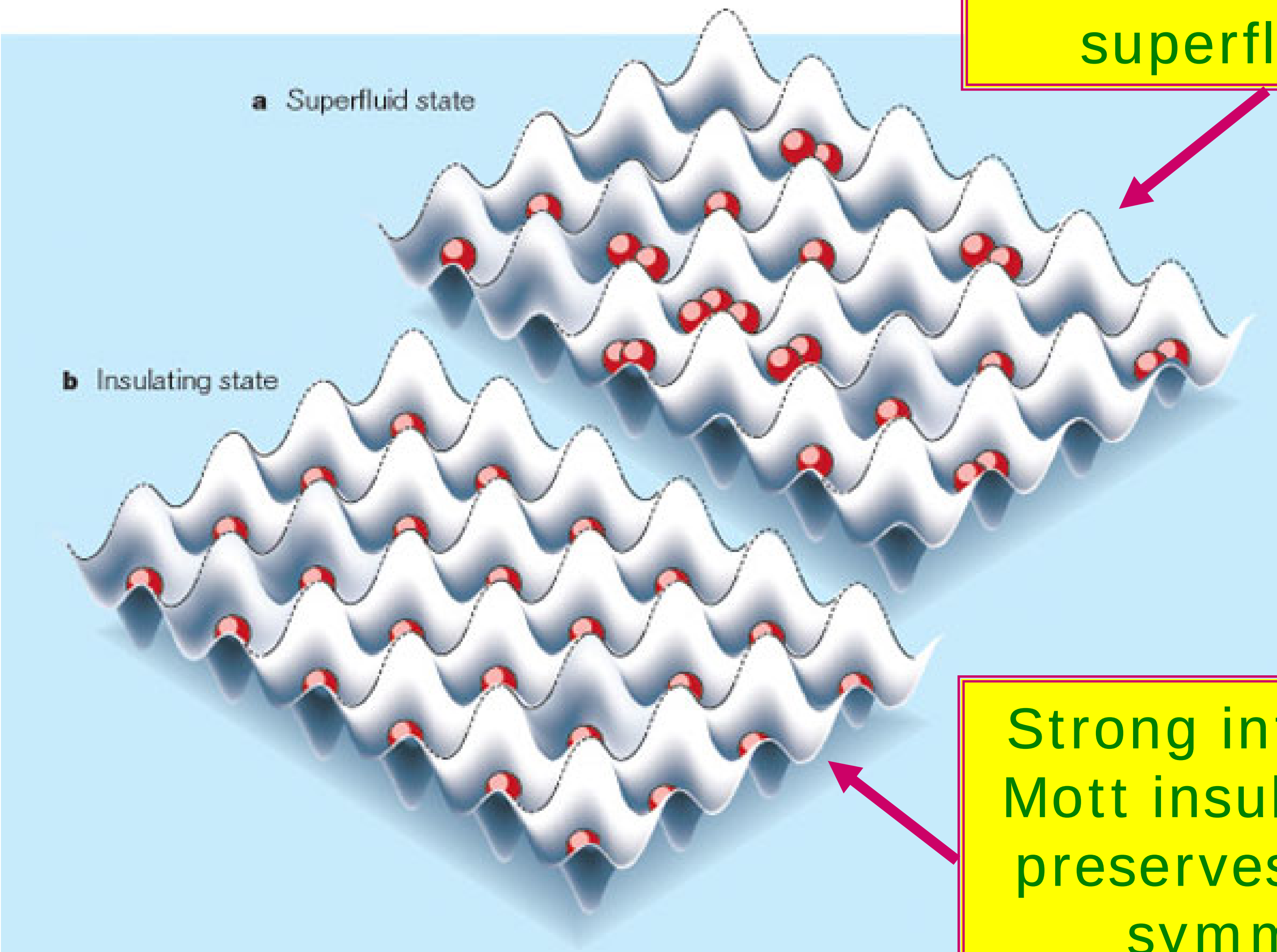
V. Bosons at fractional filling

Beyond the Landau-Ginzburg-Wilson paradigm

Bosons at density $f = 1$

Weak interactions: superfluidity



Strong interactions: Mott insulator which preserves all lattice symmetries

LGW theory: continuous quantum transitions between these states

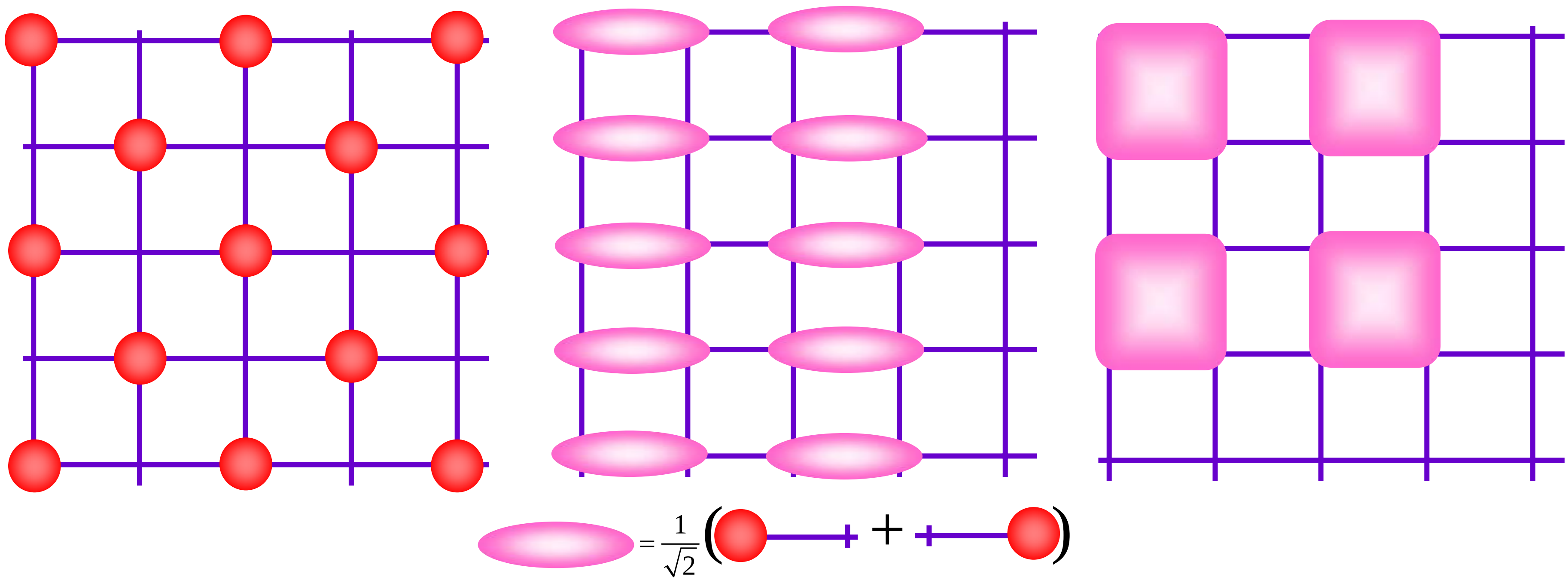
M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).

Bosons at density $f = 1/2$

Weak interactions: superfluidity

$$\langle \Psi_{sc} \rangle \neq 0$$

Strong interactions: Candidate insulating states



All insulating phases have density-wave order $\rho(\boldsymbol{r}) = \sum_{\boldsymbol{Q}} \rho_{\boldsymbol{Q}} e^{i\boldsymbol{Q}.\boldsymbol{r}}$ with $\langle \rho_{\boldsymbol{Q}} \rangle \neq 0$

C. Lannert, M.P.A. Fisher, and T. Senthil, *Phys. Rev.* B **63**, 134510 (2001)

S. Sachdev and K. Park, *Annals of Physics*, **298**, 58 (2002)

Predictions of LGW theory

Superconductor

$\langle \Psi_{sc} \rangle \neq 0, \langle \rho_{\mathbf{Q}} \rangle = 0$

First order transition

Charge-ordered insulator

$\langle \Psi_{sc} \rangle = 0, \langle \rho_{\mathbf{Q}} \rangle \neq 0$

$r_1 - r_2$

Superconductor

$\langle \Psi_{sc} \rangle \neq 0, \langle \rho_{\mathbf{Q}} \rangle = 0$

Coexistence

(Supersolid)

$\langle \Psi_{sc} \rangle \neq 0, \langle \rho_{\mathbf{Q}} \rangle \neq 0$

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$r_1 - r_2$

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$\langle \Psi_{sc} \rangle \neq 0, \langle \rho_{\mathbf{Q}} \rangle = 0$

"*Disordered*"

$\langle \Psi_{sc} \rangle = 0, \langle \rho_{\mathbf{Q}} \rangle = 0$

Charge-ordered insulator

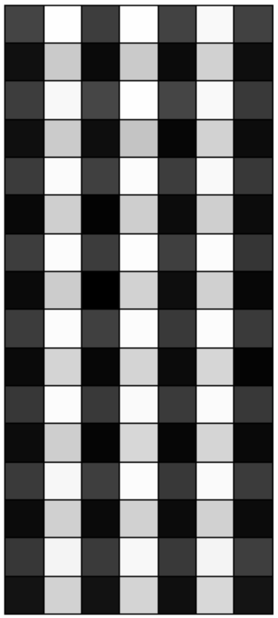
$\langle \Psi_{sc} \rangle = 0, \langle \rho_{\mathbf{Q}} \rangle \neq 0$

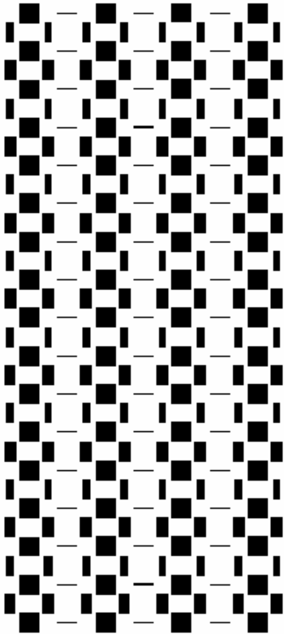
$r_1 - r_2$

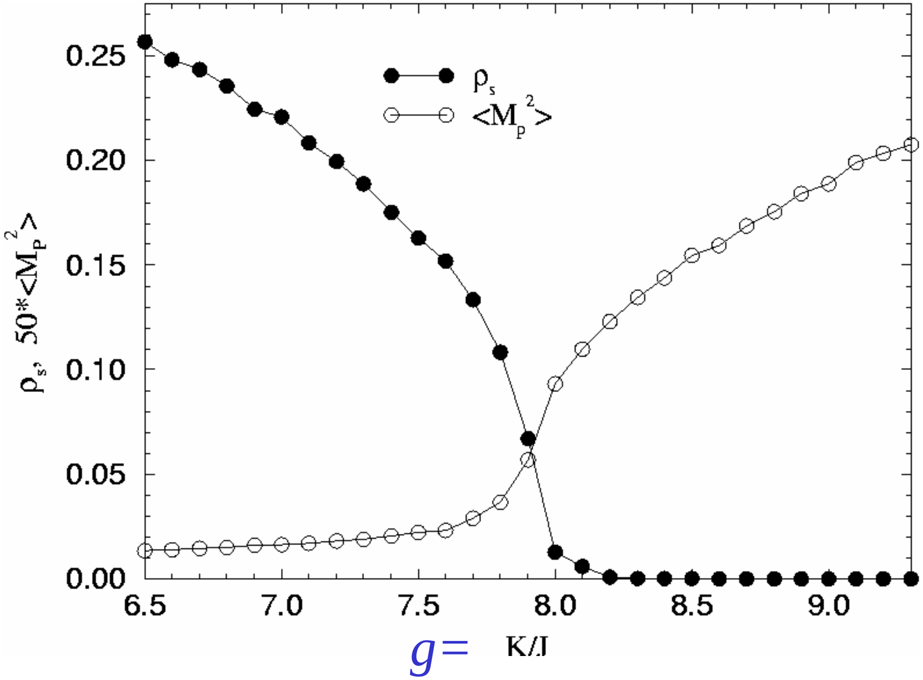
Superfluid insulator transition of hard core bosons at *f*=1/2

A. W. Sandvik, S. Daul, R. R. P. Singh, and D. J. Scalapino, *Phys. Rev. Lett.* **89**, 247201 (2002)

Large scale (> 8000 sites) numerical study of the destruction of superfluid order at half filling with full square lattice symmetry



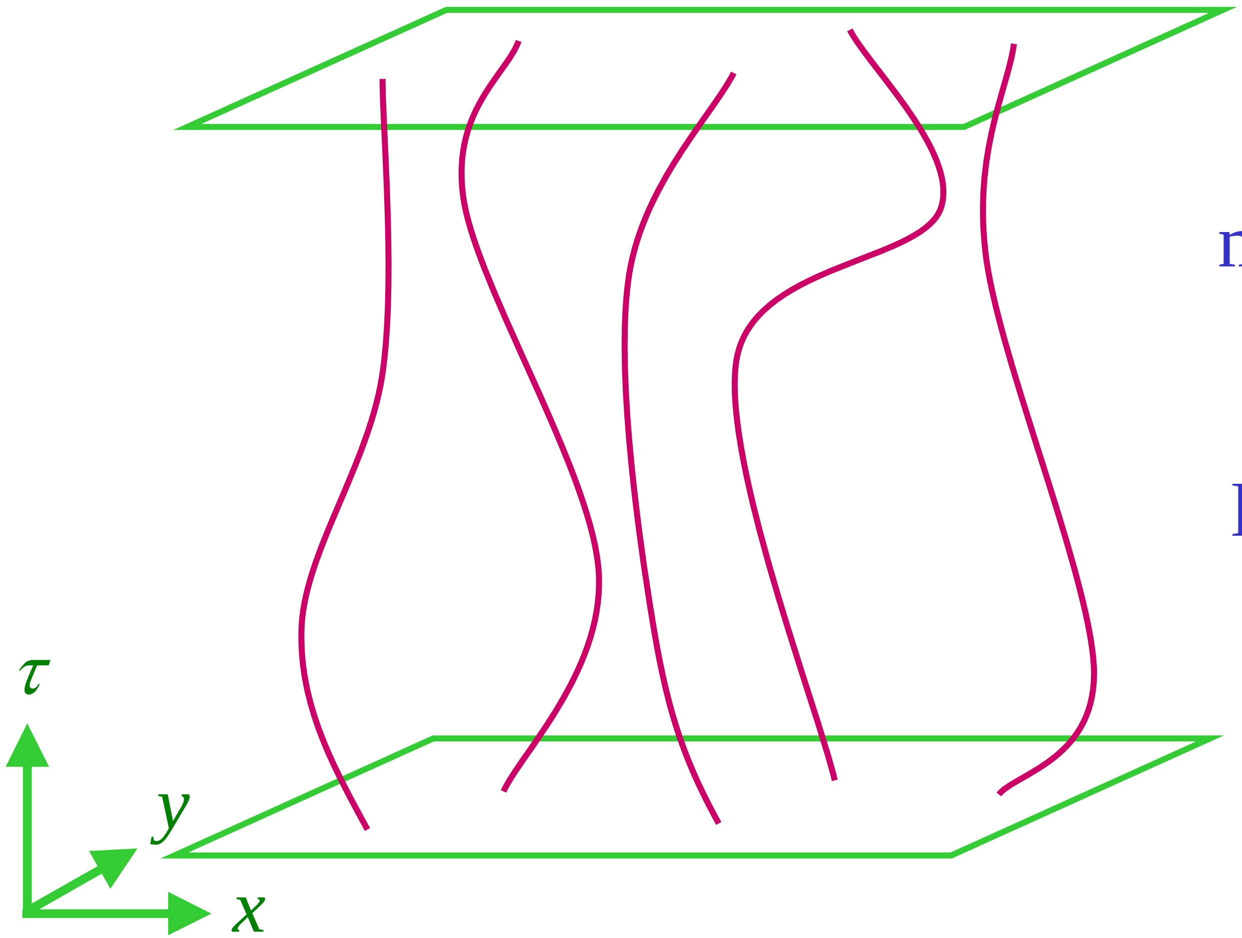




g=

$$H = J\sum_{\langle ij\rangle}\left(S_i^+S_j^- + S_i^-S_j^+\right) - K\sum_{\langle ijkl\rangle\subset\square}\left(S_i^+S_j^-S_k^+S_l^- + S_i^-S_j^+S_k^-S_l^+\right)$$

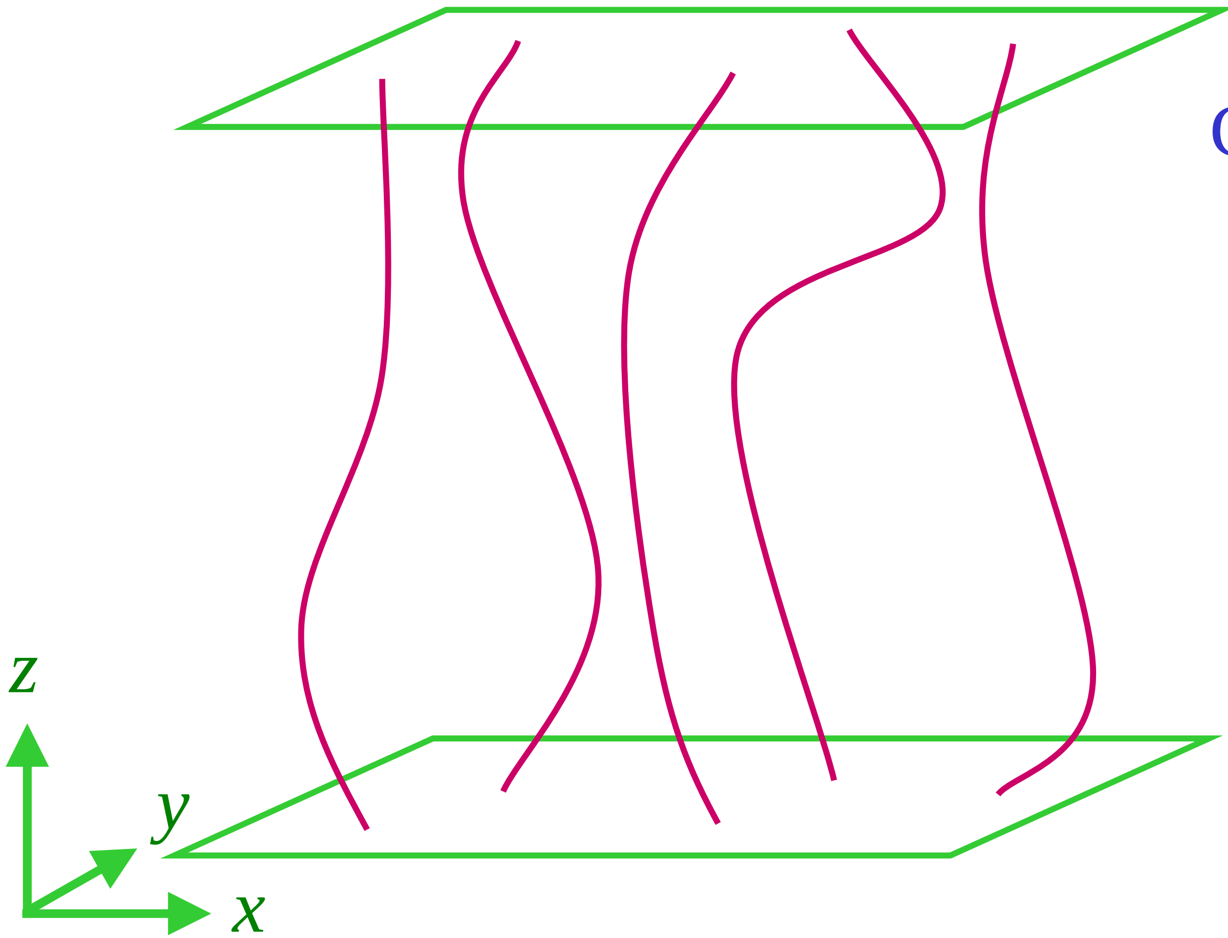
Boson-vortex duality



Quantum mechanics of two-dimensional bosons: world lines of bosons in spacetime

C. Dasgupta and B.I. Halperin, *Phys. Rev. Lett.* **47**, 1556 (1981); D.R. Nelson, *Phys. Rev. Lett.* **60**, 1973 (1988); M.P.A. Fisher and D.-H. Lee, *Phys. Rev.* B **39**, 2756 (1989);

Boson-vortex duality



Classical statistical mechanics of a "dual" three-dimensional superconductor: vortices in a "magnetic" field

Strength of "magnetic" field = density of bosons
= f flux quanta per plaquette

C. Dasgupta and B.I. Halperin, *Phys. Rev. Lett.* **47**, 1556 (1981); D.R. Nelson, *Phys. Rev. Lett.* **60**, 1973 (1988); M.P.A. Fisher and D.-H. Lee, *Phys. Rev.* B **39**, 2756 (1989);

Boson-vortex duality

Statistical mechanics of dual superconductor is invariant under the square lattice space group:

T_x, T_y : Translations by a lattice spacing in the x, y directions

R : Rotation by 90 degrees.

Magnetic space group:

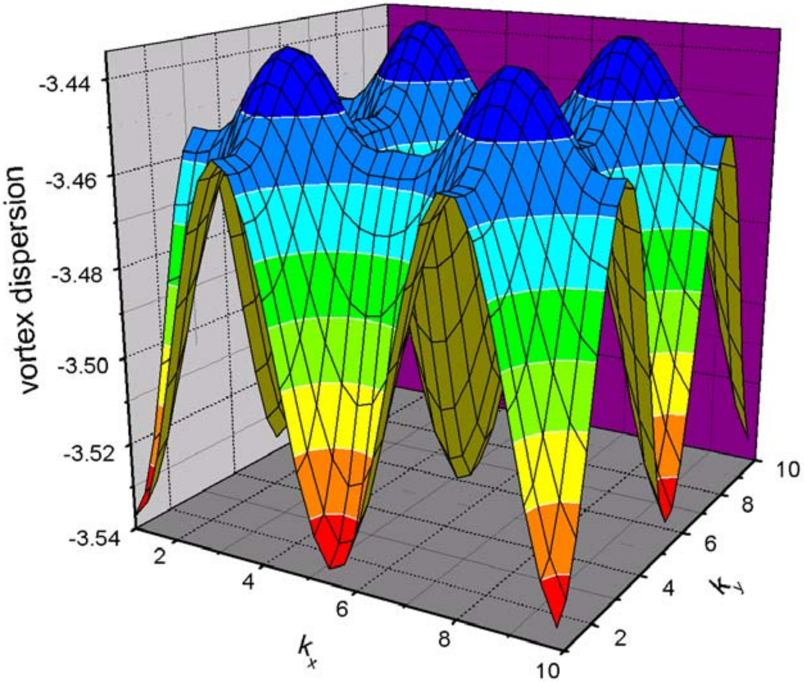
$$T_x T_y = e^{2\pi i f} T_y T_x \quad ;$$

$$R^{-1} T_y R = T_x \quad ; \quad R^{-1} T_x R = T_y^{-1} \quad ; \quad R^4 = 1$$

Strength of "magnetic" field = density of bosons
= f flux quanta per plaquette

Boson-vortex duality

Hofstädter spectrum of dual "superconducting" order



At density $f = p/q$ (p, q relatively prime integers) there are q species of vortices, φ_ℓ (with $\ell = 1 \ldots q$), associated with q gauge-equivalent regions of the Brillouin zone

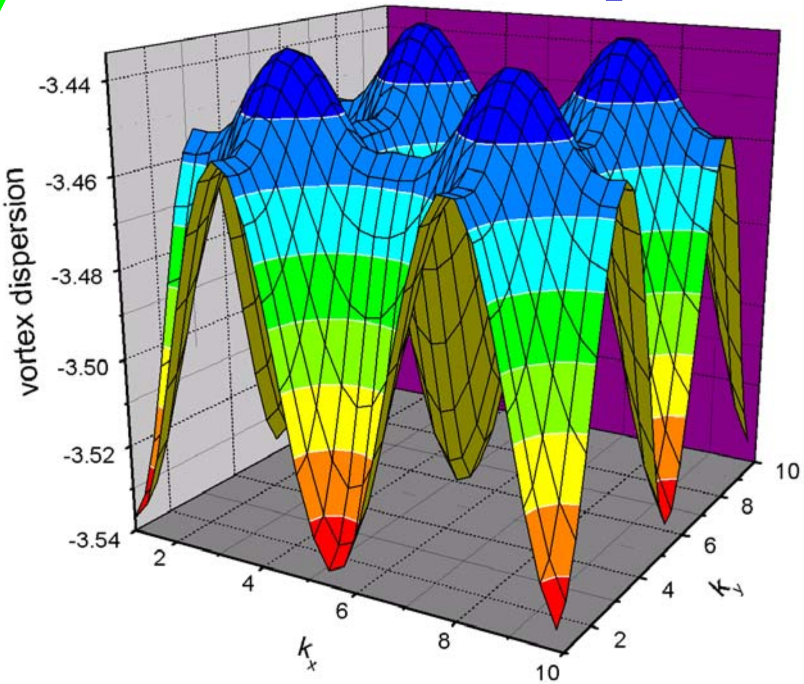
Magnetic space group:

$$T_x T_y = e^{2\pi i f} T_y T_x \quad ;$$

$$R^{-1} T_y R = T_x \quad ; \quad R^{-1} T_x R = T_y^{-1} \quad ; \quad R^4 = 1$$

Boson-vortex duality

Hofstäder spectrum of dual "superconducting" order



At density $f = p/q$ (p, q relatively prime integers) there are q species of vortices, φ_ℓ (with $\ell = 1 \ldots q$), associated with q gauge-equivalent regions of the Brillouin zone

The q vortices form a *projective* representation of the space group

$$T_x : \varphi_\ell \rightarrow \varphi_{\ell+1} \quad ; \quad T_y : \varphi_\ell \rightarrow e^{2\pi i \ell f} \varphi_\ell$$

$$R : \varphi_\ell \rightarrow \frac{1}{\sqrt{q}} \sum_{m=1}^{q} \varphi_m e^{2\pi i \ell m f}$$

See also X.-G. Wen, *Phys. Rev.* B **65**, 165113 (2002)

Boson-vortex duality

The φ_ℓ fields characterize *both* superconducting and charge order

Superconductor/insulator : $\langle \varphi_\ell \rangle = 0 / \langle \varphi_\ell \rangle \neq 0$

Charge order:

Status of space group symmetry determined by

density operators $\rho_{\mathbf{Q}}$ at wavevectors $\mathbf{Q}_{mn} = \frac{2\pi p}{q}(m,n)$

$$\rho_{\mathbf{Q}_{mn}} = e^{i\pi mnf} \sum_{\ell=1}^{q} \varphi_\ell^* \varphi_{\ell+n} e^{2\pi i \ell mf}$$

$$T_x : \rho_{\mathbf{Q}} \to \rho_{\mathbf{Q}} e^{i\mathbf{Q}\cdot\hat{x}} \quad ; \qquad T_y : \rho_{\mathbf{Q}} \to \rho_{\mathbf{Q}} e^{i\mathbf{Q}\cdot\hat{y}}$$

$$R : \rho(\mathbf{Q}) \to \rho(R\mathbf{Q})$$

Boson-vortex duality

The φ_ℓ fields characterize *both* superconducting and charge order

Competition between superconducting and charge orders:

"*Extended* LGW" theory of the φ_ℓ fields with the action invariant under the projective transformations:

$$T_x : \varphi_\ell \to \varphi_{\ell+1} \quad ; \quad T_y : \varphi_\ell \to e^{2\pi i \ell f} \varphi_\ell$$

$$R : \varphi_\ell \to \frac{1}{\sqrt{q}} \sum_{m=1}^{q} \varphi_m e^{2\pi i \ell m f}$$

Immediate benefit: There is no intermediate "disordered" phase with neither order

(or without "topological" order).

Analysis of "extended LGW" theory of projective representation

Superconductor

$\langle \Psi_{sc} \rangle \neq 0, \langle \rho_{\mathbf{Q}} \rangle = 0$

First order transition

Charge-ordered insulator

$\langle \Psi_{sc} \rangle = 0, \langle \rho_{\mathbf{Q}} \rangle \neq 0$

$r_1 - r_2$

Superconductor

$\langle \Psi_{sc} \rangle \neq 0, \langle \rho_{\mathbf{Q}} \rangle = 0$

Coexistence

(Supersolid)

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Analysis of "extended LGW" theory of projective representation

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$r_1 - r_2$

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(Supersolid)

$\langle \Psi_{sc} \rangle \neq 0, \langle \rho_{\mathbf{Q}} \rangle \neq 0$

Charge-ordered insulator

$\langle \Psi_{sc} \rangle = 0, \langle \rho_{\mathbf{Q}} \rangle \neq 0$

$r_1 - r_2$

Superconductor

$\langle \Psi_{sc} \rangle \neq 0, \langle \rho_{\mathbf{Q}} \rangle = 0$

Second order transition

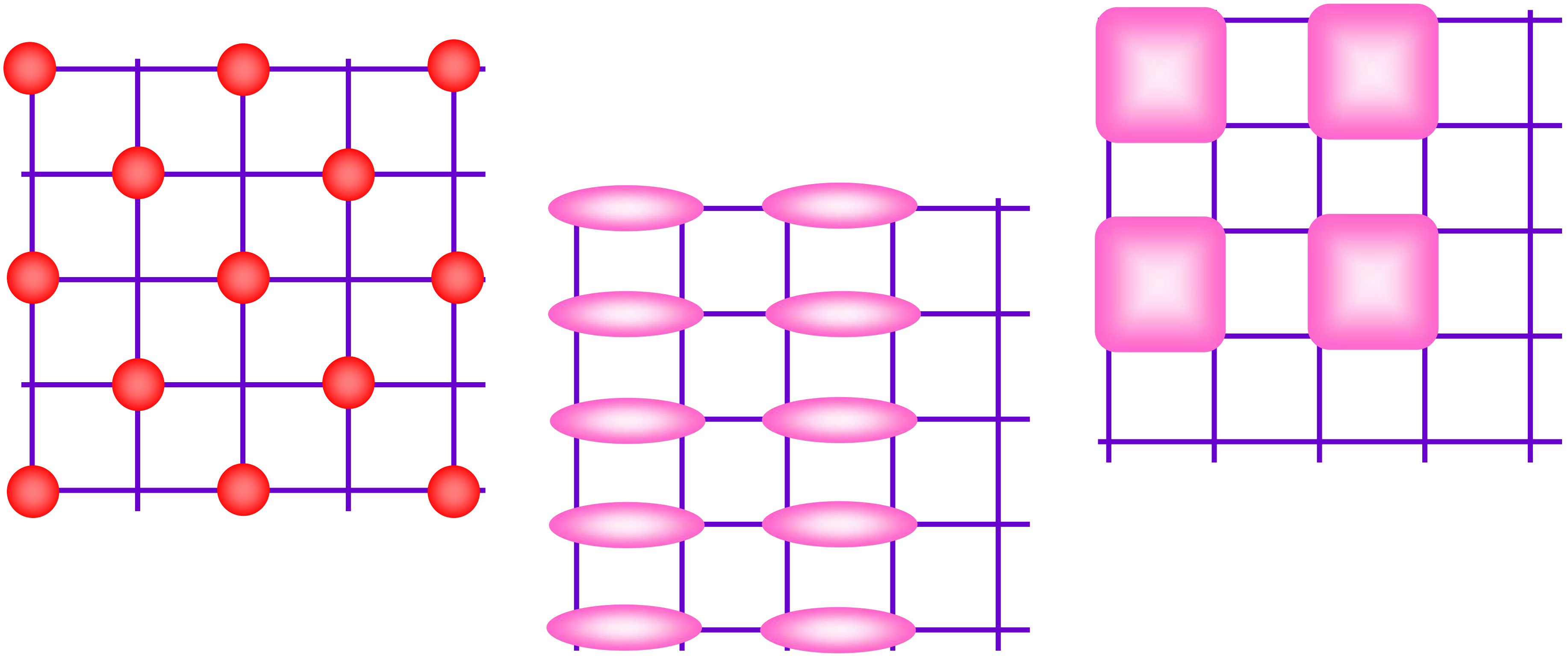
Charge-ordered insulator

$\langle \Psi_{sc} \rangle = 0, \langle \rho_{\mathbf{Q}} \rangle \neq 0$

$r_1 - r_2$

Analysis of "extended LGW" theory of projective representation

Spatial structure of insulators for q=2 (f=1/2)

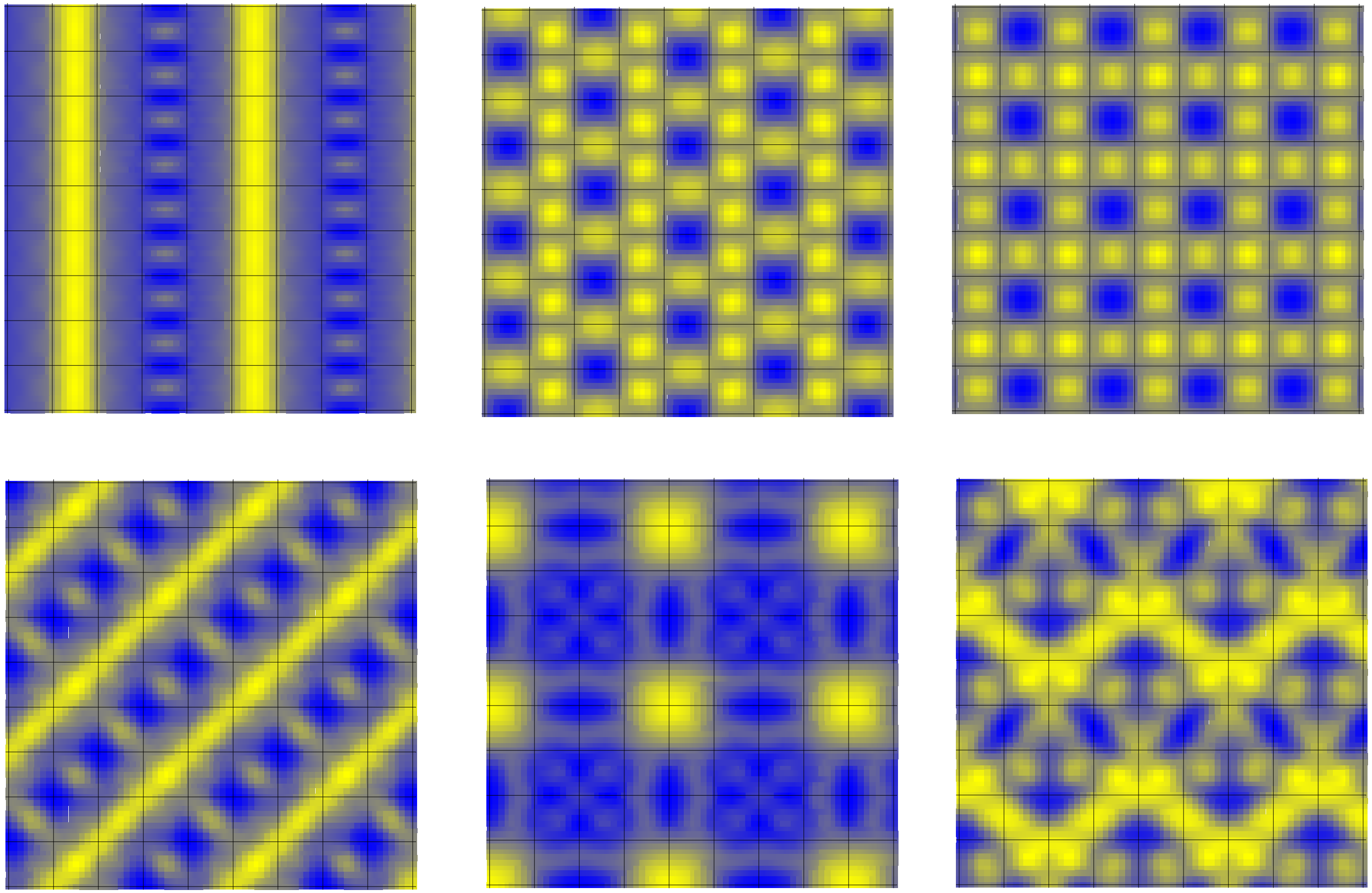


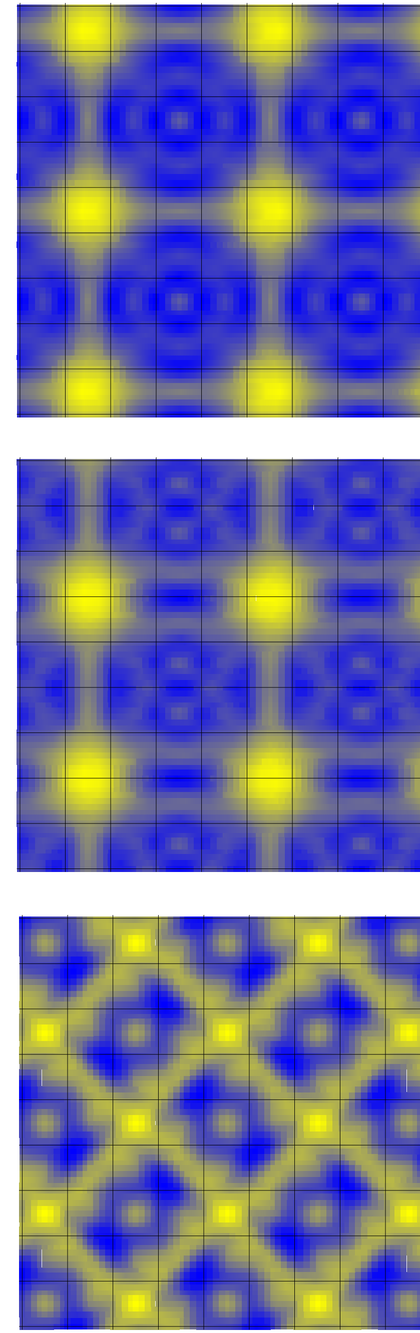
$$= \frac{1}{\sqrt{2}}\left(\bullet\!\!-\!\!\!\vdash \; + \; \dashv\!\!-\!\!\bullet \right)$$

$a \times b$ unit cells; q/a, q/b, ab/q, all integers

Analysis of "extended LGW" theory of projective representation

Spatial structure of insulators for q=4 (f=1/4 or 3/4)





$a \times b$ unit cells; ${}^{q}\!/_{a}$, ${}^{q}\!/_{b}$, ${}^{ab}\!/_{q}$, all integers