BOULDER SUMMER SCHOOL LECTURE: INTRODUCTION TO HYDRODYNAMICS

MICHAEL P. BRENNER

1. Some Philosophy

Fluid mechanics/Hydrodynamics is a remarkable subject, worthy of serious study and contemplation. The equations of motion are unambiguous, and of tremendous practical and scientific importance. They capture an enormous range of phenomena from very small scale (the motion of cells, or flow through very small, microscopic, holes), to the largest scales (flows in solar systems and galaxies), and an incredible amount in between. Understanding the equations is in many cases tantamount to understanding the phenomena themselves. For this reason, fluid mechanics is an incredibly interdisciplinary subject–practitioners routinely immerse themselves into different fields of human inquiry and try to use their knowledge of constructing solutions to the equations of motion as a means to advance basic science or engineering in these fields.

At the same time, fluid mechanics is a very humbling subject: it is *hard* to solve these equations, as we will see throughout this school. But nonetheless over the past 150 years, we have learned an enormous amount about the solutions and how to organize them.

Knowledge of how to organize and understand solutions of equations like this has had impact well beyond the field of fluid mechanics per se. Many if not most of the important approximations used routinely for solving nonlinear partial differential equations were invented in the context of fluid mechanics (among these we list the idea of the boundary layer; the idea of effective diffusivities or statistical solutions to equations; the foundations of how to solve such equations numerically, using a computer.)

Moreover, fluid mechanics gives some of the most paradigmatic examples of what it means to actually solve a problem in the face of overwhelming complexity. The solutions to the Navier Stokes equation for a turbulent flow field are incredibly complex-nobody can give a complete mathematical characterization of their solutions. But G.I. Taylor's notion of a turbulent diffusivity gives a compelling and quantitative description of how tracers actually behave in a turbulent flow. Kolmogorov's characterization of the energy spectrum is simple, but a tour de force. And it goes on.

2. The Equations

Without further adieu, lets write down the equations of motion and start to talk about what they mean and how they are organized. The equations are the celebrated Navier Stokes equation, given by:

Date: July 5, 2011.

$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \eta \nabla^2 \mathbf{u} + \zeta \nabla(\nabla \cdot \mathbf{u}) + \mathbf{f}, \tag{1}$$

together with the equation of mass conservation

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{2}$$

Here ρ is the density of material, $\mathbf{u} = \mathbf{u}(x, y, z, t)$ is the velocity, and η is the dynamic viscosity, and ζ is the so-called second viscosity. These equations must be supplemented with an equation of state for the pressure, which is usually of the form $p = p(\rho, T)$, where T is the temperature of the fluid. In situations where the temperature is non-uniform, there is an additional equation for the temperature field.

The terms have clear physical interpretations:

(1) The left hand side of the equation, $\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u})$ represents the acceleration of the fluid. Here we are representing $\mathbf{u}(x, y, z, t)$ in the coordinates of the laboratory frame (x,y,z) of reference. The first term represents explicit time dependence, whereas the second term represents the result of advection of a flow vector. This advection nonlinearity is the principal nonlinearity in fluid flows.

(2) The pressure gradient ∇p is a force due to gradients of pressures. This supplements the body force $\mathbf{f}$ which can be caused by a variety of different effects, depending on the scenario.

(3) The viscous terms, ie $\eta \nabla^2 \mathbf{u}$. These are the terms in the equation that have the highest number of spatial derivatives, a property that leads to their own unique role even in situations where the term doesnt seem like it should be important.

3. Organizing and Unravelling:The Principal of Dominant Balance

So, with the equations given–all we have to do is solve them. There is one fundamental principle that is *always* used when the equations are solved or organized. Indeed, this principal was also used for organizing this conference, as we will explain.

The principal is called *The Principle of Dominant Balance*. This is the idea that, in any given situation, the terms in the solutions to our equations will have (often wildly) different sizes and different terms will be the largest. For example, in one situation, the inertial terms might be the largest; while in another it might be the viscous terms. We categorize the solutions by which terms are the largest.

How to do this systematically? We need to estimate the sizes of the different terms in the equations. Let us suppose that we are considering a situation where the characteristic size of the velocity is U, and the flow is varying over a length scale L. This means that the typical timescale of the flow is L/U [1]. With these assumptions we can compare the sizes of the various terms.

[1] Note that there is an implicit assumption here that the velocity field itself is not varying or if it is varying it could vary on a timescale different than L/U. If this happens it could change the assertions we are about to make!

3.1. **Inertial versus Viscous: The Reynolds Number.** With these assumptions, the inertial terms have a size

$$\partial_t \mathbf{u}, \mathbf{u} \cdot \nabla \mathbf{u} \sim \rho \frac{U^2}{L},$$

whereas the viscous terms have a size

$$\eta \nabla^2 \mathbf{u} \sim \eta \frac{U}{L^2}.$$

We can compare these terms with each other by looking at the ratio of their sizes. This gives the celebrated *Reynolds number*

$$\mathrm{Re} = \frac{UL}{\nu}, \tag{3}$$

where $\nu = \eta/\rho$ is the so called kinematic viscosity of the flow. It has the dimensions of a diffusion constant.

Discuss importance of Reynolds number: Large, the flow is inertially driven; Small the flow is under damped.

3.2. **Sound versus no sound: Mach Number.** Another important approximation involves the conservation equation. If we imagine that the density of the gas is *constant*, so that we can write $\rho \approx \rho_0$, then this equation reduces to

$$\nabla \cdot \mathbf{u} = 0. \tag{4}$$

When this happens, the pressure is no longer given by the equation of state, but it is instead a Lagrange Multiplier enforcing a volume constraint.

A reasonable question to ask is: Under what conditions is the density constant? The analysis of this is slightly involved: We need to estimate the size of the perturbation to the density that is non-uniform $\rho = \rho_0 + \delta\rho$, and ask under what conditions is this density change small. For now I'll leave this as an exercise.

3.3. **Not Foolproof.** Note that there is a little bit of danger in defining dimensionless numbers and eliminating physics. This danger is the heart of the subject, and the essence of where many discoveries are made!

Let's illustrate this with a very simple example. Consider an infinite flat plate, that is oscillating side to side in an incompressible fluid (constant density) according to $U_{plate} = U_0 cos(\omega t)$. Let's assume/imagine that the velocity of the oscillation is small; if this happens we might feel justified in neglecting the inertial terms in the equations.

What are the equations: Let's let u be the component of the velocity parallel to the wall (x direction) and v the component perpindicular to the wall (the y direction). According to $\nabla \cdot \mathbf{u} = \partial_x u + \partial_y v = 0$, we see that it is perfectly consistent to pick $v = 0$, and then have $u = u(y)$;we expect the horizontal velocity to decay with distance from the plate.

Thus the momentum equation is

$$\rho(\partial_t u + u\partial_x u) = \eta \nabla^2 u - \partial_x p. \tag{5}$$

The $u\partial_x u$ term vanishes identically, as does the pressure gradient–far away from the wall there are no flows and so the pressure gradient vanishes.

We thus have

$$\partial_t u = \nu \partial_y^2 u, \tag{6}$$

the diffusion equation! To find the solution, write $u(y) = \Re e^{i\omega t} F(y)$, and find that

$$F(y) = exp(-(1+i)y/\delta),$$

where $\delta = \sqrt{2\nu/\omega}$. Thus the solution decays exponentially away from the plate. This solution is called the *Stokes* layer.

Note: We could not neglect inertia in this problem–it was critical! And this is true even if the amplitude of oscillation, U_0,is arbitrarily small.

3.3.1. *More Advanced Remark.* If we make this problem just a tad more complicated, the situation gets even worse. Lets suppose that instead of an oscillating plate we have an oscillating sphere. Now by analogy to what we have just discussed there will be a Stokes layer around the sphere, an oscillating layer where the action is, of size δ. But now the flow is not uni-directional–there is also a part of the velocity that is moving in the direction perpendicular to the plate, and correspondingly there is vertical flow. We can compute this vertical flow by just assuming that $U_0 = U_0(x)$, and that the dependence is very weak–ie the length scale over which the horizontal velocity varies is long compared to δ. We then can use continuity to say that

$$\partial_y v = -\partial_x u = -\frac{dU_0}{dx} \Re e^{i\omega t} F(y).$$

This can be directly integrated to find

$$v = -\frac{dU_0}{dx} \int dy e^{i\omega t - (1+i)/\delta y} = \frac{dU_0}{dx} \frac{\delta}{1+i} e^{i\omega t - (1+i)/\delta y}. \tag{7}$$

Now if we compute the term that we neglected before

$$\rho v \partial_y u,$$

it is no longer nonzero. It is of course small though as long as the amplitude of the oscillation is small–this is just the reynolds number criterion from above (using a length scale that is the size of the region over which U_0 varies).

But: you can show with some algebra that this term has a *nonzero time average!* The consequence of this is profound, you can think of it as a steady forcing function in the flow. Thus the term that we have neglected all of a sudden generates a steady flow. This is the most striking part of the solution! And we would have missed it if following dimensional terms too closely.

3.4. **Other Dimensionless Numbers.** The subject of fluid mechanics is rife with many many other dimensionless numbers. Here are some examples:

3.4.1. *Froude number.*

$$\mathrm{Fr} = \frac{U}{\sqrt{gL}}, \tag{8}$$

is called the *Froude number*, and represents the size of the velocity relative to what it would be if an object freefalls from rest over a distance L. This is the competition between fluid inertia and gravity.

3.4.2. *Capillary Number.*

$$\mathrm{Ca} = \frac{U}{\gamma/\eta}, \tag{9}$$

is called the *Capillary number*, and represents the size of the velocity relative to the velocity γ/η, where γ is the fluid surface tension. This dimensionless number often arises in problems of fluid wetting.

3.4.3. *Bond number.*

$$\mathrm{Bo} = \frac{\Delta\rho g L^2}{\gamma} \tag{10}$$

is called the *Bond number*, and represents the competition between gravitational forces and capillary forces (surface tension). Here γ is the surface tension, and $\Delta\rho$ is the density difference of one fluid in a background, so that $\Delta\rho g$ represents a bona fide body force.

3.5. **Dimensional Analysis.** Dimensionless numbers are quite related to another topic, called dimensional analysis. This is the idea that – if we have a sense as to what is going on in a given situation, we can often guess the form of the solution without having to solve equations at all. Of course the solution to the equations will also bear these out, assuming they are correct! But this is often extremely useful, and will come up frequently in the talks that are given. So here is a brief introduction:

There is a famous theorem that determines how many dimensionless parameters are needed to specify a given problem. The theorem summarizing this is called Buckingham's theorem, the general principle of dimensional analysis. We will present this theorem, and then present some simple examples of dimensional analysis to illustrate the power of the approach. And finally we will demonstrate in a few explicit examples how to put an equation into dimensionless form, thus illustrating how " ϵ " arises in some concrete cases.

3.5.1. *Buckingham's π Theorem.* This theorem was discovered/formalized by E. Buckingham *Phys. Rev.* , 4, 345-376, 1914. It is amusing to note that Buckingham worked at the U.S. Bureau of Standards!, building upon an earlier idea that (he credits to) Fourier that all of the terms of a meaningful equation must have the same dimensions.

The idea is the following: suppose you are given a problem which is characterized by some number of parameters $\{Q_i\}, i = 1 \ldots N$. These parameters in general have dimensions, and for this reason they are not all independent. Buckingham's theorem states that any meaningful statement about the system

$$\Phi[\{Q_i\}] = 0 \tag{11}$$

is equivalent to another statement

$$\Psi[\{\Pi_n\}] = 0 \quad n = 1...N - r., \tag{12}$$

where the variables Π_n are dimensionless. The main point is that the second relation contains r fewer variables than the first relation.

The basic reason behind Buckingham's theorem is that any problem has some number of "fundamental units" that must be specified for the problem to make sense. For example, in a problem involving Newton's laws of motion, we must specify the units of mass, length and time. Equation (**??**) contains quantities specifying this list of units, whereas equation (**??**) does not, since the variables Π_n are dimensionless. The number r is therefore just the number of fundamental units that need to be specified.

3.5.2. *Examples of Dimensional Analysis.* The main point of Buckingham's theorem is that the best way to write a mathematical relation between variables is in dimensionless form. Sometimes, by writing the problem in dimensionless form one can learn everything one wants to know about it. This lucky situation sometimes happens, and sometimes doesn't; when it doesn't, calculation is needed, hence requiring the methods of this course!

In what follows we will go through several examples of dimensional analysis, exposing both when it works and when it doesn't work.

Pythagorean Theorem: Now we try to prove the Pythagorean theorem by dimensional analysis. Suppose you are given a right triangle, with hypotenuse length L and the smallest acute angle ϕ . The area A of the triangle is clearly $A = A(L, \phi)$. Since ϕ is dimensionless, it must be that

$$A = L^2 f(\phi), \tag{13}$$

, where f is some function that we don't know.

Now, the right triangle can be divided into two little right triangles by dropping a line from the other acute angle which is perpendicular to the hypotenuse. These two right triangles have hypotenuses which happen to be the two other sides of our original right triangle, let's call them a and b . So we know that the area of the little right triangles are $a^2 f(\phi)$ and $b^2 f(\phi)$ (where, elementary geometry shows that the acute angle

ϕ is the same for the two little trianges as the big triangle.) Moreover, since these are all right triangles, the function f is the same for each! Therefore, since the area of the big triangle is just the sum of the areas of the little ones, we have

$$L^2 f = a^2 f + b^2 f \tag{14}$$

or,

$$L^2 = a^2 + b^2. \tag{15}$$

The size of atoms The size of an atom is given by the solution to the Schrodinger equation, namely

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\mathbf{r}, t) + V(\mathbf{r}) \Psi(\mathbf{r}, t), \tag{16}$$

where $V(r)$ is the interaction potential between the electrons and the nucleus. For the hydrogen atom this is particularly simple and the solution to this equation gives the probability distribution that the electron will be a distance r away from the nucleus. But we can solve for the characteristic scale even without solving the equation. The parameters in the equation are $\hbar$, Planck's constant, the mass of the electron m_e and the electric charge e. Planck's constant has the dimension of Energy-Time, whereas we must recall that e^2/Length is also an energy scale. The only way to make a length scale out of these numbers is through

$$a_0 = \frac{\hbar^2}{m_e e^2} = 0.53 \times 10^{-8}\text{cm}. \tag{17}$$

This is the famous Bohr radius.

The Density of Matter We can also extend this argument to estimate the density of matter. Roughly we say there is one proton for every bohr radius: remember that the number of protons and electrons increase together to maintain electrical neutrality. Hence the density is given by

$$\rho_0 = \frac{m_p}{a_0^3} = 1.4\text{g/cm}^3 \tag{18}$$

Fluid Viscosity Let us compute the viscosity of a fluid by dimensional analysis. The viscosity is a number ν with units of Length^2/Time that characterizes the momentum transport in a fluid. Momentum transport is roughly characterized by the molecules in a fluid bumping into each other. What numbers characterize these interactions? For a fluid at temperature T , we have the sound velocity (itself given by dimensional analysis as $c \propto \sqrt{k_B T/m}$, with m the mass of a particle, and the intermolecular scale a . Now by dimensional analysis, the viscosity must have the functional form

$$\nu \sim c \times a. \tag{19}$$

Now, the speed of sound in water (and indeed in most fluids) is about $1000 m/sec$; the characteristic distance between water molecules is about $a = 10^{-9} m$. Hence the viscosity of water should be about

$$\nu \sim 10^3\text{m/sec} \times 10^{-9}\text{m} = 10^{-6}\text{m}^2/\text{sec} = 10^{-2}\text{cm}^2/\text{sec}. \tag{20}$$

Indeed, this is essentially exactly the viscosity at room temperature! Note that our little theory also gives the temperature and molecular weight dependence of the fluid viscosity.

More serious theories of viscosity (based e.g. on the so-called Chapman Enszkog expansion of the Boltzmann equation) solve an integral equation for ν :since the parameters in the equation are necessarily those that we have used here c and a , the theory computes our answer up to a prefactor, typically of order unity.

A thought question If the argument we have given here is correct then really all simple fluids should have the same viscosity: after all, the molecular size is essentially constant, and the sound velocity is also rather constant. However if you look in the CRC handbook, you will note that there are some fluids e.g. glycerol that have much higher viscosity than that predicted by our simple argument. Why?

Atomic Energy Scale [2]What is the characteristic energy of electrical binding in atoms? We know that the interactions are electrostatic, and we have already calculated the size of an atom a_0 . Therefore the electrostatic energy is

$$\frac{e^2}{a_0} = \frac{e^4 m_3}{\hbar^2}. \tag{21}$$

Evaluating this number, it is about 27eV.

Man's Size Bill Press, when teaching a physics class at Harvard, took this argument a step further. He asked whether he could express the size of a human in terms of fundamental physical constants, and argued that the size of man is given by

$$\left(\frac{\hbar^2}{m_e e^2}\right)\left(\frac{e^2}{G m_p^2}\right)^{1/4}, \tag{22}$$

where e is the electron charge, m_e, m_p is the mass of electrons and protons, G is Newton's gravitational constant.

The Radius of the Earth Press also gives a very interesting argument for the radius of the earth. He argues that the earth's atmosphere does not contain hydrogen, and therefore the thermal velocity of hydrogen in the atmosphere must be above the escape velocity.

For a body to escape the earth's gravitional field, it needs a velocity

$$v^2 = \frac{G M_{earth}}{R_{earth}}. \tag{23}$$

On the other hand, the thermal velocity of hydrogen is given by $v^2 = \frac{k_B T}{m_p}$, , where k_B is Boltzmann's constant, and T is the temperature of our atmosphere (300K). Thus we need

$$\frac{k_B T}{m_p} = \frac{G M_{earth}}{R_{earth}}. \tag{24}$$

This gives one relationship between the mass of the earth M_{earth} and its radius. Another relationship comes from our claim that the density of matter is given by the expression above. Namely

$$\frac{M_{earth}}{R^3_{earth}} = \frac{m_p}{a_0^3}. \tag{25}$$

If we combine these two equations, we arrive at the following expression for the radius of the earth

$$R_{earth} = a_0\left(\sqrt{\frac{k_B T}{G m_p^2/a_0}}\right) \tag{26}$$

[2]Reference W. H. Press, "Man's size in terms of fundamental constants", Am. J. Phys., 48, 597(1980).

Note the number in brackets is dimensionless: this is the (square root of the) ratio of a thermal energy to the gravitational energy between two protons separated by the Bohr radius! Assuming $T = 300K$, the thermal energy $k_B T = 4.1 \times 10^{-21}$ J, whereas the gravitational attraction between two protons separated by the Bohr radius is $Gm_p^2/a_0 = 3.5 \times 10^{-54}$ J. Hence we have that

$$R_{earth} = 3.5 \times 10^{16} a_0, \tag{27}$$

which is 1.8×10^8 cm.

This is to be compared with the actual answer of 6.4×10^8 !

Another Calculation of the Earth's Radius I was very enthusiastic about the elegance of this formula, and once enthused about it to Prof. P. Chaikin, from NYU. Chaikin pointed out to me that the problem with this argument is that it assumes that one has made a measurement of the atmosphere of the earth, to check that there is no hydrogen. He argued that if we were going to make such a measurement we might as well just measure g, the gravitational acceleration of a body, and use that to make our calculation. So let's try this: We now combine

$$\frac{GM_{earth}}{R_{earth}^2} = g, \tag{28}$$

with

$$\frac{M_{earth}}{R_{earth}^3} = \frac{m_p}{a_0^3}. \tag{29}$$

These give *another* beautiful formula, namely

$$R_{earth} = a_0 \left(\frac{g}{Gm_p/a_0^2} \right), \tag{30}$$

so the earth's radius is the Bohr radius multipled by the ratio of the measured gravitational acceleration, and the acceleration that a mass feels one Bohr radius away from the center of a proton!

Evaluating numbers we find that the ratio of these accelerations is 2.28×10^{17}, so that

$$R_{earth} \sim 1.17 \times 10^9 \text{cm}. \tag{31}$$

This overestimates the radius by a factor of two!

Taylor's Blast This is a famous example, of some historical and fluid mechanical importance. The myth goes like this: In the early 1940's, there appeared a picture of the atomic blast on the cover of Life magazine. GI Taylor, an applied mathematician at Cambridge, wondered what the energy of the blast was. When he called his friends at Los Alamos and asked, they informed him that it was classified information. So he resorted to dimensional analysis. Let's imagine that the energy of the blast is E_0 . The blast starts from a spatial point. The dynamics basically just pushes the air out of the way. The speeds of this process are enormous so viscosity isn't important. The only important material parameter is the density of air ρ_0 . So, let's ask: what is the radius $R(t)$ of the blast as a function of time t

from the detonation time? We need to create an object with units of length out of E_0, ρ_0 and t .

Now E_0/ρ_0 has the dimensions of L^5/T^2 . Thus, $(E_0/\rho_0 t^2)^{1/5}$ has the units of length. Therefore

$$R(t) = c(E_0/\rho_0 t^2)^{1/5}. \tag{32}$$

Now suppose you want to estimate the energy of the blast. c is a constant we don't know, but it is probably ≈ 1 . We can measure the radius of the explosion from the cover of Life magazine. We know how long it has been since the blast, since the limits of strobe photography are around 1μ sec . We know the density of air. Thus, one can solve for E_0 .

The story is that Taylor called up his friends at Los Alamos and told them the correct number. They were deeply worried that security had been breached.

The mathematical basis for Taylor's calculation requires finding a particular solution to the equations of gas dynamics, assuming that the gas is initially at very high density.

3.5.3. *Fireworks!* Given the closeness of today to the 4th of July, perhaps it is worthwhile to try to make some estimates about fireworks. Lets ask: what sets the radius to which the fireworks explodes? ie what is the radius of the bright spheres that are so evident from a fireworks display?

Note that we cannot get this directly from Taylor's blast solution without introducing another dimensional parameter–the time that the chemical reaction leading to the light is on for. If we knew this time τ, we could use it in Taylor's formula, and find

$$R_{fireworks} = c(E_0/\rho_0 \tau^2)^{1/5}.$$

Let's try a different tact and see if it works. We might imagine that the light stays on and the explosion continues expanding until it can no longer expand. This will happen if the blast wave is no longer able to push away atmospheric pressure, p_{atm}. If this were true, then the radius then will be determined by the parameters E_0, p_{atm}; by dimensional analysis this gives

$$R_{fireworks} = \sqrt{\frac{E_0}{p_{atm}}}. \tag{33}$$

How can we determine which of our "two" theories is the correct one? The only way to do this is to try them out. The energy density of a fire work is about $5MJ/kg$ (This is the energy density of TNT). Lets imagine that the firework in question weighs 100 g; perhaps this is small enough that it can be easily projected high up in the air. The total energy is then $0.5MJ$. Atmospheric pressure is $\sim 10^5$Pa. Putting these together implies that $R_{fireworks} \sim 2 - 3$.

This is much too small!! I'm confused. Perhaps the constant in our formula is larger than order unity? This isn't entirely crazy, but....

3.6. **Back to the Equations.** With all of this as pre-amble, lets now go back to the equations of motion and see what sense can be made of them with our dimensionless numbers. Lets consider the equations with ρ constant–indeed, this approximation will be made for the rest of this school, more or less, for better or worse [3] Let's follow for a little bit the historical development of the subject, now go through the logical consequence the Reynolds number being large or small.

3.7. **Airplanes.** When the Reynolds number is large, the viscous term is much smaller than the inertial terms. This is often a sensible approximation. For example I am typing this on an airplane on the way to our summer school. The airplane is moving about 200 miles/hour or about 10 meters/second. The airplane has a size of about 50 meters. The kinematic viscosity of air is $\nu = 1 \times 10^{-5} m^2/sec$. Hence the Reynolds number is

$$\mathrm{Re_plane} = \frac{10m/sec \times 50m}{10^{-5}m^2/sec} = 5 \times 10^7.$$

Thus, the viscous terms are *seven* orders of magnitude smaller than the inertial terms in the equation.

Thus, you hopefully will not become too upset when we approximate the equations by neglecting the viscous terms. Namely we have the momentum equation

$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p, \tag{34}$$

together with $\nabla \cdot \mathbf{u} = 0$..

This doesn't seem like a huge simplification, but, as we will see in a minute, in the early part of the twentieth century, people used this with one other crucial assumption (that of potential flow) to try to compute the lift and drag on airplanes. The result was a theoretical embarassment, whose resolution eventually resulted in what was probably the most significant applied mathematical discovery of the twentieth century.

3.8. **Tiny bugs.** Before discussing this lets consider one other example: Suppose I were flying to this meeting on a tiny bug, perhaps an *e. Coli*, whose size is about 10μm. Suppose the *e. Coli* moved very quickly at about a body length/sec (this is faster than the airplane, relatively speaking!) Then the Reynolds number is

$$\mathrm{Re_plane} = \frac{10^{-5}m/sec \times 10^{-5}m}{10^{-6}m^2/sec} = 10^{-4}.$$

Here we have used the viscosity of water, which is (as we calculated above) $10^{-2}cm^2/sec$ or $10^{-6}m^2/sec$. For a tiny bug, the Reynolds number is extremely small so now the inertial terms are ten thousand times smaller than the viscous ones. It now seems quite reasonable to neglect inertia! The equations now are

$$\eta \nabla^2 \mathbf{u} = \nabla p + \mathbf{f}, \tag{35}$$

together with $\nabla \cdot \mathbf{u} = 0$. In writing these equations, I've kept the forcing function–as in practice, and e.g. in this meeting, low Reynolds number flows are only interesting when something is constantly forcing them. If you stop forcing them, they stop instantly as the equations readily demonstrate! People call these equations, without inertia, the Stokes equations.

[3] Thus, this school will ignore sound, other than the speaking of you to each other and the speakers to all of you! There is much interesting physics of sound that will be sadly neglected. Another year...

4. Meeting Organization

OK, with this extended pre-amble, I can now summarize to you what is going to happen in this school. Lets go through it by week:

4.1. **Week 1.**

This first week is partly introductory (like this lecture), and partly designed to expose you to fascinating phenomena and regimes that happen within the subject of fluid mechanics. We might organize the lectures this week as follows:

(1) Flows with surfaces and surface tension. Lohse will describe beautiful phenomona with bubbles; Stebe and Quere will describe a host of important and beautiful phenomona involving droplets and surface tension gradients and interactions between particles and lots of things like this. Most of these flows will necessarily be at low Reynolds number.

(2) Tuckerman will discuss the important subject of hydrodynamic instabilities, and the patterns that can form from these instabilities. This is usually mainly at high Reynolds number.

(3) Lohse will summarize what happens in one of these pattern forming instabilities, thermal convection, when you force it very very hard. Eventually it becomes turbulent. Lohse will describe how the different turbulent regimes will be organized. This lecture in a sense connects week 1 and 2.

(4) Goldenfeld will explain how he has used the Renormalization Group from theoretical physics to make fundamental progress on the age old problem of calculating the drag on moving bodies.

4.2. **Week 2.**

This week is on turbulence and is thus at high Reynolds number. Lathrop and Goldenfeld will give introductions to the experimental and theoretical sides of this subject; Eckhardt will discuss turbulent transition, and Pouquet will discuss geophysical examples. Voth will show beautiful examples of the importance of turbulent mixing.

4.3. **Week 3 & 4.**

These weeks get into the Low Reynolds number regime, and focus on both active matter and biological examples. These are cases where the forcing function $\mathbf{f}$ in the above is quite relevant–you will see a host of examples that are quite exciting. I won't describe these in detail–because by week 3 you will have forgotten! Peko Hosoi will give an introduction to what will happen at the beginning of week 3.

5. The two main calculations for today

Thus far, this lecture has been more words than equations. What I'd like to do now in the remaining time is to present two calculations–one at high Reynolds number and one at low Reynolds number –which represent critical ideas for the subject of fluid mechanics. The first calculation is arguably the most important applied-mathematical idea of the twentieth century (at least I would argue it), and it is Prandtl's discovery of the boundary layer. The second calculation is more prosaic, but it does come up a lot–and illustrations a sort-of application of Prandtl's main mathematical idea at low Reynolds number.

5.1. **Ludwig Prandtl and Boundary Layer Theory.** The basic ideas of boundary layer theory were invented by Ludwig Prandtl, in what was arguably the most significant contribution to applied mathematics in the 20^{th} century. Prandtl presented his ideas in a paper in 1905, though it took many years for the depth and generality of the ideas to be appreciated.

In order to give some historical context for Prandtl's discovery, it must be first understood that at the time *the* technology of the day was man -made flight. The Wright brothers were flying their airplanes, and one can only imagine the excitement in the scientific community for figuring out the mathematical principles underlying these developments.

However, there was a bit of an embarrassment for the theorists: (to paraphrase Lord Rayleigh's **1915** review of a hydrodynamics textbook in *Nature*–"Someday we might hope that theoretical hydrodynamics might be in agreement with experiments." The reason for this negativism was that as of the early 1900's, theories of airplane flight predicted

(1) The only way to get an airplane off the ground is to spin it as it lifts off. However, practically speaking, this is nonsense.
(2) Once it gets off of the ground, the drag should be identically zero. This is also nonsense–fuel is burned!

It turned out that both of these issues were the result of a fundamental theoretical issue: a perfectly reasonable estimate of the dissipation term in the equations of fluid dynamics led to the conclusion that the dissipation was unimportant, and the aforementioned conclusions were made with this in mind. However, Prandtl demonstrated that just because one estimated the dissipation to be small doesn't mean it can be neglected–hence, the idea of boundary layers.

In what follows, we will go through the historical development to both illustrate Prandtl's discovery, and document how boundary layer theory dramatically simplifies the solution to nonlinear partial differential equations.

5.1.1. *Preliminaries.* To proceed further into airplane flight, we need some fluid mechanical preliminaries.

We introduced before the Navier Stokes equation

$$\rho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla p + \nu\nabla^2\mathbf{u} \tag{36}$$

. We also pointed out that in the airplane that I am now sitting, the Reynolds number is enormous and hence we are safe in neglecting the viscous terms. We thus have

$$\rho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla p. \tag{37}$$

This equation can be written in an interesting form using the vector identity

$$\mathbf{u}\cdot\nabla\mathbf{u} = (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla(\mathbf{u})^2/2 \tag{38}$$

as

$$\rho(\partial_t \mathbf{u} + (\nabla\times\mathbf{u})\times\mathbf{u}) = -\nabla(p/\rho + \mathbf{u}^2/2). \tag{39}$$

If the flow is steady $\partial_t\mathbf{u} = 0$ and irrotational $\omega = \nabla\times\mathbf{u} = 0$ then we have *Bernoulli's law*

$$H = \frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2 \tag{40}$$

is constant. If the flow is not irrotational, then we have $\mathbf{u} \cdot \nabla H = 0$ so that the quantity H is constant along streamlines.

The constancy of H when $\omega = 0$ is a famous result, and has many simple qualitative consequences. It states that the pressure in a fluid is smaller when the velocity is larger. The classical example of this is flow through segments of pipe of varying diameter. Whenever the radius in the pipe drops, the pressure in the pipe increases.

However, for the moment, let's question this theorem: Clearly, the constancy of H only makes sense if it actually happens in practice. For it to happen in practice, it needs to be true that $\omega = 0$ for all time. This leads to the question:

If $\omega = 0$ initially in a flow, does that imply that $\omega = 0$ for all time?

5.2. **Potential Flow.** What are the consequences of $\omega = 0$? The most important is that in this case, it is possible to describe the flow by a velocity potential ϕ. Namely, there exists a scalar function ϕ so that changes in ϕ are given by

$$d\phi = \sum_i u_i dx_i. \tag{41}$$

The velocity $\mathbf{u} = \nabla\phi$. The condition for this to be possible is that

$$\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = \frac{\partial^2 \phi}{\partial x_2 \partial x_1}, \tag{42}$$

or in other words $\omega = 0$. Conversely, if $\mathbf{u} = \nabla\phi$ then $\omega = 0$.

Now, the wonderful thing about this is that if the flow is incompressible, then since $\nabla \cdot \mathbf{u} = 0$, we have $\nabla^2 \phi = 0$! The potential obeys Laplace's equation. What about the pressure? Well, we can write from the Euler equation

$$\partial_t \phi + (\nabla\phi)^2/2 = -p/\rho, \tag{43}$$

so that at each instant the pressure is given in terms of the velocity potential.

5.3. **Kelvin's Theorem.** This result is clearly only relevant if a fluid which is initially vorticity free remains this way for all time. That this is in fact true was first shown by Lord Kelvin: Consider the *circulation* around a closed loop

$$K(t) = \int \mathbf{u} \cdot d\ell, \tag{44}$$

where $d\ell$ is an element of arc length. By Stokes's theorem, $K(t) = \int \omega dA$. Thus, if $\omega = 0$, $K = 0$. What is the time evolution of K?

Let us introduce the notation

$$\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla.$$

This is called the total derivative and takes into account the rate of change of something due to both its explicit time dependence and its motion (through the laboratory coordinates where we are sitting).

The chain rule implies that

$$\frac{DK}{Dt} = \int \left(\frac{Du}{Dt} d\ell + u\frac{D\ell}{Dt}\right), \tag{45}$$

where $Dd\ell/Dt = \partial_\ell u d\ell$ so that

$$\mathbf{u} \cdot \frac{D(d\ell)}{Dt} = \mathbf{u}\partial_\ell \mathbf{u} dl = d(\mathbf{u}^2)/2. \tag{46}$$

Since integrating around the loop leaves u^2 unchanged, the second term is zero. As for the first term, it is easy to argue that it also integrates to zero using the Euler equation itself. thus, we have that

$$\frac{DK}{Dt} = 0. \tag{47}$$

Since this is true for every loop, it shows that a fluid with initially zero vorticity remains zero for all time.

One of the most interesting things about this theorem is its historical origin: Kelvin viewed it as the basis of his vortex theory of the atom!

One historical comment: imagine you were working in the end of the 19th century: you now have the wonderful situation where the most important technological problem of the day, air plane flight, is related to the most beautiful mathematical problem–potential flow, and two dimensional potential flow (conformal invariance and all) at that! Imagine how excited you would be..

5.3.1. *Drag on a wing section.* Lets now make a model for the drag and lift on a wing section. For simplicity lets consider a wing as an infinite two dimensional body with constant cross section. I'm looking out the wing of the airplane to Boulder as I'm writing this, and this is clearly not true-but it is a reasonable model. For simplicity, lets also pretend that the wing is a circle. (we will relax this in a bit). From the approximations given–Euler equation/potential flow, lets compute the lift and the drag.

Initially when the airplane is taking off, there is no vorticity in the flow. The airplane is just sitting at rest. So this must always be true.

Now lets take the wing as circle of radius R. Let's say the flow is uniform (with magnitude V) in the x direction, as $|x| \to \infty$. In polar coordinates, Laplace's equation is

$$\frac{1}{r}\partial_r(r\partial_r\phi) + \frac{1}{r^2}\partial_\theta(\partial_\theta\phi) = 0. \tag{48}$$

A uniform flow has $\phi = Vrcos(\theta)$ At the surface of the circle, we have the boundary condition that the normal component of the velocity is zero: $\partial_r\phi(r = R) = 0$. Let's find the solution of Laplace's equation consistent with these conditions.

To proceed, let's guess a solution to Laplace's equation of the form $\phi = f(r)g(\theta)$. Plugging this into Laplace's equation gives

$$r(rf')'/f + g''/g = 0.$$

The first term in this formula depends solely on r and the second term solely on θ. Hence both terms must be constants: Namely, $r(rf')' = cf$ and $g'' = -cg$. The first equation has solutions of the form $f(r) = r^\alpha$, so that $c = \alpha^2$. This implies that $g(\theta) = Asin(\alpha\theta) +$

$Bcos(\alpha\theta)$. Since g must be single valued, α must be an integer. (Call it n). Hence, the most general solution is

$$\phi = \sum_n (C_n r^n + D_n r^{-n})(A_n cos(n\theta) + B_n sin(n\theta)).$$

We need this to satisfy the boundary conditions. For $\phi \sim x$ as $x \to \infty$ we need the highest power of r in the expansion to be $n = 1$. That is

$$\phi = Vrcos(\theta) + \sum_n r^{-n}((A_n cos(n\theta) + B_n sin(n\theta))).$$

To find the other A_n and B_n we need to satisfy the boundary condition $\partial_r \phi(r = R) = 0$. This implies that

$$Vcos(\theta) - A_1 cos(\theta)/R^2 + \sum_{n>1} (A_n(cos(n\theta) + B_n sin(n\theta))) \frac{-n}{R^{n+1}}.$$

This is satisfied only if $A_1 = VR^2$ and $A_n, B_n = 0$ for $n > 1$. Hence the solution is

$$\phi = (Vr + V\frac{R^2}{r})cos(\theta).$$

Using the solution we just computed, let's figure out the force on the sphere: The velocity components at the sphere are $u_r = 0$ and $u_\theta = -2Vsin(\theta)$. By Bernoulli's law, the pressure is $p = A - 1/2\rho(u_r^2 + u_\theta^2)$. Let's fix the constant $A = V^2\rho/2$ so that the pressure is zero far from the circle. Hence the pressure is $p = 1/2\rho(V^2 - V^2 sin(\theta)^2)$. The pressure is maximal in front and back of the cylinder, and vanishes at the apex.

Now note the interesting result:

The net force on the cylinder vanishes, because the pressure is symmetrical about the equatorial plane of the cylinder. Thus, a cylinder in uniform motion in the fluid experiences no force whatsoever! This result has been known as *D'Alembert's paradox*.

Note something interesting about our solution: At the surface of the wing, the flow does not obey the no-slip boundary condition, since u_θ is not zero at the surface of the cylinder.

5.4. **Solving more arbitrary wing shapes.** There is a beautiful bit of early twentieth century applied mathematics that repeats this analysis and demonstrates that it is true for *arbitrary* wing cross sections, even those that have singular tips (like those on a real wing). The mathematics works using a little bit of complex analysis and conformal mappings. Briefly, we introduce a useful devise for thinking about two dimensional flows. Let's write $(u_x, u_y) = (\partial_y \psi, -\partial_x \psi)$, where ψ is called the *stream function* of the flow. This velocity field automatically satisfies the two dimensional incompressibility condition. Note that

$$\mathbf{u} \cdot \nabla \psi = u_x \partial_x \psi + u_y \partial_y \psi = 0.$$

Thus, ψ is constant along streamlines of the flow.

Now, besides it's physical convenience, another great thing about the stream function is the following: by definition

$$u_x = \partial_x \phi = \partial_y \psi \quad u_y = \partial_y \phi = -\partial_x \psi.$$

The second of the equations in each pair are the well known *Cauchy Riemann* equations of complex analysis! This allows us to use the power of complex analysis to think about two dimensional potential flow problems.

Namely, $w = \phi + i\psi$ is an analytic function of $z = x + iy$ (i.e. depends only on $z = x + iy$.) Using this observation, it is possible to generalize the result for the circular wing to arbitrary shape wings–the lift and drag on the wings simply vanish.

This was a tremendous embarassment for early 20th century theorists. At the dawn of quantum mechanics we couldn't even compute a nonzero drag for an airplane–this despite the fact that fuel was clearly being burned!

5.4.1. *Problems and Resolutions.* We have thus found two crucial problems with our solutions

1 They did not satisfy the no slip boundary condition known to be true at solid boundaries.

2 There was no drag on our objects. Although I didnt have time to show this, one could show that there could be lift but only if circulation was induced around the object initially. However, according to Kelvin's theorem, production of circulation is not possible.

Now we will present solutions to both of these problems, by first, focusing on the first question.

The issue, as we have remarked several times, is our *initial estimate for the importance of the viscosity*: In fact, the beautifully simple calculation that we presented at the beginning of this lecture turns out to be dead wrong. The reason is subtle, but it is one that crops up again and again. Lets review our argument:

We argued that the Reynolds number is enormous, and hence the flow should not be affected by viscosity, and we neglected it; writing down inviscid theories. Although it was by no means obvious at the time, it turns out that all of our troubles can be solved by thinking more carefully about viscosity. Is it really negligible?

The answer to this question, as first given by Prandtl in 1904, is *no*. Prandtl hypothesized that the flow consists of two regions. A region far from the solid body where viscosity really is negligible. And a region close to the body where viscosity is very important. The reason, Prandtl said, that viscosity can be important in a flow with a small viscosity is that it is only important in a layer of very thin thickness. Let's call this thickness δ. Assume the characteristic velocity of the (inviscid) flow is named U_0, and the characteristic scale over which it varies is L. Then the idea is that $\delta \ll L$ in the boundary layer, which makes viscous stresses important.

5.4.2. *The beginning.* For simplicity let's confine ourselves to two dimensional steady boundary layers. Let's suppose that the wall occurs at $y = 0$ and that the variation is along the x coordinate. It is okay if the wall is curved, as long as it isn't so curved that the characteristic curvature is much greater than the characteristic length δ of the boundary layer. We will denote the velocity field by (u, v), where u is going to vary on a scale L in the x direction [4] and v varies across the boundary layer on a scale δ.

[4]Of course, u will also vary in the y direction. We need to solve for this dependence.

The continuity equation for the flow is

$$\partial_x u + \partial_y v = 0. \tag{49}$$

Let's estimate the size of $\partial_x u$. Since u varies on a scale L, this is of size U_0/L. Similarity, $\partial_y v \sim V/\delta$, where V is the size of velocity in the y direction. Hence we have found that

$$V \sim U_0 \frac{\delta}{L}.$$

The y velocity is much smaller than the x velocity!

Onwards to the navier stokes equation. The x component is

$$u\partial_x u + v\partial_y v = -\partial_x p + \nu(\partial_x^2 u + \partial_y^2 u). \tag{50}$$

The y component is

$$u\partial_x v + v\partial_y v = -\partial_y p + \nu\nabla^2 v. \tag{51}$$

Estimate sizes of terms: In the first equation $u\partial_x u \sim U_0^2/L$. $v\partial_y u \sim U_0 V/\delta \sim U^2/L$. The important point is that $\nu\partial_y^2 u \gg \nu\partial_x^2 u$, since the former is of order $\nu U_0/\delta^2$. Thus we neglect that derivatives of u in the x direction in the viscous stresses. What about the pressure gradient? Since it is causing the flow, we expect it will be of order the inertial terms, and so it is not negligible.

Now you should see the point of the boundary layer: it is not legitimate to neglect the viscous term because ν is small because this could be compensated as long as δ is small as well! In fact, if

$$\nu U_0/\delta^2 \sim U_0^2/L \tag{52}$$

then these two terms are exactly of the same size. Indeed, we will argue that this is exactly what happens.

Before doing this, let's look at the y equations. You can convince yourself that the biggest term in this equation is $\partial_y^2 v$ so to leading order we should just set this equal to zero. This implies that $\partial_y v = f(x)$. By the continuity equation, we know that $f(x) = -\partial_x u$, so that $v = -y\partial_x u$. We will not really use this form in what follows but it is good to keep in mind.

The equations we must solve for the structure of the boundary layer are

$$\mathbf{u}\partial_x u + v\partial_y u = -p(x)'/\rho + \nu\partial_y^2 u, \tag{53}$$

and the continuity equation. That the pressure $p(x)$ depends only on x comes from the following reasoning: far from the boundary layer, the flow is parallel to the plate, and the pressure is just $p(x)$. Near the plate there is indeed a pressure variation induced by the y component of the flow but since the y component of the flow is small this correction to the pressure will be small. And moreover, the pressure in this equation will be given by the pressure variation far away from the plate!

Far from the plate of course, viscosity is not importnat, there are no substantial variations of the flow in the y direction, and the equation for the velocity field $u_0(x)$ is just

$$u_0 u_0' = -p'. \tag{54}$$

So the pressure far away can be represented in terms of the velocity far away. Our goal is now to solve these equation under the conditions that the velocity vanishes at the solid wall ($y = 0$) and goes to u_0 far away from the wall.

5.4.3. *Back to boundary layers.* It is clear that since the equation for the boundary layer depends on U_0, there will be many different cases for the types of boundary layers that can occur. Roughly speaking, one can divide them into three cases, depending on the pressure gradient:

(1) $dp/dx > 0$. Here the external flow is decelerating.
(2) $dp/dx < 0$. External flow is accelerating.
(3) $dp/dx = 0$ no acceleration.

It turns out that the first situation is rather unstable, and the second is rather stable.

Let's first do the simplest case first, called the Blasius boundary layer in which $p = 0$, and the velocity asympotes to a constant far away. We will consider a situation where in the left half plane there is no solid, and in the right half plane there is an infinite solid at $y = 0$. The question is what does the boundary layer look like in the right side.

We will sketch a few high points of the derivation:

The pressure far from the boundary layer is uniform. Thus, we need to solve the equations

$$u\partial_x u + v\partial_y u = \nu\partial^2 u \tag{55}$$

$$0 = \partial_x u + \partial_y v, \tag{56}$$

with a boundary layer for $x > 0$ (where the plate exists). What will the structure of the boundary layer be? Before going through the derivation let's anticipate. We have already emphasized that the length scale in the x direction is L and the length scale in the y direction is δ, where balancing the viscous term against the others we find that $\delta = \sqrt{\nu L/U_0}$. Now, in the present case, there IS no horizontal length scale L! What does dimensional analysis say?

The quantity u/U_0 must be a function of x/L and y/δ, which is independent of L. The only combination that has this property is $y/\delta/\sqrt{x/L} = y\sqrt{U_0/(\nu x)}$. Hence, we anticipate

$$u = U_0 F(y\sqrt{U_0/(\nu x)}). \tag{57}$$

Let's check that this actually comes out of the calculation. Guess that $u = U_0 F(y/g(x))$. To satisfy continuity, take a stream function $(u, v) = (\partial_y\psi, -\partial_x\psi)$, so that $\psi = Ug(x)\int^\eta F(\eta) + k(x)$. For the plate itself to be a streamline, we need $k = 0$. Let's express $\psi = g(x)f(\eta)$, with $f(0) = 0$. Now plug this into the boundary Layer equations, and proceed.

5.4.4. *Computing Drag.* Now that we have the correct structure near the flat plate, we can return to the strange problems we were having with the drag. The stress on the plate is just

$$t_x = \sigma \cdot (0, 1, 0) = \mu(\partial_y u + \partial_x v)_{y=0} = \mu\partial_y u = \mu U(U/(2\nu x))^{1/2} f''(0).$$

Hence, the total drag on the plate is $2\int t_x = 2\sqrt{2}f''(0)\rho U^2 L/\sqrt{R}$.

Note two features: (a) The drag scales with $\sqrt{L}$, NOT L as you might have guessed. (b) as $\nu \to 0$, the drag is supposed to vanish.

5.5. **A Low Reynolds number Example: Lubrication Theory.** Let's consider a droplet spreading on a solid surface. Assume the liquid is very viscous and that it therefore spreads very slowly. We want to find out how it moves.

In this limit, the fluid obeys the Stokes equations, namely

$$\eta \nabla^2 \mathbf{u} = \nabla p, \tag{58}$$

together with $\nabla \cdot \mathbf{u} = 0$. Lets parameterize the droplet shape by a function $h(x,t)$. Typically in such problems, the thickness of the droplet is much smaller than its extent–ie $h \ll R$, where R is the radius of the droplet. In such a case, we can use an approximation, similar to that of Prandtl, to derive a tractable equation of motion. Indeed, we can use it to derive the equations for droplets as they spread.

Here it goes, as a sketch. Lets call the components of the velocity field (u, v), respectively. The continuity condition then becomes

$$\partial_x u + \partial_y v. \tag{59}$$

If we call the typical horizontal velocity U and the typical vertical velocity V, this equation then implies that

$$V \sim \frac{h}{R} U,$$

so that vertical velocities are smaller than horizontal velocities by a factor h/R.

Now lets write out the two components of the Stokes equations:

$$\eta(\partial_{xx} + \partial_{yy})u = \partial_x p, \tag{60}$$

and

$$\eta(\partial_{xx} + \partial_{yy})v = \partial_y p. \tag{61}$$

Now since $v \ll u$, the second equation implies that $\partial_y p \ll \partial_x p$. We therefore can assume that at first approximation, the pressure in the droplet $p \approx p(x)$; ie , it does not depend on the vertical (y) direction.

Now lets turn to (**??**): because of the scale separation, the $\partial_{xx} \ll \partial_{yy}$. Thus we have that

$$\eta \partial_{yy} u = \frac{dp}{dx}. \tag{62}$$

Since dp/dx is only dependent on x we can integrate the equation twice with respect to y, to arrive at

$$u = \frac{1}{\eta} \frac{dp}{dx} \left(\frac{y^2}{2} + Ay + B \right). \tag{63}$$

Here A and B are constants. We can determine then using boundary conditions. No slip at the solid surface ($y = 0$) implies that $B = 0$. If we require there is no tangential stress at the top surface $y = h$, this implies that $\partial_y u(y = h) = 0$, which means that $A = -h$. We thus have

$$u = \frac{1}{\eta} \frac{dp}{dx} \left(\frac{y^2}{2} - hy \right). \tag{64}$$

Now to proceed and arrive at an equation, we compute the total flux in the x direction, given by

(65) $$Q = \int_0^h dy\ u = -\frac{h^3}{12\eta}\frac{dp}{dx}.$$

Mass conservation

(66) $$\partial_t h = -\partial_x Q$$

then implies the equation of motion

(67) $$\partial_t h = \partial_x\left(\frac{h^3}{12\eta}\frac{dp}{dx}\right).$$

Other remarks:

(1) What sets the pressure? For droplets this is generally a combination of surface tension and gravity. For surface tension, there is a boundary condition across the fluid interface that says the pressure jump is the product of the surface tension and mean curvature. This gives that $p = \gamma\partial_{xx}h$.
(2) Comment about gravity
(3) Tanner's Law

School of Engineering and Applied Sciences, Harvard University