Four Lectures on Non-Equilibrium Statistical Physics

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Contents

Chapter 4

COMPLEX NETWORKS

Complex networks provide a rich playground for non-equilibrium statistical physics. It might seem surprising, at first glance, that the tools of this field could describe geometrical properties of such networks. However, by viewing a static network as a single snapshot of a dynamically evolving network, we can determine many geometrical properties from a master equation approach. Our perspective here qualitatively parallels that in chapter 2 on irreversible adsorption, where the hard-to-calculate final coverage of a surface — an ostensibly static quantity — can be obtained for free by solving, in a technically simpler way, for the full time-dependent coverage. We use such a kinetic approach to understand many properties of complex networks.

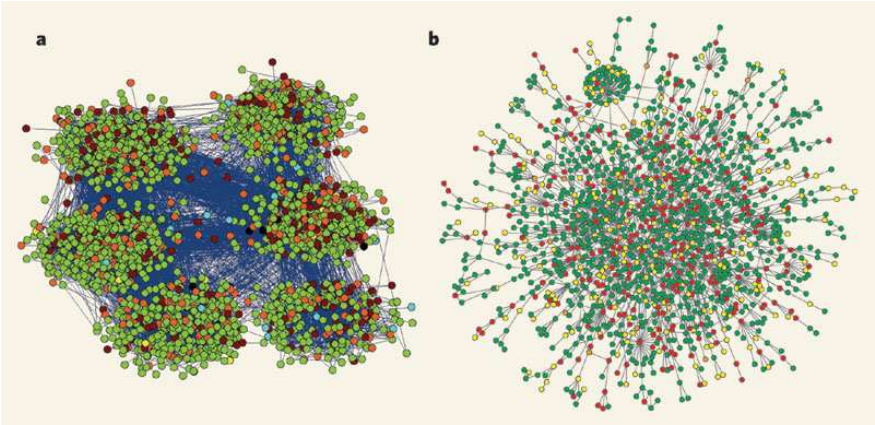

Figure 4.1: Examples of complex networks.

4.1 Erdös-Rényi Random Graph

We start by studying the classic *Erdös-Rényi (ER) random graph* (Fig. 4.2). There are two generic, and closely related, forms of this model. In the probabilistic version, the graph consists of N nodes, in which the probability that each node pair is connected by a link is $\frac{p}{N}$, with $0 \leq p \leq N$. The factor $\frac{1}{N}$ is conventionally included because it leads to a percolation transition at $p = 1$ (see below). Since each node has $N - 1$ neighbors, the average number of links attached to a node (also known as the *node degree*) is $p(N-1)/N$, which approaches p as $N \to \infty$. In contrast to regular lattices, the degree (or coordination number) of each node is not the same on the ER graph (and indeed for any complex network). Thus the *degree distribution* provides a new characterization of a complex network.

An alternative, but equivalent, model is to randomly connect the $N(N-1)/2$ pairs of nodes using a fixed

number of links[1] L. Consequently, the fraction of connected links is $2L/[N(N-1)]$. Equating this fraction with $\frac{p}{N}$, the fixed-link rule is essentially the same, in the limit $N, L \to \infty$, as the probabilistic ER random graph with $p = \frac{2L}{N}$.

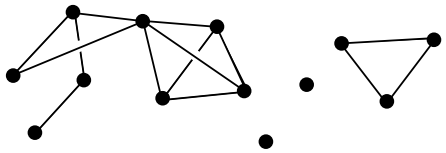

Figure 4.2: A realization of the ER graph of 13 nodes and 14 links. The network is partitioned into four clusters: one of size 8, one of size 3, and two of size 1. There are also one node of degree 5, four of degree 3, four of degree 2, one of degree 1, and two of degree 0.

While the ER random graph construction is exceedingly simple, its geometric properties are amazingly rich. Perhaps the most striking feature is the existence of a percolation transition, as p passes through a critical value $p_c = 1$, in which the nature of the cluster-size distribution change dramatically. A cluster is defined as the maximal set of nodes that are connected by links, so that the graph consists of the union of disjoint clusters (Fig. 4.2). For $p < p_c$, clusters are small and tree-like. For $p > p_c$, there exists a "giant" cluster that consists of a non-zero fraction of all the nodes. At $p = p_c$, the cluster size distribution decays algebraically with size. Many of these properties have been elucidated using probabilistic approaches. In this section, we present an alternative, kinetic approach for unraveling the structure of the ER graph.

Kinetic formulation

We recast the ER graph kinetically by starting with N isolated nodes and introducing links one by one between randomly-selected node pairs. The two nodes could be the same and also more than one link may be created between a pair of nodes. However, these two processes occur with a vanishingly small probability, when $N \to \infty$, and may be ignored. For convenience, we define the rate at which each link is introduced as $\frac{N}{2}$. Consequently, the total number of links at time t is $Nt/2$, so that the average degree $2L/N$ equals t. Thus the average degree evolves by a stochastic process in which the average degree $k \to k+1$ at rate 1.

Let's first determine the degree distribution of the ER graph. It is convenient to work with the normalized degree distribution n_k, defined as the fraction of nodes of degree k. Nodes of degree k are created at rate 1 by introducing a link that attaches to a node of degree $k-1$; similarly nodes of degree k are lost at rate 1 by linking to create nodes of degree $k+1$. The degree distribution therefore satisfies the master equation of the Poisson process

$$\frac{dn_k}{dt} = n_{k-1} - n_k \,, \tag{4.1}$$

which applies for all $k \geq 0$ if we impose the additional condition $n_{-1} \equiv 0$. For the initial condition of N isolated nodes, $n_k(0) = \delta_{k,0}$, the solution to this master equation gives the Poisson distribution

$$n_k = \frac{t^k}{k!}\, e^{-kt} \ . \tag{4.2}$$

Thus the mean degree $\langle k \rangle = t$, while the standard deviation is $\sqrt{\langle k^2 \rangle - \langle k \rangle^2} = \sqrt{t}$.

We now investigate the time evolution of the cluster size distribution, from which we can probe the percolation transition of the ER graph. Initially the network consists of N isolated single-site clusters. As links are added, clusters can only merge, so that the number of clusters systematically decreases and their mean size grows. With the addition of a single link, which requires a time $2/N$, the probability that two disconnected clusters of sizes i and j join to create a cluster of size $k = i + j$ equals $(i\, N_i/N) \times (j\, N_j/N)$;

[1] In this formulation, it is possible that a node pair could be joined by more than one link; however, the probability of this event is negligible for $N \to \infty$

here N_j is the number of clusters of size j. Following the same reasoning that led to Eq. (1.20) in section 1.1, the master equation for $c_k(t) = N_k(t)/N$, the density of clusters with k nodes at time t, is

$$\frac{dc_k}{dt} = \frac{1}{2}\sum_{i+j=k} (ic_i)(jc_j) - k\,c_k. \tag{4.3}$$

Since the network starts with all nodes isolated, the initial condition is $c_k(0) = \delta_{k,1}$. As given in Eq. (1.30), the cluster size distribution is

$$c_k(t) = \frac{k^{k-2}}{k!}\,t^{k-1}\,e^{-kt}\ . \tag{4.4}$$

We now exploit this basic result, together with various relevant results of section1.1 to determine a variety of geometrical features for the ER graph.

The percolation transition

The first basic consequence of Eq. (4.4) is that there is a transition as t passes through 1. Applying Stirling's approximation to this equation, the asymptotic behavior of c_k is given by (see also Eq. (1.33)) is:

$$c_k(t) \simeq \frac{1}{\sqrt{2\pi}\,k^{5/2}}\,e^{-k(t-\ln t-1)} \simeq \frac{1}{\sqrt{2\pi}\,k^{5/2}}\,e^{-k(1-t)^2/2} \qquad t\to 1. \tag{4.5}$$

For $t < 1$, the cluster size distribution c_k decays exponentially with k, so that the mean cluster size is finite. Normally, the mean size is defined as $\sum k\,c_k$. By construction, however, this sum equals 1, so that the appropriate measure of the mean cluster size is the second moment, $M_2 \equiv \sum_k k^2 c_k$. At $t = 1$, the cluster size distribution has a power-law tail with exponent $-5/2$, and this implies that the mean cluster size diverges in an infinite network. Therefore a percolation transition occurs at $t = 1$.

For an infinite ER graph, the singular behavior near the percolation transition is best appreciated by studying the first few moments of the cluster size distribution, $M_n \equiv \sum_k k^n c_k$ (see page 8). It is important to note that the sum does not include the contribution of the infinite cluster, whenever it happens to exist. For the zeroth moment, we have

$$M_0 = \begin{cases} 1 - t/2 & t \le 1; \\ 1 - t/2 + 2(t-1)^3/3 + \ldots & t \downarrow 1; \\ e^{-t} + (t/2)\,e^{-2t} + \ldots & t\to\infty. \end{cases} \tag{4.6}$$

Thus there are of the order of N finite clusters below and at the percolation threshold, and this number goes to zero as the infinite cluster engulfs the entire network. For the first moment, it is more useful to instead consider the *gel fraction* $g \equiv 1 - M_1$, namely, the fraction of nodes that are part of the infinite cluster. The evolution for the gel fraction is implicitly determined by $g = 1 - e^{-gt}$ (Eq. (1.35)), and the limiting behaviors of this gel fraction are given by

$$g = \begin{cases} 0 & \text{for} \quad t < 1 \\ 2(t-1) - 8(t-1)^2/3 + \ldots & t \downarrow 1 \\ 1 - e^{-t} - te^{-2t} + \ldots & t\to\infty. \end{cases} \tag{4.7}$$

Finally, for the initial condition $M_2(0) = 1$, the second moment is given by

$$M_2(t) = \begin{cases} (1-t)^{-1} & t < 1; \\ (e^{gt} - t)^{-1} & t > 1. \end{cases} \tag{4.8}$$

Thus an infinite cluster forms at $t = 1$ on an infinite ER graph. As t increases far beyond the percolation point, the mean size of the remaining finite clusters goes to zero.

Since one normally studies a large, but finite ER graph, finite-size scaling is an integral part of the description of the percolation transition. To appreciate the role of finite-size effects, let's first determine the size M^* of the largest cluster on a finite graph. This size may be estimated by the extremal criterion

$$N\sum_{k\ge M^*} c_k = 1, \tag{4.9}$$

which states that there will be a single cluster whose size is in the range $[M^*, \infty]$. Using the asymptotic forms in Eqs. (4.5) for $c_k(t)$ and approximating the above sum by an integral, the extremal criterion gives

$$M^* \sim \begin{cases} \dfrac{\ln N}{(1-t)^2} & t < 1; \\ N^{2/3} & t = 1; \\ gN & t > 1. \end{cases} \tag{4.10}$$

For a finite network, the sum for the mean cluster size should be cut off at the largest cluster size of M^*, so that

$$M_2 \approx \sum^{M^*} k^2 c_k \sim \int^{M^*} k^2 \; k^{-5/2}\, dk \sim \sqrt{M^*} \sim N^{1/3}.$$

For a finite ER graph, the mean cluster size $M_2 = (1-t)^{-1}$ will be truncated when it becomes of the order of $N^{1/3}$. Consequently, the sharp percolation point becomes smeared into a critical regime that is defined by the inequality $|1-t| \ll N^{-1/3}$

Geometrical properties

We can understand many geometrical properties of a large ER graph by exploiting the logical consequences of the kinetic formulation. First consider the regime below the percolation threshold. Since there are of the order of N small clusters for $t < 1$ (Eq. (4.6)), a newly-introduced link will typically appear between two disjoint clusters rather than between two nodes in the same cluster. Consequently, clusters are predominantly trees. Generally, clusters can be categorized as *trees, unicyclic*, or *complex*. A unicyclic cluster contains a single closed path, while a complex cluster contains at least two independent closed paths. Below the percolation threshold, there is typically a single unicyclic cluster in the entire graph and no complex clusters.

Above the percolation threshold, the size of the largest cluster equals gN, but the size of the second largest cluster is governed by the same exponential decay in c_k as in the largest cluster below percolation. Thus the size of this second-largest cluster scales as $\ln N/(t-1)^2$. When the average node degree p is larger than, but of the order of one, the structure of the ER graph above percolation is roughly that of a branching tree. We can exploit this picture to give a cheap estimate of the diameter of the network. For $p \gtrsim 1$, then an arbitrary node is connected, on average, to p first-generation nodes; each first-generation node is connected to $p-1$ second-generation nodes, *etc.*, and each $(k-1)^{\rm st}$-generation node is connected to $p-1$ nodes in generation k. Because $\frac{p}{N} \ll 1$, the probability of forming closed loops is of the order of $\frac{1}{N}$ in early generations of this branching tree. Thus the network diameter increases by one each time a new generation is added. However, when the total number of nodes $p(p-1)^k$ in a tree of k generations becomes of the order of N, then additional generations must lead to closed loops and the diameter saturates at this point. Thus the diameter of the ER graph is of the order of $\ln N/\ln p$. This logarithmic dependence of diameter on N is a generic feature of many complex networks.

Above the percolation threshold, the ER graph possesses some intriguing features that cannot occur for percolation on a regular lattice because of the possibility of long-range connections. Perhaps the most striking is the existence of a threshold for the disappearance of all finite-size clusters. This transition can be inferred directly from the expression for c_k in Eq. (4.4). The number of monomers N_1 is given by $N_1 = Ne^{-t}$. Thus there is of the order of a single monomer remaining when $t = \ln N$. In the same spirit, the number of k-mers $N_k = Nc_k$ first increases but later reaches 1 when $t \simeq \frac{1}{k}\ln N$. Thus there exists a series of transitions times, given by $t_k \simeq \frac{1}{k}\ln N$, where the last k-mer disappears.

4.2 Random Recursive Tree (RRT)

A prototypical and illustrative example of a growing network is the *random recursive tree* (RRT). In this network, nodes are added one by one, and each new node attaches to a single "target" node with an attachment rate A_k that is the same for every target. By the restriction to a single point of attachment, the

resulting network has a tree structure[2], a feature that simplifies analysis of the system. The growth rules of the RRT thus are:

1. Pick one of the nodes of the RRT — the target — with uniform probability.
2. Introduce a new node that links to the target node.

Starting with an initial state that consists of a single node, these steps are repeated until the tree reaches a desired number of nodes N. Since each newly-introduced node has a single link, closed loops cannot be generated.

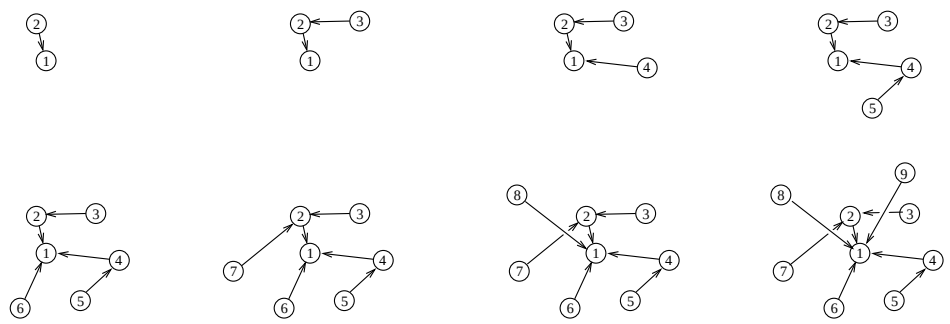

Figure 4.3: Evolution of a representative random recursive tree.

An appealing aspect of the RRT is that many of its geometrical properties are readily soluble. Moreover, there are many physically-motivated and simple ways to generalize the RRT that illustrate a variety of rich structural properties of networks.

The degree distribution

A basic characterization of the RRT is its degree distribution. We first outline the steps needed to derive the exact degree distribution and then turn to a more complete analysis of the degree distribution in the limit of large N. The degree state of any network may be generally characterized by $\mathbf{N} \equiv \{N_1, N_2, \ldots\}$, where N_k denotes the number of nodes of degree k. For the case of the RRT, when a new node is introduced, the network state $\mathbf{N}$ evolves by:

$$\begin{aligned} &\text{attach to node of degree } 1\text{:} && (N_1, N_2) \to (N_1, N_2 + 1) \\ &\text{attach to node of degree } k > 1\text{:} && (N_1, N_k, N_{k+1}) \to (N_1 + 1, N_k - 1, N_{k+1} + 1). \end{aligned} \tag{4.11}$$

While one can in principle write the master equation for the probability distribution $P(\mathbf{N})$, this equation provides "too much" information. Typically we are interested in the average number of nodes of a given degree, $\langle N_k \rangle$ (the degree distribution), or perhaps the two-body correlations functions $\langle N_i\, N_j \rangle$. Here the angle brackets denote an average over all possible growth histories of the network. As we now discuss, the master equation approach is ideally-suited to determine such quantities.

As suggested by Eq. (4.11), we need to separately consider nodes of degree 1 and nodes of degree greater than 1. The number of nodes of degree 1, $N_1(N)$, is a random variable that changes according to

$$N_1(N+1) = \begin{cases} N_1(N) & \text{prob.} \quad \frac{N_1}{N} \\ N_1(N) + 1 & \text{prob.} \quad 1 - \frac{N_1}{N} \end{cases} \tag{4.12}$$

after the addition of each new node. Namely, with probability N_1/N, the new node attaches to a node of degree 1 and the number of such nodes does not change. Conversely, with probability $(1 - N_1/N)$, the new

[2]If a new node attaches to more than one pre-existing node, closed loops can form. The degree distribution of such a network is modified only slightly compared to growing trees, but other features, such as the network diameter, are strongly influenced by the existence of loops.

node attaches to a node of degree $k > 1$ and N_1 thus increases by 1. The resulting evolution equation for the average number of nodes of degree 1 is therefore given by

$$\begin{aligned}\langle N_1(N+1)\rangle &= \left\langle N_1(N) \times \frac{N_1(N)}{N}\right\rangle + \left\langle (N_1(N)+1) \times \left(1-\frac{N_1}{N}\right)\right\rangle \\ &= 1 + \left(1-\frac{1}{N}\right)\langle N_1(N)\rangle. \end{aligned} \tag{4.13}$$

By similar reasoning, the number of nodes of degree $k \geq 2$, $N_k \equiv N_k(N)$ evolves according to

$$N_k(N+1) = \begin{cases} N_k - 1 & \text{prob. } \frac{N_k}{N} \\ N_k + 1 & \text{prob. } \frac{N_{k-1}}{N} \\ N_k & \text{prob. } 1 - \frac{N_{k-1}+N_k}{N} \end{cases} \tag{4.14}$$

after each node addition. Following the same states that led to Eq. (4.13), N_k evolves by

$$\langle N_k(N+1)\rangle = \langle N_k(N)\rangle + \left\langle \frac{N_{k-1}(N) - N_k(N)}{N}\right\rangle. \tag{4.15}$$

Equations (4.13) and (4.15) provide the basis for computing the exact degree distribution.

In what follows, we restrict ourselves to the much simpler leading $N \to \infty$ behavior of the degree distribution. To minimize notation, we drop the angle brackets and write N_k for the average number of nodes of degree k in a network that consists of N total nodes. Now we replace the discrete differences with derivatives in Eqs. (4.13) and (4.15), so that the asymptotic degree distribution evolves according to

$$\frac{dN_k}{dN} = \frac{N_{k-1} - N_k}{N} + \delta_{k1}\ . \tag{4.16}$$

This master equation similar to that for the ER graph, Eq. (4.1), except for the additional delta-function term that accounts for the introduction of the new node of degree 1.

To get a feeling for the solution, let's solve the master equations (4.16) one by one. With the understanding that $N_{-1} = 0$, the master equations are:

$$\begin{aligned} \frac{dN_0}{dN} &= -\frac{N_0}{N} \\ \frac{dN_1}{dN} &= \frac{N_0 - N_1}{N} + 1 \\ \frac{dN_2}{dN} &= \frac{N_1 - N_2}{N} \\ \frac{dN_3}{dN} &= \frac{N_2 - N_3}{N}, \end{aligned}$$

etc. The solution to the first equation is $N_0 = 1/N$. Substituting this result into the equation for N_1 and solving gives the asymptotic solution $N_1 \sim N/2$. Following this same approach, $N_2 \sim N/4$ and, in fact, all the N_k are proportional to N. It therefore is convenient to work with the density of nodes of degree k, $n_k \equiv N_k/N$, in terms of which Eq. (4.16) reduces to

$$n_k = n_{k-1} - n_k + \delta_{k1}. \tag{4.17}$$

Starting with $n_0 = 0$, we obtain $n_1 = \frac{1}{2}$, $n_2 = \frac{1}{4}$, *etc.*, and the general solution is $n_k = 2^{-k}$. From this distribution, The average degree $\langle k\rangle = 2$, while the largest degree $k_{\max}$, which follows from the extremal criterion $\sum_{k_{\max}}^{\infty} N\, 2^{-k} = 1$, gives $k_{\max} \simeq \ln N/\ln 2$. Qualitatively, the degree distribution of the RRT is rapidly decaying with k and even the largest degree is, in some sense, small.

Redirection

We now generalize the RRT to incorporate *redirection*. One motivation for this construction is that redirection is leads to linear preferential attachment, a fundamental growth mechanism for complex networks that will be discussed in detail in the next section. In redirection, the network is built by repeatedly applying the following steps (Fig. 4.4):

1. Pick a pre-existing node **x** from the network uniformly at random.
2. With probability $1-r$, the new node **n** attaches to **x**,
3. With probability r, node **n** attaches to an *ancestor* node **y** of node **x**.[3]

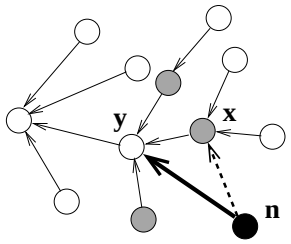

Figure 4.4: Illustration of redirection. The rate at which attachment occurs to the ancestor **y** of node **x** by redirection is proportional to the number of upstream neighbors to **y** (shaded).

According to redirection, the degree distribution now evolves according to

$$\begin{aligned}\frac{dN_k}{dN} &= \frac{1-r}{N}\left[N_{k-1}-N_k\right]+\delta_{k1}+\frac{r}{N}\left[(k-2)N_{k-1}-(k-1)N_k\right]\\ &= \frac{r}{N}\left\{\left[k-1+\left(\frac{1}{r}-2\right)\right]N_{k-1}-\left[k+\left(\frac{1}{r}-2\right)\right]N_k\right\}+\delta_{k1},\end{aligned} \tag{4.18}$$

where the second line is obtained by trivial rearrangement of terms. In the first line, the first three terms correspond to the RRT, whose master equation (4.16) is recovered for redirection probability $r=0$. The last two terms account for the change in N_k due to redirection. To understand their origin, consider the gain term. Since the initial node is chosen uniformly, if redirection does occur, then the probability that a node of degree $k-1$ receives the newly-redirected link is proportional to the number of its upstream neighbors (shaded nodes in Fig. 4.4), which equals $k-2$. A similar argument applies for the redirection-driven loss term. Thus uniform attachment, in conjunction with redirection, generates *linear preferential attachment* because rate at which a node attaches to a node of degree k is a linear function of k, $A_k = k+(\frac{1}{r}-2)$. We will study preferential attachment networks systematically in the next section, from which the degree distribution in redirection follows easily.

Genealogical tree and the diameter

A revealing feature of the RRT is its underlying genealogical structure. We define the initial node as being in generation $g=0$. Nodes that attach to those in generation g form generation $g+1$, irrespective of when attachment occurs. For example, in the final network of Fig. 4.2 (reproduced below left), node 1 is the ancestor of 2, while nodes 3 and 7 are the descendants of 2. There are 5 nodes in generation $g=1$ and 3 in $g=2$, leading to the genealogy on the right of Fig. 4.5.

How many generations are in a tree of N nodes? What is the size $L_g(N)$ of the g^{th} generation? To answer these questions, note that $L_g(N)$ increases by 1 when a new node attaches to a node in generation $g-1$. For uniform attachment, this event occurs with probability L_{g-1}/N. Consequently L_g evolves as

$$\frac{dL_g(N)}{dN} = \frac{L_{g-1}}{N}\ , \tag{4.19}$$

with solution $L_g(\tau)=\tau^g/g!$, where $\tau=\ln N$. Using Stirling's approximation, $L_g(N)$ initially grows with g for $g<\tau$, and then decreases and becomes of order 1 when $g=e\,\tau$. The genealogical tree therefore contains $e\,\tau$ generations for a tree of N nodes. Since the diameter is twice the distance from the root to the last generation (also the maximum distance between any pair of nodes), the diameter scales as $2e\,\tau\approx 2e\,\ln N$.

[3]There is a technical subtlety because redirection requires that every node has an ancestor. To ensure this condition always holds, the initial state should consist of at least two nodes and one link, with each node defined as the ancestor of the other.

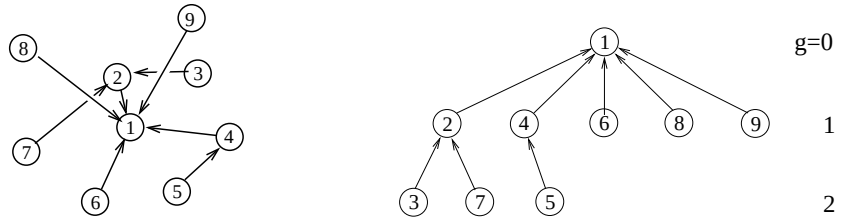

Figure 4.5: A random recursive tree of 9 nodes (from Fig. 4.2) and its corresponding genealogical tree.

The genealogical tree construction can also be applied to the RRT with redirection. In this case, there are two separate mechanisms by which the size of the g^{th} generation can change. One way is for the new node to attach to one of the nodes in generation $g-1$. On the other hand, L_g can grow when the new node provisionally attaches to one of the nodes in generation g and the attachment point is redirected to generation $g-1$. With these two mechanisms, the evolution equation for L_g is now

$$\frac{dL_g(N)}{dN} = \frac{(1-r)L_{g-1} + rL_g}{N} \ , \tag{4.20}$$

where the factors $1-r$ and r respectively account for attachment without and with redirection. The solution to this equation is best obtained by the Laplace transform method. A basic main feature of this solution is that the network diameter is again of order $\ln N$ times a constant that is of the order of 1. Thus the genealogy of the RRT is robust as it is not qualitatively affected by redirection.

4.3 Preferential Attachment Networks

Master equation

In *preferential attachment*, the rate A_k at which the new node attaches to a pre-existing node of degree k is an increasing function of k. Such a rule encapsulates the notion of the "rich get richer" in which being advantaged now confers the benefit of gaining future advantage at a higher rate. As an example in the context of scientific citations, preferential attachment corresponds to a currently well-cited paper continuing to be well cited in the future merely be virtue of having been well cited. A ubiquitous feature of preferential attachment networks is that the degree distribution has a broad tail. This discovery was initially surprising because traditional network models, such as the ER random graph discussed previously, have a much steeper Poisson degree distribution. Networks with a broad-tailed degree distribution are therefore far from random.

To solve for the degree distribution of such networks, we need to specify the attachment rate A_k and we focus on the general case of $A_k = k^\gamma$, with $\gamma \geq 0$. The master equation for the degree distribution for an arbitrary attachment rate is (compare with Eq. (4.16) for the RRT):

$$\frac{dN_k}{dN} = \frac{A_{k-1}N_{k-1} - A_kN_k}{A} + \delta_{k1}. \tag{4.21}$$

The first term on the right accounts for the new node connecting to a node that already has $k-1$ links, thereby increasing N_k by one. Since there are N_{k-1} nodes of degree $k-1$, the total rate at which such processes occur equals to $A_{k-1}N_{k-1}$. The factor $A_{k-1}N_{k-1}/A$, where $A(N) \equiv \sum_{j\geq 1} A_jN_j$ is the total rate for any event, is then the probability that the new node attaches to a node of degree $k-1$. A corresponding role is played by the second term on the right-hand side. The overall amplitude of A_k is immaterial, since only the ratio A_k/A appears in the master equation. The last term accounts for the new node that has one outgoing link and no incoming links. As we shall see, fundamentally different behaviors arise for sublinear ($\gamma < 1$), superlinear ($\gamma > 1$), and linear ($\gamma = 1$) attachment rates.

Moments and the degree distribution

As a first step to solve for the degree distribution, it is instructive to study the moments $M_\alpha(N) \equiv \sum_j j^\alpha N_j$. The zeroth and first moments of this distribution have a particularly simple evolution with N:

$$\frac{dM_0}{dN} = \sum_j \frac{dN_j}{dN} = 1; \qquad \frac{dM_1}{dN} = \sum_j j\frac{dN_j}{dN} = 2. \tag{4.22}$$

The equation for M_0 states that the total number of nodes of any degree increases by 1 each time a new node is introduced. Similarly, the addition of a single link increases the total degree of the network, $\sum jN_j$, by 2. Thus both the zeroth and first moments of the degree distribution increase linearly with N. For attachment rate $A_k = k^\gamma$ with $0 \le \gamma \le 1$, the total rate $A = \sum_j j^\gamma N_j$ also grows linearly with N, because A is intermediate to the zeroth and first moments. Thus we write $A \sim \mu N$, with μ an as yet undetermined amplitude that varies smoothly between 1 and 2 as γ increases from 0 to 1.

Solving for the first few N_k from Eq. (4.21), it becomes clear that each N_k is also proportional to N. Thus substituting $N_k(N) = n_k N$ and $A = \mu N$ into the master equations, the overall N dependence cancels, leaving behind the recursion relations

$$n_k = \frac{A_{k-1}n_{k-1} - A_k n_k}{\mu} \quad k > 1, \qquad \text{and} \quad n_1 = -\frac{A_1 n_1}{\mu} + 1\,,$$

whose formal solution is

$$n_k = \frac{\mu}{A_k}\prod_{j=1}^{k}\left(1 + \frac{\mu}{A_j}\right)^{-1}. \tag{4.23}$$

To make this solution explicit, we need the amplitude μ in $A(N) = \mu N$. Using the definition $\mu = \sum_{j\ge 1} A_j N_j$ in Eq. (4.23), gives the condition

$$\sum_{k=1}^{\infty}\prod_{j=1}^{k}\left(1 + \frac{\mu}{A_j}\right)^{-1} = 1. \tag{4.24}$$

Thus the amplitude μ depends, in general, on the functional form of the attachment rate. Equations (4.23) and (4.24) represent the formal solution for the degree distribution of preferential attachment networks. To extract the physical meaning of this solution, we examine its asymptotic behavior for different values of the exponent γ in the attachment rate.

`Sublinear attachment rate:` For $A_K = k^\gamma$ with $\gamma < 1$, we rewrite the product in Eq. (4.23) as the exponential of a sum, convert the sum to an integral, and then expand the logarithm inside the integral in a Taylor series. These straightforward steps lead to

$$n_k \sim \begin{cases} k^{-\gamma}\exp\left[-\mu\left(\frac{k^{1-\gamma}-2^{1-\gamma}}{1-\gamma}\right)\right] & \frac{1}{2} < \gamma < 1, \\ k^{(\mu^2-1)/2}\exp\left[-2\mu\sqrt{k}\right] & \gamma = \frac{1}{2}, \\ k^{-\gamma}\exp\left[-\mu\,\frac{k^{1-\gamma}}{1-\gamma} + \frac{\mu^2}{2}\,\frac{k^{1-2\gamma}}{1-2\gamma}\right] & \frac{1}{3} < \gamma < \frac{1}{2}, \end{cases} \tag{4.25}$$

etc. The leading behavior is a universal stretched exponential decay, $\exp(-\text{const.} \times k^{1-\gamma})$ that is modified by subdominant corrections whose form changes whenever γ decreases below $1/m$, with m a positive integer.

`Superlinear attachment rate:` For $\gamma > 1$, an analog of gelation occurs in which nearly all links condense onto a single node. Ultra singular behavior occurs for $\gamma > 2$ in which there is a non-zero probability of a "bible" — a single node that links to *every other* node in an infinite network, while only a finite number of links exist between all other nodes. Suppose that there is a bible in a network of $N + 1$ nodes (Fig. 4.6). The probability that the next node links to the bible is then $N^\gamma/(N + N^\gamma)$, and the probability that this pattern of connections continues indefinitely is $\mathcal{P} = \prod_{N\ge 1}(1 + N^{1-\gamma})^{-1}$. The asymptotic behavior of this product is $\mathcal{P} = 0$ for $\gamma \le 2$ and $\mathcal{P} > 0$ for $\gamma > 2$. Thus for $\gamma > 2$, there a non-zero probability for a bible

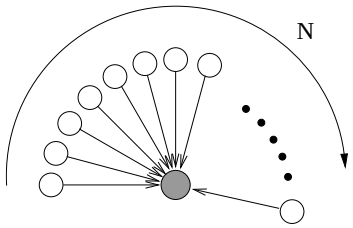

Figure 4.6: Creation of a "bible" in which each new node attaches only to the bible (shaded).

to exist in an infinite network. When $1 < \gamma < 2$, the attachment pattern of low-degree nodes is less trivial than in Fig. 4.6, but there continues to exist a single node that is linked to nearly all nodes.

`Linear attachment rate:` We distinguish between *strictly linear* attachment, $A_k = k$, and *asymptotically linear* attachment, $A_k \simeq k$. In the former case, the total event rate is $A = \sum_k A_k N_k = \sum_k k N_k = 2N$. Substituting this value of $\mu = 2$ in Eq. (4.23) immediately leads to the discrete power-law form

$$n_k = \frac{4}{k(k+1)(k+2)} = \frac{4\Gamma(k)}{\Gamma(k+3)} \sim \frac{4}{k^3} \ , \tag{4.26}$$

where Γ is the Euler gamma function. Because this distribution has no characteristic degree scale, such networks have been dubbed *scale free*, and they stand in stark contrast to the delta-function degree distribution of regular lattices and the Poisson degree distribution of the Erdös-Rényi random graph.

For asymptotically linear attachment, the surprising feature is that the degree distribution exponent is *non-universal*. This non-universality is counter to statistical-physics dogma that the absence of a characteristic scale at a phase transition leads to universal power-law scaling laws. The non-universal behavior for the degree distribution can be easily derived for asymptotically linear attachment rates, $A_k \sim k$. Then Eq. (4.23) becomes

$$\begin{aligned} n_k &= \frac{\mu}{A_k} \prod_{j=1}^{k} \left(1 + \frac{\mu}{A_j}\right)^{-1} \\ &\sim \frac{\mu}{k} \exp\left[-\int_1^k \ln\left(1 + \frac{\mu}{j}\right)\right] dj \\ &\sim \mu\, k^{-(1+\mu)} \ . \end{aligned} \tag{4.27}$$

Thus the degree distribution exponent can take *any* value larger than 2 merely by tuning the amplitude in the total rate $A = \mu N$.