Boulder Summer School 2015: elasticity, marginally stable solids, and topological phonons

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I. OVERVIEW: ELASTIC AND VIBRATIONAL PROPERTIES OF MAXWELL LATTICES

These lectures will treat a particular class of marginally stable lattices. In 1864, James Clerk Maxwell wrote a paper entitled "On the calculation of the equilibrium properties of frames" in which he outline the conditions under which frames (lattices) of rigid rods (connections) connected at sites (points) by frictionless pins are mechanically stable. He was interested in the number, N_0, of independent distortions of the frame that do not change the length of any of rods. These are the zero modes of the frame. He reasoned that each point in d dimensions has d translations degrees of freedom that cost no energy if there are no bonds. Each bond introduces one constraint and reduces the number of zero modes by one. Thus, if there are N_B bonds and N sites, there are

$$N_0 = dN - N_B \tag{1.1}$$

zero modes so long as $dN - N_B$ is greater than or equal to the number of trivial zero modes of rigid translation and rotation [$d(d+1)/2$ for free frames and d for frames under periodic boundary conditions]. Equation (1.1). is often called the *Maxwell relation* Since each bond is shared by 2 sites, $N_B = (1/2)zN$, where z is the average coordination number or average number of neighbors per site. Thus a good estimate of the minimun condition for a lattice or frame to be stable is for $N_B - dN = \frac{1}{2}(z-2d)N > 0$. This type or reasoning has proven useful in analysis of systems as varied a bridge, glasses, as jamming, rigidity percolation.

Show figures of lattices near $z = 2d$ - Slides 1-7

The Maxwell relation ignores what are called redundant bonds or, equivalently, what are called *States of Self Stress*. These are configurations in which bonds can be put under stress while leaving the total forces as all sites equal to zero. Consider The configurations shown in Figs. 1 (b), (c), and (d), which show how a frame with six sites and either eight or nine bonds. With eight bonds, the Maxwell relation give $N_0 = 2 \times 6 - 8 = 4$ corresponding to two rigid translations, one rigid rotation, and one internal *floppy* mode that involves internal displacement of sites without changing the lengths of any bonds. In (c), there are nine bonds, and Maxwell would say there are three zero modes, as there are. In (d), however, there are also nine bonds and Maxwell would say again that there are three zero modes, but there are clearly four: the same four as (c) with only eight bonds. So the Maxwell relation appears to break down. But (d) has a special feature: the left hand square with the two diagonal bonds can exist in a state in which its for sides are under tension and its two diagonal are under compression or vice versa. This is a state of self stress. So we can modify the Maxwell relation to read,

$$N_0 - N_S = dN - N_B, \tag{1.2}$$

where N_S is the number of states of self-stress. It turns out that this relation is rigorous at least for linearized displacements. We will have a closer look at this in the second lecture.

We will call lattices in which $dN = N_B$ *Maxwell lattices*. These lattices include the familiar square, kagome, cubic, and pyrochlore and lattices under periodic boundary at the jamming transition studied numerically also under periodic boundary conditions. These periodic lattices both under periodic boundary conditions in which case $N_0 = N_S$ free boundary conditions in form cut from those under periodic boundary conditions are the primary subject of these lectures. They have a remarkably rich phenomenology with complex elastic and vibrational behavior, and their surface or edge modes with topological characterizations similar to those found in quantum systems like polyacetylene and topological insulators.

We begin with a review of classical Lagrangian elasticity, which we present in its full nonlinear form. Then in lecture II, we will introduce lattice models and formalisms for describing phonon behavior, and we look at some specific examples of Maxwell lattices with different degrees of self stress. Finally in Lecture III, we will consider topological lattices and their protect zero-energy edge modes.

Parts of lecture II and all of lecture III follow a recent review coauthored by me that will be made available to you. The relevant sections of this review are signaled in this text by RA -sec. There is a separate set of figures in Power point, labeled P.1, P.2, etc.

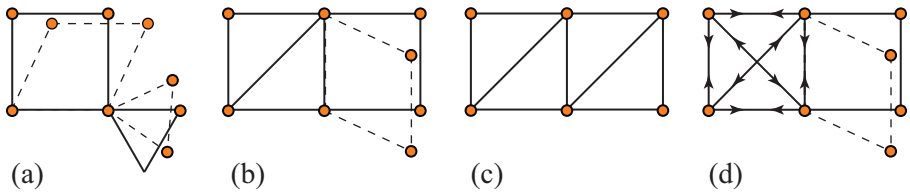

FIG. 1: (a) to (c) Frames satisfying the Maxwell rule. (a) has 6 sites, 7 bonds, 5 zero modes, and two mechanisms indicated by the dotted bonds. (b) has 6 sites, 8 bonds, 4 zero modes, and one mechanism. (c) and (d) are constructed from (b) by adding an additional diagonal bond. (c) satisfies the Maxwell rule with only the three trivial zero modes. (d) has 4 zero modes and one state of self stress indicated by the arrows on the bonds in the left square.

II. LECTURE I: NON-LINEAR ELASTICITY AND LINEAR PHONON MODES

Elastic materials resist changes of state. If they are stretched, compressed, or sheared, they will return to their original shape when external forces causing these deformations are removed. The distorted state has higher energy than the original undistorted one. Crystalline solids, glasses, and rubbers and elastomers are all elastic solids, at least at small enough shape or size distortions. Many solids will undergo irreversible plastic shape changes if deformations are large enough. Here we will consider only elastic media that return do to their original shape.

A. The deformation and nonlinear strain tensors

In the absence of external forces, an elastic material has an equilibrium preferred shape and size. In addition each mass point has a preferred position relative to other mass points. We represent the positions of these mass points by a vector $\mathbf{x}$ of dimension equal to the dimension d of the material. Thus $\mathbf{x}$ is of dimension $d = 2$ for an elastic membrane, and of dimension $d = 3$ for a $3D$ solid. The set of points $\mathbf{x} = (x_1, x_2, ...) \equiv (x, y, ...)$ constitute what we call the *reference space*. When the material is distorted, points $\mathbf{x}$ are mapped into points $\mathbf{X}(\mathbf{x})$ in what we will call the *target space* of dimension D as depicted schematically in Fig. (P8). We will restrict our attention to situations in which the dimensions of the reference and target spaces are the same, i.e., $D = d$. In an undistorted material, that is neither translated or rotated, $\mathbf{X}(\mathbf{x})$ is identical to $\mathbf{x}$. Deviations from this state are described by the *displacement vector* $\mathbf{u}(\mathbf{x})$ defined through

$$\mathbf{X}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}). \tag{2.1}$$

Under uniform translation through $\mathbf{x}_0$, $\mathbf{u}(\mathbf{x}) = \mathbf{x}_0$. Under uniform rotation, $X_i \to O_{ij}x_j$ where O_{ij} is a rotation matrix and $u_i = (O_{ij} - \delta_{ij})x_j$, where as usual, the summation convention on repeated indices is understood.

Uniform translations produce no deformation and cost no energy. Deformations thus require that $\mathbf{X}(\mathbf{x})$ have non-vanishing derivatives with respect to $\mathbf{x}$. The elastic description of materials is usually understood to deal only with slow spatial variations of $\mathbf{X}(\mathbf{x})$ and involves only the first derivatives of $\mathbf{X}(\mathbf{x})$ with respect to $\mathbf{x}$ [1] that constitute what is called the *deformation tensor*:

$$\Lambda_{ij} = \frac{\partial X_i}{\partial x_j} = \delta_{ij} + \eta_{ij}, \tag{2.2}$$

where i and j are Cartesian indices (x, y, z) and $\eta_{ij} = \partial u_i/\partial x_j$. [2] Note that there is no requirement that Λ_{ij} be symmetric; it describes both rotations and deformations. In addition, the indices i and j transform under different group operations: The right index i, associated with $\mathbf{X}$, transforms under operations in the target space, whereas the left index j, associated with $\mathbf{x}$ transforms under operations in the reference space. Thus, the j part of Λ_{ij} is invariant under the symmetry operations of the reference space, whereas the i part transforms with $\mathbf{X}$. If Λ_{ij} is spatially uniform as is often the case, $X_i(\mathbf{x}) = \Lambda_{ij}x_j$. A uniform orthogonal (thus anti-symmetric) Λ_{ij} corresponds to a pure rotations, whereas a symmetric Λ_{ij} corresponds to a deformation. A Λ_{ij} that is neither symmetric nor anti-symmetric describes a deformation followed by a rotation or vice versa. The *polar decomposition theorem* of matrix algebra says that any square real matrix can be written as a product of an orthogonal rotation matrix times a symmetric deformation matrix (or vice versa). The jacobian of the transformation form $\mathbf{x}$ to $\mathbf{X}(\mathbf{x})$ is simply the determinant of the deformation matrix. In three dimensions,

$$d^3X = \det \mathbf{\Lambda}\, d^3x, \tag{2.3}$$

where $\mathbf{\Lambda}$ is the matrix with components Λ_{ij}, and the volume (in three dimensions) in the target space is $V = \det \mathbf{\Lambda} V_0^3$, where V_0 is the volume in the reference space, and

$$\frac{\delta V}{V_0} = \det \mathbf{\Lambda} - 1 \approx \eta_{ii}, \tag{2.4}$$

where $\delta V = V - V_0$ and the final expression is valid in the small η_{ij} limit. Deformations in which $\det \Lambda = 1$ clearly preserve volume.

Special names are given to certain simple deformations. A uniform *compression* or *dilation* is described by an isotropic deformation tensor: $\Lambda_{ij} = \Lambda \delta_{ij}$, where $\Lambda < 1$ for compression and $\Lambda > 1$ for dilation. Deformations that preserve volume are called *shear*, and two versions of shear deformations are given special names; *pure shear* in which

$$\mathbf{\Lambda}_p = \begin{pmatrix} \Lambda^{-1/2} & 0 & 0 \\ 0 & \Lambda^{-1/2} & 0 \\ 0 & 0 & \Lambda^{-1/2} \end{pmatrix} \tag{2.5}$$

and simple shear in which

$$\mathbf{\Lambda}_s = \begin{pmatrix} 1 & 0 & \Lambda_{xz} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.6}$$

These two deformations are shown in Fig. (P9)

The uniform deformation matrix provides a complete description of uniform shape distortions of an elastic medium. It does not, however, provide a description of the stretching between individual mass points under distortion from $\mathbf{x}$ to $\mathbf{X}(\mathbf{x})$. It is this stretching that raises the elastic energy of the medium, just as increasing the distance between two masses connected by a spring increases the energy of the spring. The distance between two nearby points, which is determined by the metric tensor, determines the energy of a distortion. In the reference space, the distance between a point $\mathbf{x} + d\mathbf{x}$ and $\mathbf{x}$ is $dx^2 = dx_j dx_j$, and the distance between the same two points in the target space is

$$dX^2 = dX_i dX_i = \frac{\partial X_i}{\partial x_j}\frac{\partial X_i}{\partial x_k} dx_i dx_k \equiv g_{jk} dx_j dx_k, \tag{2.7}$$

where

$$g_{ij} = \Lambda_{ki}\Lambda_{kj} = \Lambda^T_{ik}\Lambda_{kj} \tag{2.8}$$

is the metric tensor tensor ($\mathbf{\Lambda}^T$ is the transpose of the deformation tensor). Note that the contraction with respect to the target-space index k, leaving only the reference-space indices i and j, implies that g_{ij} is a tensor of the reference space and is invariant under target-space rotations. This is as it should be – the distance between two target space points does not change if the vectors in that space are rigidly rotated. The metric tensor of the reference space is simple $g^R_{ij} = \delta_{ij}$. Thus a measure of changes in distance between points in the deformed target space relative to those in the undeformed reference space is provided by the *Lagrangian* nonlinear strain tensor,

$$\begin{aligned} \varepsilon_{ij} &= \tfrac{1}{2}(g_{ij} - \delta_{ij}) \equiv \tfrac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i u_k \partial_j u_k) \\ &= \tfrac{1}{2}(\eta_{ij} + \eta_{ji} + \eta_{ki}\eta_{kj}), \end{aligned} \tag{2.9}$$

which is a tensor in the reference space and which is invariant under rotations in the target space ($\partial_i f(\mathbf{x}) \equiv \partial f(\mathbf{x})/\partial x_i$ for any function $f(\mathbf{x})$). ε_{ij} is symmetric tensor, and it has $a_d = d(d+1)/2$ independent components in d dimensions. In two dimensions, the three independent components are

$$\boldsymbol{\varepsilon}_V = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}) \tag{2.10}$$

and in three dimension, the six independent components are $\boldsymbol{\varepsilon}_V = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{yz}, \varepsilon_{xz}, \varepsilon_{xy})$. [3]

In the linearized limit in which $\eta_{ij} \ll 1 = \frac{1}{2}(\eta_{ij} + \eta_{ji})$, and

$$\delta V/V_0 = \varepsilon_{ii} \equiv \mathrm{Tr}\,\boldsymbol{\varepsilon}, \tag{2.11}$$

where $\mathrm{Tr}\boldsymbol{\varepsilon}$ is the trace of the linearized Lagrangian strain matrix, often called the *Volumetric strain*. The symmetric, traceless part of of the strain tensor,

$$\tilde{\varepsilon}_{ij} = \varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}, \tag{2.12}$$

is called the *deviatoric* strain in the linearized limit. Because its trace is by definition zero, it represents deformations that do not change volume in the linearized limit, i.e., it measures strain distortions.

B. Elastic energy

The energy associated with small, slowly varying deformations of an elastic medium should be proportional to the square of local distortions integrated over all points in the lattice, i.e., it should be a quadratic form in ε_{ij}: $F[\varepsilon_{ij}] = \int d^d x\, \mathcal{F}_{\rm el}(\varepsilon_{ij})$, where

$$\mathcal{F}_{\rm el}(\varepsilon_{ij}) = \tfrac{1}{2} K_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \tag{2.13}$$

is the elastic energy density, where K_{ijkl} is the elastic tensor. In a purely mechanical, zero temperature system, $\mathcal{F}_{\rm el}$ is simply the energy density of elastic distortions; at nonzero temperature, $\mathcal{F}_{\rm el}(\varepsilon_{ij}, T)$ is the elastic Helmholtz free-energy density. Alternatively, it could be the energy density expressed as a function of ε_{ij} and entropy S. The elastic tensor is of fourth-rank, and it is invariant under the replacements $\{ij\} \to \{ji\}$ and $\{kl\} \to \{lk\}$ because ε_{ij} is symmetric and under $\{ijkl\} \to \{klij\}$ because interchanging ε_{ij} and ε_{kl} does not change the energy. Because of these symmetries, the off-diagonal components of ε_{ij} (e.g. ε_{xy}) always appear with a factor of two rather than a factor of one. Thus in two-dimensions,

$$\begin{aligned}\mathcal{F}_{\rm el}(\varepsilon_{ij}) = \tfrac{1}{2}[&K_{xxxx}\varepsilon_{xx}^2 + K_{yyyy}\varepsilon_{yy}^2 + 2K_{xxyy}\varepsilon_{xx}\varepsilon_{yy} \\ &+ 4K_{xxxy}\varepsilon_{xx}\varepsilon_{xy} + 4K_{yyxy}\varepsilon_{yy}\varepsilon_{xy} + 4K_{xyxy}\varepsilon_{xy}^2]\end{aligned} \tag{2.14}$$

A similar but longer expression applies in three dimensions. To see in more detail how this expression arises, consider first the term in $\varepsilon_{xx}\varepsilon_{xy}$. The factor of 4 arises because of K_{xxxy}, K_{xxyx}, K_{xyxx}, and K_{yxxx} are all equal to K_{xxxy} and contribute collectively a term $(1/2) \times 4K_{xxxy}\varepsilon_{yy}\varepsilon_{xy}$ to the energy density. Similarly, K_{xyxy}, K_{yxxy}, K_{xyyx} and K_{yxyx} are all equal to K_{xyxy}, and they collectively contribute a term $(1/2) \times 4K_{xyxy}\varepsilon_{xy}^2$ to the energy density.

The free energy simplifies considerably in media that have a high symmetry. The highest possible symmetry is that of an isotropic fluid, in which all tensors must be composed of the Kronecker-δ, δ_{ij}. There is no crystal with this precise symmetry, but glasses and gels, which are homogenous at the macroscopic scale but random at microscopic scales are effectively isotropic. In addition, in hexagonal lattices in two-dimensions, the lowest-rank tensor tensor not constructed entirely from Kronecker-δs is of rank 6, and the 4th-rank elastic constant tensor is isotropic. There are two 4th rank tensors with the symmetry of K_{ijkl} that can be constructed from Kronecker-δs, and the elastic constant tensor for an isotropic medium has two independent elastic constants:

$$K_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{2.15}$$

where λ and μ are the Lamé coefficients. The elastic energy density is then

$$\begin{aligned}\mathcal{F}_{\rm el} &= \tfrac{1}{2}[\lambda\varepsilon_{ii}^2 + 2\mu\varepsilon_{ij}\varepsilon_{ij}] \\ &= \tfrac{1}{2}[B\varepsilon_{ii}^2 + G\tilde{\varepsilon}_{ij}\tilde{\varepsilon}_{ij}]\end{aligned} \tag{2.16}$$

where $G \equiv \mu$ sets the energy scale of strains in the linearized limit and is called the *shear modulus* and $B = \lambda + 2(\mu/d)$ sets the energy scale for volumetric deformations an is called the *Bulk* modulus.

$\mathcal{F}_{\rm el}$ can be expressed in matrix bilinear form in terms of the independent components of the strain tensor:

$$\mathcal{F}_{\rm el} = \frac{1}{2}\boldsymbol{\varepsilon}_\alpha \mathbb{K}_{\alpha,\beta}\boldsymbol{\varepsilon}_\beta \tag{2.17}$$

where α and β run over the a_d components of the strain tensor and $\mathbb{K}_{\alpha,\beta}$ is an $a_d \times a_d$ matrix of elastic elastic constants. In two dimensions,

$$\begin{aligned}\mathbb{K} &= \begin{pmatrix} K_{xxxx} & K_{xxyy} & 2K_{xxxy} \\ K_{xxyy} & K_{yyyy} & 2K_{yyxy} \\ 2K_{xxxy} & 2K_{yyxy} & 4K_{xyxy} \end{pmatrix} \\ &\xrightarrow{\text{isotropic}} \begin{pmatrix} B+G & B-G & 0 \\ B-G & B+G & 0 \\ 0 & 0 & 4G \end{pmatrix},\end{aligned} \tag{2.18}$$

where the final form is the isotropic limit, where $B = \lambda + \mu$ is the bulk modulus and $G = \mu$ is the shear modulus with λ and μ the standard Lamé coefficients. Thermodynamic stability requires that all of the eigenvalues of of $\mathbb{K}$ be positive.

C. Stress

Stress is defined to be a force per unit area. Since both the direction of surface normal and the direction of the force relative to that normal can vary, stress described by a second-rank tensor. Physicists usually think of the *Cauchy Stress Tensor*, σ^C_{ij}, which is the force in direction i across a surface oriented along j in normal Euclidean space, which in the case of Lagrangian elasticity is the target space, e.g. σ^C_{zx} is the force per area in the z direction on a surface with normal along the X direction. The force per unit volume f_i exerted on a small volume element δD by the stresses exerted by surrounding matter on the boundary $\partial\delta D$ of δD is the divergence of the Cauchy stress tensor:

$$f_i == \frac{\partial \sigma^C_{ij}}{\partial X_j}, \tag{2.19}$$

where X_i is the position in Euclidean space. In an elastic medium, the stress tensor represents forces exerted across boundaries by matter within an elastic medium, and as a result, σ^C_{ij} is zero outside the medium. Internal forces, measured by σ^C_{ij} generate no macroscopic net force F_i or torque τ_i on a sample. The absence of net force follows from the fact σ^C_{ij} is zero outside the medium:

$$F_i = \int_D d^3X f_i = \int_D d^3X \frac{\partial \sigma^C_{ij}}{\partial X_j} = \int_{\partial D} dS_j \sigma^C_{ij} = 0, \tag{2.20}$$

where the boundary ∂D lies entirely outside the medium in question. The final relation follows because σ^C_{ij} is zero on ∂D. The second constraint requires σ^C_{ij} to be symmetric:

$$\begin{aligned}\tau_i = \int_D d^3X \epsilon_{ijk} X_j f_k = \int_D d^3X \epsilon_{ijk} X_j \frac{\partial \sigma^C_{kl}}{\partial X_l} = \\ \int_{\partial D} dS_l \epsilon_{ijk} X_j \sigma^C_{kl} - \int_D d^3x \epsilon_{ijk} \sigma^C_{kj} = -\int_D d^3x \epsilon_{ijk} \sigma^C_{kj} = 0,\end{aligned} \tag{2.21}$$

where again the final relation, which sets the anti-symmetric part of σ^C_{kj} to zero, follows because σ_{kl} is zero on the boundary ∂D.

In elasticity, there are as we have seen, two separate spaces: the references space coinciding with the undistorted elastic medium and the target space coinciding with the space we live in. This leads to the possibility of different stress tensors depending on whether forces are measured across surfaces in the reference or target space.

Forces exerted on an infinitesimal volume of mass in an elastic medium by the matter surrounding it are transmitted across the surfaces of that volume. We can take the surfaces to be either those surrounding the volume in the undeformed reference space or in the target space. The latter gives us the Cauchy stress tensor. The former give the first Piola-Kirchhoff (PK) stress tensor, σ^I_{ij}, and a force density $f_i = \partial_j \sigma^I_{ij}$, where the force is what we measure in the target space, and as before $\partial_i = \partial/\partial x_i$. Unlike σ^C_{ij}, σ^I_{ij} does not have to be symmetric, and it in general is not so.

To see how σ^I_{ij} is related to the elastic free energy, we consider the work done on an elastic medium by internal forces f_i. The change in total (free) energy, δF is equal to minus the work done by internal forces. As before, we consider a domain D' whose boundary $\partial D'$ lies entirely outside the volume D occupied by the medium. Thus

$$\begin{aligned}\delta F &= -\int_{D'} f_i \delta X_i = -\int_{D'} d^d x \partial_j \sigma^I_{ij} \delta X_i \\ &= \int d^d x \sigma^I_{ij} \delta \Lambda_{ij},\end{aligned} \tag{2.22}$$

where the surface term on $\partial D'$ vanishes because σ^I_{ij} is zero outside matter and $\delta\Lambda_{ij} = \partial_j \delta R_i$. Recall that $\Delta x_i = \delta u_i$ so that the force density is

$$f_i = -\frac{\delta F}{\delta u_i(\mathbf{x})}. \tag{2.23}$$

Equation (2.22) implies that

$$\sigma^I_{ij} = \frac{\delta F}{\delta \Lambda_{ij}} = \frac{\partial \mathcal{F}}{\partial \Lambda_{ij}} = \frac{\partial \mathcal{F}}{\partial \varepsilon_{kl}} \frac{\partial \varepsilon_{kl}}{\partial \Lambda_{ij}} \equiv \Lambda_{ik} \sigma^{II}_{kj}, \tag{2.24}$$

where

$$\sigma_{kj}^{II} = \frac{\partial \mathcal{F}}{\partial \varepsilon_{kj}} \tag{2.25}$$

is the second PK stress tensor which is symmetric because ε_{kj} is, invariant under rotations in the target space, and a second-rank tensor with respect to rotations in the reference space. In deriving Eq. (2.24), we used $\partial \varepsilon_{kl}/\partial \Lambda_{ij} = \frac{1}{2}(\delta_{jk}\Lambda_{il} + \Lambda_{ik}\delta_{jl})$ and the fact that σ_{ij}^{II} is symmetric. Finally, the Cauchy stress tensor is obtained by transforming the volume integral over $\mathbf{x}$ in Eq. (2.22) to an integral over $X_i = \Lambda_{ij}x_j$ and the derivative with respect to ξ in the same equation to one with respect to R_i via $\partial/\partial x_j = (\partial X_l/\partial x_j)\partial/\partial X_l$:

$$\delta F = \int d^d X \frac{1}{\det \mathbf{\Lambda}} \sigma_{ij}^I \Lambda_{lj} \frac{\partial \delta X_i}{\partial X_l} = -\int d^d X \frac{\partial \sigma_{ij}^C}{\partial X_j} \delta X_j, \tag{2.26}$$

implying

$$\sigma_{ij}^C = \frac{1}{\det \mathbf{\Lambda}} \sigma_{il}\Lambda_{jl} = \frac{1}{\det \mathbf{\Lambda}} \Lambda_{ik}\sigma_{kl}^{II}\Lambda_{lj}^T. \tag{2.27}$$

As required σ_{ij}^C is symmetric.

The second PK stress tensor in terms of nonlinear strains is easily calculated from the elastic energy:

$$\sigma_{ij}^{II} = \frac{\partial \mathcal{F}}{\partial \varepsilon_{ij}} = K_{ijkl}\varepsilon_{kl} \tag{2.28}$$

which reduces in the isotropic limit to

$$\begin{aligned} \sigma_{ij}^{II} &= \lambda \delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \\ &= B\delta_{ij}\varepsilon_{kk} + 2\mu\left(\varepsilon_{ij} - \tfrac{1}{d}\delta_{ij}\varepsilon_{kk}\right). \end{aligned} \tag{2.29}$$

In the linearized limit, all of the three stress tensors are equal, and we will denote them all as σ_{ij}.

Under isotropic pressure, $\sigma_{ij} = -\delta_{ij}$ (linearized)

$$\epsilon_{kk} = \frac{\delta V}{V} = -p/B. \tag{2.30}$$

Under uniaxial tension T along x in two dimensions, $\sigma_{ij} = T\delta_{ix}\delta_{iy}$ and

$$\begin{aligned} T &= \lambda(\varepsilon_{xx} + \varepsilon_{yy}) + 2\mu\varepsilon_{xx} \\ 0 &= \lambda(\varepsilon_{xx} + \varepsilon_{yy}) + 2\mu\varepsilon_{yy} \end{aligned} \tag{2.31}$$

and

$$\epsilon_{yy} = -\frac{\lambda}{\lambda + 2\mu}\varepsilon_{xx} = -\frac{B-\mu}{B+\mu} \equiv -\sigma_p \varepsilon_{xx}, \tag{2.32}$$

where $\sigma_p = \lambda/(\lambda+\mu)$ is The Poisson ratio. If σ_p is positive, the sample stretches in the x direction and contracts in the y direction. But if σ_p is negative, the sample will expand in the y direction when stretched along the x direction. This unusual behavior is called *auxetic* response. The compression modulus has to lie between 0 and ∞, implying that σ_p lies between -1 and 1, reaching maximally auxetic response at $B = 0$. Similar behavior occurs in three-dimensional systems.

Classical elasticity theory assumes that strains strains produced in response to uniform stress at surfaces will be constant throughout the sample. In random materials and in periodic crystals with more than one site per unit cell, the response is more complex. Sites generally under go displacements relative to uniform strain to minimize elastic energy. If response is uniform, it is said to be *affine*; if it is not, it is *nonaffine*

D. Acoustic Modes

Dynamical properties of elastic media are determined by the continuum version of Newton's equations. We have already introduced the force density $f_i(\mathbf{x}) = -\delta F_{\rm el}/\delta u_i(\mathbf{x})$. Newton's equation for the acceleration of the mass $\rho d^d x$, where ρ is the mass density, in a volume $d^d x$ at point $\mathbf{x}$ is $f_i(\mathbf{x})d^d x$, or after removal of the common factor of $d^d x$,

$$\begin{aligned} \rho\ddot{u}_i &= f_i = \partial_j \sigma_{ij}^I \to \partial_j \sigma_{ij} \\ &= K_{ijkl}\partial_j\partial_k u_l \end{aligned} \tag{2.33}$$

where the final form applies in the linearized limit, which is all that concerns us here. In an isotropic material,

$$f_i = (\lambda + \mu)\partial_i \partial_j u_j + \mu \nabla^2 u_i \tag{2.34}$$

These relations determine the frequencies of bulk normal modes in which $u_i(\mathbf{x}, t) \sim e^{i\mathbf{q}\cdot\mathbf{x} - i\omega t}$ where ω is the angular frequency and $\mathbf{q}$ is the wavevector. The Fourier transform of $f_i(\mathbf{q})$ is

$$f_i(\mathbf{q}) = (\lambda + 2\mu) q_i q_j u_j + \mu q^2 \delta^T_{ij} u_j, \tag{2.35}$$

where $\hat{q} = q_i/q$ is the unit vector along $\mathbf{q}$ and $\mathbf{q}^T_{ij} = \delta_{ij} - \hat{q}_i \hat{q}_j$ is the projection operator onto directions perpendicular to $\mathbf{q}$. $\mathbf{u}(\mathbf{q})$ can be decomposed into a longitudinal part $\mathbf{u}_L$ with vanishing curl: $\mathbf{q} \times \mathbf{u}_L = 0$, and a transverse part $\mathbf{u}_T$ with vanishing divergence: $\mathbf{q} \cdot \mathbf{u}_T = 0$, which imply, $\mathbf{u}_L = \hat{q} u_L$ and $\mathbf{u}_{Ti} = \delta^T_{ij} u_j$. Then,

$$f_i(\mathbf{q}) = (\lambda + 2\mu) q^2 u_{Li} + \mu q^2 u_{Ti} \tag{2.36}$$

and

$$\rho \omega^2 u_i = (\lambda + 2\mu) q^2 u_{Li} + \mu q^2 u_{Ti}, \tag{2.37}$$

yielding longitudinal and transverse sound modes with respective frequencies,

$$\omega_L^2 = c_L^2 q^2 \qquad \omega_T^2 = c_T^2 q^2, \tag{2.38}$$

where $c_L^2 = (\lambda + 2\mu)/\rho$ and $c_T^2 = \mu/\rho$. In two dimensions with $\mathbf{q} = (q_x, q_y)$, the longitudinal and transverse parts of $\mathbf{u}$ can be represented at

$$\begin{aligned} u_{Lx} &= \hat{q}_x u_L, \qquad u_{Ly} = \hat{q}_y u_L \Rightarrow u_{Ly} = (q_y/q_x) u_{Lx} \\ u_{Tx} &= -\hat{q}_y u_T, \quad u_{Ty} = \hat{q}_x u_T \Rightarrow u_{Ty} = -(q_x/q_y) u_{Tx} \end{aligned} \tag{2.39}$$

More generally,

$$\omega^2 u_i = D_{ij}(\mathbf{q}) u_j, \tag{2.40}$$

where $D_{ij}(\mathbf{q}) = (1/\rho) K_{ikjl} q_k q_l$. In two-dimensions,

$$D_{ij}(\mathbf{q}) = \frac{1}{\rho} \begin{pmatrix} (\lambda + 2\mu) q_x^2 + \mu q_y^2 & (\lambda + \mu) q_x q_y \\ (\lambda + \mu) q_x q_y & (\lambda + 2\mu) q_y^2 + \mu q_x^2 \end{pmatrix} \tag{2.41}$$

in and isotropic system, and

$$D_{ij}(\mathbf{q}) = \frac{1}{\rho} \begin{pmatrix} K_1 q_x^2 + K_4 q_y^2 & (K_2 + K_4) q_x q_y \\ (K_2 + K_4) q_x q_y & K_1 q_y^2 + K_4 q_x^2 \end{pmatrix} \tag{2.42}$$

in a system with square symmetry, where $K_1 = K_{xxxx} = K_{yyyy}$, $K_2 = K_{xxyy}$, and $K_4 = K_{xyxy}$. The square of the normal-model frequencies are the eigenvalues of od D_{ij} and the normal-mode displacements are proportional to the eigenvectors of D_{ij}.

E. Surface Rayleigh Waves

In addition to bulk modes with $\omega \sim q$ just discussed, there are also surface normal modes, also with a linear dispersion in q, exponentially localized at free surfaces. The boundary condition at a free surface separating the elastic material from the vacuum is that all stress on that surface be zero, i.e., that $\sigma_{ij} \hat{n}_j = 0$ for all components i, where $\hat{n}_j$ is the unit vector normal to the surface. This relation is easily generalized to cases where the interface separates the material from an isotropic fluid of from another elastic medium. These surface modes are called *Surface Rayleigh waves* if mass motion occurs in planes perpendicular to the surface and parallel to the surface wavevector and *Rayleigh-Lamb waves* if mass motion occurs in the planes perpendicular to both the surface and the surface wavenumber.

To keep our discussion as simple as possible, we consider here only Rayleigh waves in an isotropic medium in two dimensions, and we consider a surface parallel to the x-axis separating the vacuum in the half-plane $y < 0$ from the elastic medium in the half-plane $y > 0$. In this case, the boundary conditions are

$$\sigma_{yy} = (\lambda + 2\mu)\varepsilon_{yy} + \lambda \varepsilon_{xx} = 0 \tag{2.43}$$

$$\sigma_{xy} = 2\mu \varepsilon_{xy} \tag{2.44}$$

The mode amplitude is expected to decay to zero exponentially with depth $y > 0$ into the elastic medium for each wavenumber q_x parallel to the surface, and we seek solutions of the form

$$u_i(x, y) = Ae^{iq_x x - \kappa y}, \tag{2.45}$$

where $\kappa > 0$ is the inverse penetration (which in the most general of cases could be complex, but here it will turn out to be purely real). Thus the form of $u_i(\mathbf{x})$ for surface modes has exactly the same form as that of bulk modes except that $e^{iq_y y}$ is replaced by $e^{-\kappa y}$, i.e., q_y is replaced by $i\kappa$. Thus, we can obtain the eigenvectors and the dependence of κ on ω directly from our solutions for the bulk modes.

Replacing $q_x^2 + q_y^2$ by $q_x^2 - \kappa^2$, we find that κ, which depends on ω, has two possible solutions arising from the difference sound velocities of longitudinal and transverse sound:

$$\kappa_L^2(\omega) = q_x^2 - \frac{\omega^2}{c_L^2} \qquad \text{and} \qquad \kappa_T^2(\omega) = q_y^2 - \frac{\omega^2}{c_T^2}. \tag{2.46}$$

There are thus, two possible decay rates for a given frequency, and displacements associated with the surface mode can be expressed as a sum of the the two decaying functions:

$$\begin{aligned} u_x(\mathbf{x}) &= e^{iq_x x}\left(A_{Lx}e^{-\kappa_L y} + A_{Tx}e^{-\kappa_T y}\right) \\ u_y(\mathbf{x}) &= e^{iq_x x}\left(\frac{i\kappa_L}{q_x}A_{Lx}e^{-\kappa_L y} + \frac{iq_x}{\kappa_T}A_{Tx}e^{-\kappa_T y}\right) \end{aligned} \tag{2.47}$$

where we used Eqs. (2.39) relating the x and y components of $\mathbf{u}_L$ and $\mathbf{u}_T$. Inserting Eq. (2.47) into the stress boundary conditions of Eq. (2.44) the yields

$$\begin{aligned} \sigma_{yy} &= \frac{e^{iq_x x}}{q_x}\left[\left(-(\lambda+2\mu)\kappa_L^2 + \lambda q_x^2\right)A_{Lx} - 2\mu q_x^2 A_{Ly}\right] = 0 \\ \sigma_{xy} &= -\frac{\mu}{\kappa_T}e^{iq_x x}\left[2\kappa_L\kappa_T A_{Lx} + (\kappa_T^2 q_x^2)A_{Ty}\right] = 0. \end{aligned} \tag{2.48}$$

These two equations can be written in matrix form (after removing common prefactors and dividing the first equation by ρ) as

$$\begin{pmatrix} c_L^2\kappa_L^2 - (c_L^2 - 2c_T^2)q_x^2 & 2c_T^2 q_x^2 \\ 2\kappa_L\kappa_T & \kappa_T^2 + q_x^2 \end{pmatrix}\begin{pmatrix} A_{Lx} \\ A_{Tx} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.49}$$

Recall that κ_L and κ_T depend on frequency through Eq. (2.46) so that the frequency of the surface mode is determined by setting the determinant Δ of the matrix in Eq. (2.49) to zero

$$\begin{aligned} &c_T^2\Delta = 0 \\ &= (2c_T^2 q_x^2 - \omega^2)^2 - 4c_T^4 q_x^2\sqrt{[q_x^2 - (\omega^2/c_L^2)][q_x^2 - (\omega^2/c_T^2)]}. \end{aligned} \tag{2.50}$$

Putting the square root on the right hand side of the equation and squaring leads to a fourth-order polynomial in ω^2. The determinant, however, is zero when $\omega^2 = 0$ because in this limit $\kappa_L^2 = \kappa_T^2 = q_x^2$, and the two boundary equations for the stress are strictly proportional to each other in this case. Thus, when that mode is removed, the polynomial is only third order in ω^2. Δ can be simplified by setting $\omega = c_T q_x s^2$ producing a common factor of q_x^4. The resulting equation for s is

$$s^6 - 8s^4 + 8s^2[3 - (c_T^2/c_L^2)] - 16[1 - (c_T^2/c_L^2)] = 0. \tag{2.51}$$

There are three solution to this equation for s^2. The physical solutions must be greater than zero to ensure that ω is real and less than 1 to ensure that κ_T^2 is positive. Only one of the three solutions to Eq. (2.51) satisfies both of the criteria producing a unique solution for the surface mode velocity $c_S = c_T g(c_T^2/c_L^2)$. This solution has some interesting features: When the bulk modulus $B = \lambda + \mu$ is zero, $c_L^2 = c_T^2$, and $c_S = 0$. This means that the Rayleigh mode has zero frequency for all q_x (or course the continuum approximation breaks down at as q_x increases, but we will find that this result holds for all q_x in lattice models. In the opposite limit in which either $\mu \to 0$ or $\lambda \to \infty$, $c_S = 0.955 c_T$.

III. LATTICE MODELS

A. Defining the model

For our purposes a lattice (or in engineering language, a *frame*) is a collection of mass points l at positions $\mathbf{X}(l)$ connected pairwise by bonds β. Each bond β connects two sites s_β and s'_β at respective positions $\mathbf{X}(s_\beta)$ and $\mathbf{X}(s'_\beta)$. In mechanical equilibrium (no net force at any site), site l is at positions $\mathbf{R}(l)$, and

$$\mathbf{X}(l) = \mathbf{R}(l) + \mathbf{u}(l), \tag{3.1}$$

where $\mathbf{u}(l)$ is the displacement for equilibrium. The bond vector connecting sites s'_β and s_β is

$$\mathbf{X}_\beta = \mathbf{X}(s'_\beta) - \mathbf{X}(s_\beta) = \mathbf{R}_\beta + \Delta\mathbf{u}_\beta, \tag{3.2}$$

where $\mathbf{R}_\beta = \mathbf{R}(s_{\beta'}) - \mathbf{R}(s_\beta)$ is the equilibrium vector and $\Delta\mathbf{u}_\beta = \mathbf{u}(s'_\beta) - \mathbf{u}(s_\beta)$. Deviations of the length of bond β from its rest length R_β are conveniently measured by the scalar quantity v_β defined via

$$v_\beta = \tfrac{1}{2}(X_\beta^2 - R_\beta^2) = \mathbf{R}_\beta \cdot \Delta\mathbf{u}_\beta + \tfrac{1}{2}(\mathbf{u}_\beta \cdot \mathbf{u}_\beta). \tag{3.3}$$

By construction, v_β, like ε_{ij} in the continuum limit, is invariant with under rotation of $\mathbf{X}_\beta$. Changes δX_β in the length of bond β are easily expressed in terms of v_β:

$$\delta R_\beta = \frac{v_\beta}{R_\beta} - \frac{1}{2}\frac{v_\beta^2}{R_\beta^3} + O(v_\beta^2) \to \hat{\boldsymbol{b}}_\beta \cdot \Delta\mathbf{u}_\beta, \tag{3.4}$$

where the final form is the linearized limit and $\hat{\boldsymbol{b}}_\beta = \mathbf{R}_\beta/R_\beta$ is the unit vector along bond β.

We now associate with each bond a central-force, rotationally invariant potential energy $V_\beta(X_\beta)$, where $X_\beta = |\mathbf{X}_\beta \cdot \mathbf{X}_\beta|$, which can be expanded in powers of δX_β. The total potential energy $V_{\rm el}[\{X_\beta\}]$ is simply the sum over all bonds of $V_\beta(X_\beta)$, and the energy associated with bond deformations as

$$\begin{aligned} V_{\rm el} &\equiv V_{\rm el}[\{X_\beta\}] - V_{\rm el}[\{R_\beta\}] \\ &= \sum_\beta [V'_\beta(R_\beta)\delta X_\beta + \tfrac{1}{2}V''_\beta(R_\beta)\delta X_\beta^2 + \cdots] \qquad (3.5) \\ &\to \frac{1}{2}\sum_\beta [V''_\beta \Delta u_{\beta||}^2 + R_\beta^{-1} V'_\beta(R_\beta)(\Delta\mathbf{u}_{\beta\perp})^2]. \qquad (3.6) \end{aligned}$$

where the final form is the linearized harmonic limit where

$$\Delta u_{\beta||} = \hat{\boldsymbol{b}} \cdot \Delta\mathbf{u}_\beta \equiv e_\beta \tag{3.7}$$

is the linearized bond stretch, e_β, and

$$\Delta u_{\beta\perp,i} = (\delta_{ij} - \hat{b}_{\beta i}\hat{b}_{\beta j})u_{\beta,j} \tag{3.8}$$

is the relative displacement of sites s'_β and s_β perpendicular to $\hat{\boldsymbol{b}}_\beta$. We left out the contribution from terms $V'_\beta \Delta u_{\beta||}$ because they give zero when summed over β because the net force at each site is zero when $\mathbf{X}_\beta = \mathbf{R}_\beta$ for all β. The first term in Eq. (3.6) is the familiar term arising from stretching or compressing springs away from their equilibrium. The second term reflects the fact that springs under tension will resist motion perpendicular their axes; it be present in networks whose bonds are not at their rest length in their equilibrium configuration as often occurs in random systems for example. In what follows, we will assume that $V'_\beta(R_\beta)$ is zero for all β, in which case, the elastic energy is simply

$$V_{\rm el} = \tfrac{1}{2}\sum_\beta k_\beta e_\beta^2, \tag{3.9}$$

a result that follows directly from a network connected Hooke's-law springs with individual spring constants $k_\beta = V''_\beta$.

B. Compatibility and Equilibrium Matrices

Dealing with zero modes and states of self-stress is facilitated by the introduction of a more compact vector-matrix notation. To this end, we first introduce the N_B-dimensional vector of bond displacements $\mathsf{E}^T = (e_1, e_2, \cdots)$, where the T superscipt refers the row vector that is the transposes of E, and the $N_B \times N_B$ diagonal matrix k of spring constants. Then

$$V_{\rm el} = \frac{1}{2}\mathsf{E}^T\mathsf{k}\mathsf{E}, \tag{3.10}$$

where k is the diagonal $N_B \times N_B$ matrix of spring constants. The bond stretches are linearly proportional to site displacements via Eq. refeq:bond-stretch), which can be written as a matrix equation,

$$\mathsf{E} = \mathsf{C}\mathsf{U}, \tag{3.11}$$

where U is the dN dimensional vector of site displacements and C is the $N_B \times N_B$ *compatibility* matrix with components

$$C_{\beta,li} = \hat{b}_{\beta i}(\delta_{l,s'_\beta} - \delta_{l,s_\beta}), \tag{3.12}$$

where l specifies the site and i the Cartesian coordinates (x, y, z). These relations provide us with an expression for the elastic energy in terms of displacements U rather than bond-stretches E:

$$V_{\rm el} = \frac{1}{2}\mathsf{U}^T\mathsf{K}\mathsf{U}, \tag{3.13}$$

where

$$\mathsf{K} = \mathsf{Q}\mathsf{k}\mathsf{Q}^T = \mathsf{C}^T\mathsf{k}\mathsf{C} \tag{3.14}$$

is the $dN \times dN$ *stiffness* matrix.

When bonds are stretched or compressed, they are under positive or negative tension and exert forces on sites they connect. In our harmonic theory, the tension of bond β is simply

$$t_\beta = k_\beta e_\beta, \tag{3.15}$$

which can be expressed in matrix form with the introduction of the N_B dimensional vector of bond tensions $\mathsf{T}^T = \{t_1, t_2, \cdots\}$:

$$\mathsf{T} = \mathsf{k}\mathsf{E}. \tag{3.16}$$

The force $f_i(l)$ at site l is simply

$$f_i(l) = -\frac{\partial V_{\rm el}}{\partial u_i(l)} = -\mathsf{K}_{ij}(l, l')u_j(l'), \tag{3.17}$$

which, with the introduction of the dN dimensional force vector F, can be written as

$$\mathsf{F} = -\mathsf{K}\mathsf{U} = -\mathsf{C}^T\mathsf{k}\mathsf{C}\mathsf{U} = -\mathsf{C}^T\mathsf{k}\mathsf{E} = -\mathsf{C}^T\mathsf{T}. \tag{3.18}$$

This defines the *compatibility* matrix $\mathsf{Q} = \mathsf{C}^T$ relating forces at exerted by internal bonds ont sites via

$$\mathsf{F} = -\mathsf{Q}\mathsf{T} = -\mathsf{L}, \tag{3.19}$$

where $\mathsf{L} \equiv -\mathsf{F}$ is the vector of loads, which are the external forces needed to balance the internal forces F. If all of the masses are the same, then the dynamical matrix is $\mathsf{D} = m\mathsf{K}$. If there are different masses at each site, $\mathsf{D} = \mathsf{m}^{-1/2}\mathsf{K}\mathsf{m}^{-1/2}$ where m is the $dN \times dN$ diagonal matrix of masses (with the same entry for each of the d directions of displacement for a give site).

C. The Rank-Nullity Theorem and The Generalized Maxwell Relation

Sec. 2.2 of RA

The compatibility matrix C gives bond stretches in terms of lattice displacements. The set of displacements U that satisfy $\mathsf{C}\mathsf{U} = 0$ constitute the *Null Space* or *kernel*, ker(C), of C. These are displacements that leave all bonds at their rest length and, therefore, cost no energy. The dimension of this space, referred to as Nullity(C) or as dim ker(C) is simply the number of independent zero modes, N_0. Similarly the the null space of Q is the set of tensions than leave the forces at all sites equal to zero, i.e., the states of self stress, and Nullity(Q) is the number N_S of states of self stress. The *Rank-Nullity* theorem of linear algebra then yields the Generalized Maxwell relation $N_0 - N_S = dN - N_B$.

Calculation of C and Q for sample lattice: Sec. 2.2 of RA

D. Elastic limit and States of Self Stress

Sec. 3.2 and Appendices A and B of RA

Important result: The elastic free energy density depends on states of self-stress. This stands to reason: under external stress, sites are in mechanical equilibrium but bonds are under stress. The free energy density is

$$f_{\rm el} = \frac{1}{2V}\mathsf{E}_{{\rm aff},s}^T[(\mathsf{k}^{-1})_{ss})]^{-1}\mathsf{E}_{{\rm aff},s} \xrightarrow{\mathsf{k}\to k\mathsf{I}} \frac{k}{2V}\sum_\alpha (\mathsf{E}_{\rm aff}\cdot\hat{\mathsf{t}}_\alpha)^2 \tag{3.20}$$

where $\mathsf{E}_{\rm aff}$ is the vector of affine bond stretches, $\mathsf{E}_{{\rm aff},s}$ and $(\mathsf{k}^{-1})_{ss}$ are the projections of $\mathsf{E}_{\rm aff}$ and k^{-1} onto $\ker(\mathsf{Q})$, and $\hat{\mathsf{t}}_\alpha$ is the αth orthonormal basis vector of $\ker(\mathsf{Q})$. Thus, only the projections of the affine displacement vectors onto states of self-stress contribute to the elastic energy.

Important consequence: because $f_{\rm el}$ it is a sum of squares of linear combinations of strain, one for each state os self-stress, it shows that there must be at least $a_d = d(d+1)/2$ load-bearing SSSs to produce an elastically stable system with an elastic matrix with a_d positive eigenvalues. If number of load-bearing states of SSSs is less than a_d, there will be zero-energy macroscopic elastic strains that cost zero energy. These distortions are called *Guest Modes*.

E. Periodic Lattices

Sec. 4 of RA

Site and bond variables in lattices of periodically repeated unit cells can be expressed in terms of Fourier transforms:

$$\boldsymbol{u}_\mu(\ell) = \frac{1}{N_c}\sum_{\boldsymbol{q}} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}_\ell+\boldsymbol{r}_\mu)}\boldsymbol{u}_\mu(\boldsymbol{q}), \tag{3.21}$$

$$\boldsymbol{u}_\mu(\boldsymbol{q}) = \sum_{\ell} e^{-i\boldsymbol{q}\cdot(\boldsymbol{R}_\ell+\boldsymbol{r}_\mu)}\boldsymbol{u}_\mu(\ell) \tag{3.22}$$

$$t_\beta(\ell) = \frac{1}{N_c}\sum_{\boldsymbol{q}} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}_\ell+\boldsymbol{r}_\beta)}t_\beta(\boldsymbol{q}) \tag{3.23}$$

$$t_\beta(\boldsymbol{q}) = \sum_{\ell} e^{-i\boldsymbol{q}\cdot(\boldsymbol{R}_\ell+\boldsymbol{r}_\beta)}t_\beta(\ell), \tag{3.24}$$

$$\mathbf{Q}_{\sigma\beta}(\boldsymbol{q}) = \sum_{\ell} e^{-i\boldsymbol{q}\cdot(\boldsymbol{R}_{\ell,\sigma}-\boldsymbol{R}_{0,\beta})}\mathbf{Q}_{\sigma\beta}(\ell,0), \tag{3.25}$$

where $\sigma = (\ell,\mu,i)$.

There are a separate equilibrium and compatibility relations for each wavenumber $\boldsymbol{q}$:

$$\mathbf{Q}(\boldsymbol{q})\,\mathbf{t}\,(\boldsymbol{q}) = -\mathbf{f}(\boldsymbol{q}) \qquad \mathbf{C}\,(\boldsymbol{q})\,\mathbf{u}(\boldsymbol{q}) = \mathbf{e}(\mathbf{q}), \tag{3.26}$$

$\mathbf{Q}(\boldsymbol{q})$ is a $dn \times n_b$- matrix and $\mathbf{C}(\boldsymbol{q})$ is an $n_b \times dn$ matrix, where n is the number of sites pand n_b is the number of bonds per unit cell. There are separate generalized Maxwell relations for each $\boldsymbol{q}$:

$$n_0(\mathbf{q}) - n_s(\mathbf{q}) = dn - n_b, \tag{3.27}$$

where n the number of sites per unit cell and n_b.

The harmonic energy is

$$V_{\rm el} = \frac{1}{2N_c}\sum_{\boldsymbol{q}} \mathbf{e}^\dagger(\boldsymbol{q})\,\mathbf{k}\,\mathbf{e}(\boldsymbol{q}) = \frac{1}{2N_c}\sum_{\boldsymbol{q}} \mathbf{u}^\dagger(\boldsymbol{q})\mathbf{K}(\mathbf{q})\mathbf{u}(\boldsymbol{q}), \tag{3.28}$$

where $\mathbf{k}$ is the $n_b \times n_b$ diagonal matrix of spring constants, and

$$\mathbf{K}(\mathbf{q}) = \mathbf{Q}(\boldsymbol{q})\,\mathbf{k}\,\mathbf{Q}^\dagger(\boldsymbol{q}) \equiv m\mathbf{D}(\mathbf{q}) \tag{3.29}$$

is the stiffness matrix. We will usually set $m = 1$ so that the stiffness matrix $\mathbf{K}$ and the dynamical matrix $\mathbf{D}$ are the same. In periodic systems, nearest-neighbor (NN), next-nearest neighbor (NNN), and further-neighbor

bonds are well defined, and bond vectors can be expressed as the direct sum of NN and NNN components, e.g. $\mathbf{e} = \mathbf{e}_{NN} \oplus \mathbf{e}_{NNN}$, and elastic constant matrix and dynamical matrices can be decomposed into NN and NNN:

$$\mathbf{D}(\mathbf{q}) = \mathbf{D}_{NN}(\mathbf{q}) + \mathbf{D}_{NNN}(\mathbf{q}). \tag{3.30}$$

Under affine strain, the strain of equivalent bonds in different unit cells are identical, and we can describe affine strain in terms of the n_b dimensional vector $\mathbf{e}_{\mathrm{aff},s}$. As a result, the elastic energy depends only on the projection of the affine strain onto the $\mathbf{q} = 0$ SSSs:

$$f_{\mathrm{el}} = \frac{1}{2v_c}\mathbf{e}_{\mathrm{aff},s}^T(\mathbf{k}_{ss}^{-1})^{-1}\mathbf{e}_{\mathrm{aff},s} \to \frac{1}{2v_c}k\sum_{\alpha}(\mathbf{e}_{\mathrm{aff}}\cdot\hat{\mathbf{t}}_\alpha)^2, \tag{3.31}$$

F. Important of Properties of Periodic Maxwell Lattices

In periodic Maxwell lattices under periodic boundary conditions, $n_b = dn$, and $n_0(\boldsymbol{q}) = n_s(\boldsymbol{q})$ for every $\boldsymbol{q}$ in the BZ. Under periodic boundary conditions.

1. There are zero modes at any $\boldsymbol{q} \neq 0$ only if there are states of self-stress at that $\boldsymbol{q}$. This means that the phonon spectrum has no zero modes, i.e., it is fully gapped if and only if there are not states of self stress at any $\mathbf{q} \neq 0$.

2. Under periodic boundary conditions, there are at least d $\boldsymbol{q} = 0$ zero modes in d-dimensions corresponding to uniform rigid translations of the lattice. This requires at least d $\mathbf{q} = 0$ SSSs implying that there are at least d load-bearing states of self-stress. If there are no $\mathbf{q} = 0$ modes and thus no $\mathbf{q} = 0$ SSSs beyond the d required ones, there are d positive eigenvalues of the Voigt elastic-constant matrix and $d(d+1)/2 - d = d(d-1)/2$ zero-energy Guest modes, i.e., one Guest mode in two dimensions and three in three dimensions.

G. Examples of Periodic Maxwell Lattices: 5 of RA

1. Square Lattice: RA Sec. 5.1 -Fig. P10

1. N_x columns and N_y rows of straight lines of bonds, each carries a state of self-stress: There are therefore $N_x + N_y$ zero modes, which correspond to independent uniform displacements of each row and column. This translates upon Fourier transformation to zero modes along the lines $q_x = 0$ and $q_y = 0$ in the BZ. Since there are N_x values of q_x and N_y values of q_y, these modes exhaust the generalized Maxwell relation. Phonon frequencies rise linearly from the two lines creating a sort of "knife-edge".

2. There are two $\mathbf{q} = 0$ SSSs and, thus, two stable elastic distortions and one Guest mode. The latter is clearly a simple shear. The elastic energy density is

$$f_{\mathrm{el}} = \tfrac{1}{2}k\,(\varepsilon_{xx}^2 + \varepsilon_{yy}^2). \tag{3.32}$$

   There are two independent lattice distortions, ε_{xx} and ε_{yy}, that cost energy and one ε_{xy} that does not.

3. The addition of next-nearest-neighbor (NNN) bonds adds more constraints and gaps the phonon spectrum (except of course at $\boldsymbol{q} = 0$.

4. Lattices of finite width (or height) can be constructed from ones under periodic boundary conditions on a torus by cutting bonds. A strip with finite width along y but periodic along x involves cutting N_x vertical bonds (one per surface unit cell) thereby removing N_x states of self-stress in the y direction. Thus, the number of bonds and the number of states of self stress are diminished by the same abound and the number of zero modes remains unchanged. In fact the zero modes of the finite lattice are identical to those of the periodic lattice in that they correspond to displacements of individual lines of bonds in the y direction; in effect, they are surface modes with infinite penetration depth in that the influence of distortions of one surface are transmitted all of the way to the opposite surface. There are no zero modes corresponding to surface Rayleigh waves.

2. Kagome lattice: RA Sec. 5.2 - Figs. P11, P12

1. There are now sets of straight lines of bonds oriented along the three directions of an equilateral triangle, implying one state of self stress per line and associated zero modes that occur along the q_y axis and two symmetry related directions of the BZ. Show zero modes.

2. There are now three states of self-stress at $\mathbf{q} = 0$. The result is that the Voigt elastic matrix now has three stable eigenvalues, and the elastic energy density is that of an isotropic $2D$ solid with $\lambda = \mu = \sqrt{3}k/8$.

3. Again, the addition of NNN bonds fully gaps the phonon spectrum.

4. As in the square lattice, the zero modes of the cut lattice are identical those of the uncut lattice, and there are not surface modes.

3. The twisted kagome lattice: RA 5.3 - Figs. P13, P14, P15

The kagome lattice can be distorted so as to keep all bond lengths unchanged by counter rotating corner sharing triangles through an angle α as shown in the Figure. This process removes all load-bearing straight lines of bonds and all states of self-stress except the two trivial ones at $\boldsymbol{q} = 0$.

1. There are no SSSs at $\mathbf{q} \neq 0$. As a result there are no zero modes at $\mathbf{q} \neq 0$ in the BZ: The spectrum is fully gapped!. The simple geometric "twisting" operation completely changes the spectrum. The gap at the zone edge is proportional to $|\sin\alpha|$.

2. There are now only the two states of self-stress at $\mathbf{q} = 0$ required by translational invariance, and there is one Guest mode. The elastic energy is stable with respect to shear but the bulk modulus vanishes. This leads to a maximally auxetic lattice in which the Poisson ratio is equal to -1! (Show movie of experimental system)

3. Cutting a strip from aa periodic torus now has a new effect. Because there are no ($\boldsymbol{q} \neq 0$) SSSs under period BCs, no states of self stress are removed upon cutting to produce a finite strip. This means that the number of zero modes at $\mathbf{q} \neq 0$ must equal the number of bonds cut: two per surface unit cell for strips along periodic along the x-axis and 4 per unit cell for strips periodic along the y-axis. These modes are Rayleigh surface waves, not present in the bulk spectrum.

IV. LECTURE III: SURFACE MODES AND TOPOLOGICAL LATTICES

A. Surface Modes in the Twisted kagome Lattice: 5.3 of RA Fig. P15

1. Choice of unit cells: There are many different ways of collecting the three sites and six bonds together to create a periodically repeated unit cell. In some sense the most obvious choice of cells is a symmetric one such as those in fig. P18. But to fully treat surface states, it is necessary to choose cells whose boundaries match the boundaries of the particular free surface as shown in Fig. 19.

2. Gauge choice: in taking Fourier transforms, the position vector of a given site (or bond) can either include the basis vectors relative to the unit-cell origin or not. Changes of gauge lead to changes in the phase of $\mathsf{C}(\boldsymbol{q})$ and $\mathsf{Q}(\boldsymbol{q})$.

3. Once a surface-compatible unit cell has been chosen, the compatibility matrix takes the form,

$$\mathsf{C} = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & 0 & \dots & 0 & 0 \\ 0 & \mathbf{C}_{11} & \mathbf{C}_{12} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathbf{C}_{11} & \mathbf{C}_{N-1,N} \\ 0 & 0 & 0 & \dots & 0 & \mathbf{C}_{NN} \end{pmatrix}, \tag{4.1}$$

which only connects unit cells in the "forward" direction perpendicularly into the sample, where $\mathbf{C}_{11}$ is that par of the C that only connects bonds in s single cell to sites in a single cell and $\mathbf{C}_{12}$ connects bonds in one cell to sites in the next cell deeper in the sample. Both $\mathbf{C}_{11}$ and $\mathbf{C}_{12}$ are $n_b \times n_b$ matrices, and both depend on the

wavenumber $q_{||}$ parallel to the surface. Since we expect displacement amplitudes to decay into the bulk, we set $\mathbf{u}(q_{||}, n+1) = \lambda \mathbf{u}(q_{||}, n)$. Then the eigenvalue equation for λ becomes

$$\mathbf{C}_{11}\mathbf{u}_n + \mathbf{C}_{12}\mathbf{u}_{n+1} = 0, \tag{4.2}$$

for $n = 1, ...N-2$. These equations are solved by $\mathbf{u}_{n+1} = \lambda \mathbf{u}_n$ and

$$\det(\mathbf{C}_{11} + \lambda \mathbf{C}_{12}) = 0 \tag{4.3}$$

subject to the boundary conditions that $\mathbf{C}_{11}\mathbf{u}_{N-1} + \mathbf{C}_{N-1,N}\mathbf{u}_N = 0$ and $\mathbf{C}_{NN}\mathbf{u}_N = 0$.

4. Fig P19 shows the results of these calculations for various surfaces. Note that there is always one zero mode per wavenumber on each surface. The Maxwell rule gives the total number of zero modes per wavenumber in a strip cut from a lattice under periodic BC but not where they are. In the twisted kagome lattice, the surface zero modes are always equally distributed on the two surfaces. But is it possible to change that? Experience with quantum systems such as the Su-Schrieffer-Heeger (SSS) model for polyacetylene and topological insulators suggest that it is possible to do so. The phonon spectrum of the twisted kagome lattice has a gap that vanishes as the twist angle goes to zero, much as the electron spectra of topological quantum systems does. As we shall see, is is possible to create lattice with topological properties that can move zero surface modes from one side of the lattice to the other.

B. A one-dimensional model: 6.1 of RA P.16

The one-dimensional Su-Schrieffer-Heeger model is the simplest quantum system that has topological surface zero-energy surface and boundary states. It is depicted in Fig. P16. Rather than review the properties of that system directly, we will instead look at a one-dimensional classical model that is essentially identical to the (SSS) model. In this model depicted in Fig. P16, rigid rods of length r are anchored on frictionless pivots a distance a apart on a rigid beam. The orientation of rods alternate in direction, making angles, θ_s relative to the normal to the beam. The ends of the rods are then connected via central force springs. Thus each site s has one degree of freedom θ_s, and in a system of finite length, there are N sites and $N_B = N + 1$ bonds. In a free system, there are no states of self stress and $N_S = N - N_B = 1$zero mode. The rods are anchored so that free translation is not a zero mode.

The compatibility matrix is easily calculated: We proceed now to a more detailed analysis of the our model. The components of the compatibility matrix at rest angle $\bar{\theta}$ are

$$C_{\beta s}(\bar{\theta}) = -c_1(\bar{\theta})\delta_{\beta,s} + c_2(\bar{\theta})\delta_{\beta+1,s}, \tag{4.4}$$

where it is understood that rotations of "upward" ("downward") pointing rods are clockwise (clockwise) and

$$c_{1(2)} = \frac{(a \pm 2r\sin\bar{\theta})r\cos\bar{\theta}}{\sqrt{a^2 + 4r^2\cos^2\bar{\theta}}}. \tag{4.5}$$

Thus $|c_1| > |c_2|$ for all $0 < \bar{\theta} < \pi$, and $|c_1| < |c_2|$ for all $-\pi < \bar{\theta} < 0$. The energy of the system (contained entirely in the stretching of the springs) is then

$$E = \tfrac{1}{2}k\sum_\beta (\delta l_\beta)^2 = \tfrac{1}{2}k\sum_s (c_1\delta\theta_s - c_2\delta\theta_{s+1})^2. \tag{4.6}$$

The Fourier transform of $C_{\beta,s}$ is

$$C(q) = -c_1 + e^{iqa}c_2, \tag{4.7}$$

and bulk phonon modes have frequency

$$\omega(q) = \pm|C(q)| = \pm\sqrt{(c_1 - c_2)^2 + 4c_1c_2\sin^2(qa/2)}, \tag{4.8}$$

(for unit mass), where $-\pi/a < q \leq \pi/a$. When $\bar{\theta} = 0$ (vertical rods), $c_1 = c_2$, the energy becomes invariant with respect of $\delta\theta_s \to \delta\theta_s + \delta$ for every s, and there is necessarily a bulk zero mode at $q = 0$ - this in spite of the fact that the bases of the rods are anchored, breaking translational invariance. For other values of $\bar{\theta}$, the phonon spectrum is fully gapped.

1. There is a zero surface mode when $C(q) = 0$, i.e., when $\lambda = e^{iqa} = c_1/c_2$. Thus if $|c_1| < |c_2|$, $\lambda < 1$ and θ_n decays with increasing n. In other words, there is a surface modes localized on the left end. But if $|c_1| > |c_2|$, $\lambda > 1$, and there is a surface mode localized on the right surface because $\lambda^n = \lambda^{-|n|} < 1$ for $n < 0$. $|c_1| < |c_2|$ ($|c_1| > |c_2|$) when $\bar{\theta} < 0$ ($\bar{\theta} > 0$). Thus the zero mode is a left surface mode when $\bar{\theta} < 0$ and a right surface modes when $\bar{\theta} > 0$. We now have an example of how zero modes can be moved from one side of the system to the other. Note when $\bar{\theta} = 0$, $\lambda = 1$ and there is a bulk zero mode extending across the sample. This is analogous to what happens in the untwisted kagome lattice.

2. There is a topological characterization of the two states. This can be seen using what is sometimes called the *Cauchy argument principle* provides a relative count of the number of zeros and poles of any meromorphic function $F(z)$ of a complex number z in the interior of a contour $\mathcal{C}$:

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{F'(z)}{F(z)}dz = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{d\ln F(z)}{dz}dz = n - p, \tag{4.9}$$

   where n is the number of zeros and p the number of poles counted with their order (e.g., if $F(z) = z^{-2}$, $p = 2$) in the region bounded by $\mathcal{C}$. If the contour $\mathcal{C}$ is the unit circle, the magnitude of z at any zero of $F(z)$ is less than unity. Thus, the argument principle applied to $z = e^{iqa}$, which has no poles, will count the number of zeros with $|z| < 1$. Thus the number of zeros within the unit contour of $C(z)$

$$n = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{d\ln C(z)}{dz} = \frac{1}{2\pi i}\int_0^{2\pi/a} dq\frac{d}{dq}\mathrm{Im}\ln C(q) \tag{4.10}$$

   provides a topological invariant that does not change so long as $\bar{\theta}$ does not change sign. $n = 1$ ($n = 0$) for $\bar{\theta} < 0$ ($\bar{\theta} > 0$). n is often referred to as a *winding number*.

3. There is a zero mode localized at a boundary between a $n = 0$ lattice on the left and a $n = 1$ on the right as shown in Fig. P 16. There are formal ways of proving this, but a brute-force procedure gives the general idea: we have surface zero modes that decay to the right for $n = 1$ and to the left for $n < 1$, Each comes with an independent amplitude, one of which can be chosen arbitrarily because we are dealing with a linearized theory. Figure P16(f) shows a odmain wall consisting of a site connected by two bonds. The site carries one degree of freedom, so the amplitudes of the surface modes on either side of the wall along with the displacement of the site in the wall gives three independent amplitudes to accommodate the constraint that the two bond lengths do not change. Since the overall amplitude of the zero modes is not fixed, there are in effect only two independent amplitudes, which is enough go guarantee that neither bond changes its length. Alternatively, we can look at the effective 2×3 dimensional compatibility linking the elongations of the two bonds to the displacement of the central site and the two zero-mode amplitudes. Since this configuration of bonds has no states of self stress, there is $3 - 2 =!1$ zero mode localized at the domain wall. Alternatively, the two sites terminating the left and right sides of the domain wall could be connected by a single bond, and again there is a single zero mode. (HW - can derive this single condition?)

C. Topological lattices: 6.2 of BA

We have seen that unlike electron systems, the strips of the fully gapped Maxwell lattice we are considering have a number of edge states equal to the number of bonds that are cut to produce the strip from the lattice under periodic boundary conditions. We have also seen in a one-dimensional example without acoustic phonons that different topological states, characterized by different winding numbers of the compatibility matrix, can be constructed and that the two topological lattices have zero edge modes on different sides of the lattice. The question then is, can we construct lattices that both obey the Maxwell count of zero edge modes and that have topological characterizations that upon change can move zero modes from one free surface to another. The answer is of course yes. Fig P17 shows three lattices constructed from the kagome lattice. The center lattice (b) has one set of straight segments parallel to the x-axis. These give rise to states of self-stress and associated bulk zero modes along $q_x = 0$ in the Brillouin zone. This lattice is the analog of the state with $\bar{\theta} = 0$ in the one-dimensional example. Figure (a) shows the three-fold symmetric twisted kagome lattice and (c) shows a new lattice that has a different topological characterization from the twisted kagome lattice. Both lattices (a) and (c) have a fully gapped spectrum, though lattice (c) has phonon modes with frequency proportional along certain directions to q^2 rather than q, and both can be obtained from lattice (b) by continuous distortion. Thus (a) and (c) are the analogs of the $\theta = \pm\bar{\theta}$. They have gaps that continuously approach zero as lattice (b) is approached.

1. Let $\nu^S = N_0 - N_S$ be the number of zero modes minus the number of states of self stress in a subsystem, which could include a free surface or a buried interface. If the bulk modes are gapped, the the zero modes are necessarily at at free surfaces or buried surfaces. It a particular surface has zero modes, it will not have states of self stress and vice-versa. ν_L can be decomposed into a local part and ν_L^S and a topological part ν_T^S:

$$\nu^S = \nu_L^S + \nu_T^S \tag{4.11}$$

To evaluate these quantities, it is useful to introduce a kind of dipole moment. Associate with each site a charge 2 (charge d in d dimensions) and with each bond a charge -1. Because there are three sites and six bonds in each unit cell, the total charge of each unit cell is zero. There can, however, be a nonzero dipole moment, depending both on the shape of the unit cell and on the gauge (recall - assignment of positions to bonds and sites).

2. Symmetric unit cells (Fig. P18) in the kagome lattice have a dipole moment of zero. The dipole moment is

$$\mathbf{R}_L = d \sum_{\text{sites } \mu} \tilde{\mathbf{r}}_\mu - \sum_{\text{bonds } \beta} \tilde{\mathbf{r}}_\beta. \tag{4.12}$$

Because the dipole moment of the symmetric cell is zero, the moments of surface compatible cells can be calculated by calculating the change in moment brought about by moving bonds an sites from the symmetric cell to the surface cell yielding displacements $\Delta\tilde{\mathbf{r}}_\mu$ and $\Delta\tilde{\mathbf{r}}_\beta$ so that

$$\mathbf{R}_L = d \sum_{\text{sites } \mu} \Delta\tilde{\mathbf{r}}_\mu - \sum_{\text{bonds } \beta} \Delta\tilde{\mathbf{r}}_\beta. \tag{4.13}$$

Note, because the shifts $\Delta\tilde{\mathbf{r}}_\mu$ and $\Delta\tilde{\mathbf{r}}_\beta$ are necessarily multiples of lattice vectors $\mathbf{R}_L$ is a lattice vector. Examples of shifted cells are shown in the figure. The local count per site is

$$\tilde{\nu}_L \equiv \nu_L^S / N_{\text{cell}} = \boldsymbol{G} \cdot \mathbf{R}_L / 2\pi, \tag{4.14}$$

where $\boldsymbol{G}$ is the reciprocal lattice vector associated with the surface cut point away from the surface (Fig. P 19).

3. The topological count is similar to the index for the one-dimensional system. It too can be expressed in terms of a lattice vector $\mathbf{R}_T$, which can be viewed as a topological polarization.

$$\tilde{\nu}_T^S = \nu^S / N_{\text{cell}} = \boldsymbol{G} \cdot \mathbf{R}_T / (2\pi), \tag{4.15}$$

where N_{cell} is the number of surface unit cells and $\mathbf{R}_T$, a generalization of the one-dimensional winding number, is a lattice vector

$$\mathbf{R}_T = \sum_i n_i \boldsymbol{a}_i, \tag{4.16}$$

where $\boldsymbol{a}_i$ are the primitive translation vectors and

$$n_i = \frac{1}{2\pi i} \oint_{C_i} d\mathbf{q} \cdot \text{Tr}[\mathbf{Q}(\mathbf{q})^{-1} \nabla_{\mathbf{q}} \mathbf{Q}(\mathbf{q})]. = \frac{1}{2\pi} \oint_{C_i} d\mathbf{q} \cdot \nabla_{\mathbf{q}} \phi(\mathbf{q}), \tag{4.17}$$

where $\phi(\mathbf{q})$ is the phase of $\det \mathbf{Q}(\mathbf{q})$ ($\mathbf{Q}(\mathbf{q}) = |\mathbf{Q}(\mathbf{q})|^{i\phi(\mathbf{q})}$). Here C_i is a cycle of the BZ connecting $\mathbf{q}$ and $\mathbf{q} + \boldsymbol{B}_i$, where $\boldsymbol{B}_i$ is a primitive reciprocal vector satisfying $\boldsymbol{a}_i \cdot \boldsymbol{B}_j = 2\pi\delta_{ij}$ ($\boldsymbol{B}_1 = -\boldsymbol{G}_2$ and $\boldsymbol{B}_2 = \boldsymbol{G}_1$. The n_i are winding numbers of the phase of $\det \mathbf{Q}(\mathbf{q})$ around the cycles of the BZ, where $\mathbf{Q}(\mathbf{q})$ is the equilibrium matrix in a Bloch basis.

Examples from Fig. P19

4. The derivation of theses results, which is fairly technical, provided in the supplementary material for the Nature Phys. article.

5. On domain walls, separating regions with different polarizations, the local count is zero because there are no broken bonds, and the total count is due only to the polarization charge.

$$\tilde{\nu}_T = \boldsymbol{G} \cdot (\mathbf{R}_T^1 - \mathbf{R}_T^2) / 2\pi. \tag{4.18}$$

If this is positive, there are zero modes. If it is negative, there are states of self stress. Under periodic boundary conditions, $N_0 = N_S$, and if there is a domain wall with N_0 zero modes, there must be one with $N_S = N_0$ states of self-stress. If there is a domain wall connecting two finite strips, the number of zero modes per cell must still equal the number of bonds that were cut. Thus for example, there could be one zero mode on the bottom surface, one zero mode at the domain wall, and no zero modes on the top surface.

Fig. P20

6. Alternative domain wall count: There are cases (e.g., when there are Weyl mode (see later)) when it is not possible to cleanly define a polarization, but it is always possible to determine the number of zero modes for each free surface at each $\mathbf{q}$. From this information and the number of bonds per cell that have to be cut to break up the domain wall that joins two topologically distinct lattices, it is always possible to calculate the number of zero modes at the domain wall. Let n_{0R} and n_{0L} be the number of zero modes per unit cell (or equivalently per surface wavenumber) of the right and left free surfaces that are joined by n_B bonds per cell. There is a zero mode at the domain wall whenever the zero modes of the free surfaces can be joined without stretching the added bonds creating the domain wall. There is a kind of compatibility matrix relating the stretches of the n_B bonds to the $n_{0R} + n_{0L}$ amplitudes of the surfaces modes on the two sides. The dimension of the null space of this matrix is the number of zero modes (per wavenumber) at the domain wall. Since there are no states of self-stress, the dimension of the range of this matrix is equal to n_B implying that the number of zero modes at the domain wall is

$$n_{0D} = n_{0R} + n_{0L} - n_B. \tag{4.19}$$

Thus in the example of Fig. 23, $n_{0L} = 2$, $n_{0R} = 1$, $n_B = 2$ and $n_{0D} = 2 + 1 - 2 = 1$. On the other hand if the lattice in fig. 23 (a) were joined with its mirror image, $n_{0R} = n_{0l} = 4$, $n_B = 4$ and $n_{0D} = 8 - 4 = 4$.

7. Elastic properties of topological Maxwell lattices: Each of fully gapped two-dimensional Maxwell lattice has one Guest mode. In the twisted kagome lattice, the mode is simply isotropic compression. It is more complicated in the family of distorted kagome lattices. Those with zero polarization generally have a negative and those with nonzero polarization have a positive Poisson ratio. See figure and Mathematic animation. The different in Guest modes lead to different long-wavelength phonon behavior with zero polarization having a linear dispersion and positive polarization having lines with quadratic dispersion. See Fig. 21 of RA.
Fig. P21

D. Other systems

1. Quasicrystal - RA Fig. 17 - like jammed system
2. Square lattices - Weyl modes
3. Pyrocholore - Weyl lines.

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[1] An exception is two-dimensional sheets in three dimensions that require second derivatives to describe bending and twisting of the sheet.

[2] λ_{ij} is sometimes called the strain tensor in the engineering literature. As we shall see below, we will define a slightly different quantity to be the strain tensor.

[3] This is a variation of the Voigt notation in which the off-diagonal components of ε_{ij} appear doubled in $\boldsymbol{\varepsilon}_V$, i.e., in two-dimensions, $\boldsymbol{\varepsilon}_V = (\varepsilon_{xx}, \varepsilon_{yy}, 2\varepsilon_{xy})$.