An Introduction To Bosonization And Some Of Its Applications

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Abstract

We discuss the technique of bosonization for studying systems of interacting fermions in one dimension. After briefly reviewing the low-energy properties of Fermi and Luttinger liquids, we present some of the relations between bosonic and fermionic operators in one dimension. We use these relations to calculate the correlation functions and the renormalization group properties of various operators for a system of spinless fermions. We then apply the methods of bosonization to study the Heisenberg antiferromagnetic spin 1/2 chain, the Hubbard model in one dimension, and transport in clean quantum wires and in the presence of isolated impurities.

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1 Fermi and Luttinger liquids

In two and three dimensions, many systems of interacting fermions at low temperatures are described by the Fermi liquid theory developed by Landau (see Ref. 1 for a brief review). According to this theory, at zero temperature, the ground state of each species of fermions has a Fermi surface in momentum space located at an energy called the Fermi energy E_F, such that all the states within that surface (*i.e.*, with energies less than E_F) are occupied while all the states outside it are unoccupied. An elementary low-energy excitation is one in which a particle is added (annihilated) in a state just outside (inside) the Fermi surface; these are called particle and hole excitations respectively. In an interacting system, these one-particle excitations are accompanied by a cloud of particle-hole pairs, and they are more commonly called quasiparticles; these carry the same charge as a single particle (or hole). If the particle number is held fixed, the low-energy excitations of the system consist of particle-hole pairs in which a certain number of particles are excited from states within the Fermi surface to states outside it. A few of these excitations have both low wave numbers and low energies with the energy being proportional to the wave number; such excitations can be thought of as sound waves. But most of the particle-hole excitations do not have such a linear relationship between energy and wave number; in fact, for most such excitations, a given energy can correspond to many possible momenta.

Another interesting property of a Fermi liquid in two and three dimensions is that the one-particle momentum distribution function $n(\vec{k})$, obtained by Fourier transforming the one-particle equal-time correlation function, has a finite discontinuity at the Fermi surface as shown in Fig. 1 (a). This discontinuity is called the quasiparticle renormalization factor $z_{\vec{k}}$; it is also equal to the residue of the pole in the one-particle propagator. For non-interacting fermions, $z_{\vec{k}} = 1$; but for interacting fermions, $0 < z_{\vec{k}} < 1$ because a quasiparticle is a superposition of many states, only some of which are one-particle excitations. To compute $z_{\vec{k}}$, we consider the one-particle Green's function $G(\vec{x}, t)$ defined as the expectation value of the time-ordered product of the fermion operator $\psi(\vec{x}, t)$ in the ground state $|0\rangle$, namely,

$$G(\vec{x}, t) \;=\; \langle 0|\; T\psi(\vec{x}, t)\psi^\dagger(\vec{0}, 0)\; |0\rangle \;. \tag{1}$$

(We will ignore the spin label here). The Fourier transform of this function can be written as

$$\mathcal{G}(\vec{k},\omega) \;=\; \frac{i}{\omega - \epsilon_{\vec{k}} - \Sigma(\vec{k},\omega)} \;, \tag{2}$$

where $\epsilon_{\vec{k}}$ is the dispersion relation for the non-interacting theory; we absorb the chemical potential μ in the definition of $\epsilon_{\vec{k}}$ so that $\epsilon_{\vec{k}} = 0$ for $\vec{k}$ lying on the Fermi surface. (We will set $\hbar = 1$). The self-energy $\Sigma(\vec{k},\omega)$ contains the effects of all the interactions as well as any prescription necessary to shift the pole slightly off the real axis in ω. For a Fermi liquid, $\mathcal{G}(\vec{k},\omega)$ has a pole near the real axis of ω for any value of $\vec{k}$ on the Fermi surface. In addition, Σ is sufficiently analytic at all such points so that the derivative $\partial\Sigma/\partial\omega$ has a finite value. The quasiparticle renormalization factor $z_{\vec{k}}$ is then given by the residue at the pole, *i.e.*,

$$z_{\vec{k}} \;=\; \Big(1 \;-\; \frac{\partial\Sigma}{\partial\omega} \Big)^{-1} \;. \tag{3}$$

This gives the discontinuity in $n(\vec{k})$ at the Fermi surface.

Finally, in a Fermi liquid, the various correlation functions decay asymptotically at long distances as power laws, with the exponents being independent of the strength of the interactions. Thus non-interacting and interacting systems have the same exponents and there is a universality.

The discussion above does not apply if the ground state of the system spontaneously breaks some symmetry, for instance, if it is superconducting, or forms a crystal or develops charge or spin density ordering.

In contrast to a Fermi liquid, interacting fermion systems in one dimension behave quite differently [2, 1, 3]; we will assume again that the ground state breaks no symmetry. Such systems are called Luttinger liquids and they have the following general properties. First of all, there are no single particle or quasiparticle excitations. Thus *all* the low-energy excitations can be thought of as particle-hole excitations; further, all of these take the form of sound waves with a linear dispersion relation. (As we will see below, there are also excitations of another kind possible which correspond to adding a small number of particles N_R and N_L to the right and left Fermi points. However, these correspond to only two oscillator degrees of freedom, and therefore do not contribute to thermodynamic properties like the specific heat). Secondly, there is no discontinuity in the momentum distribution function at the Fermi momentum, as indicated in Fig. 1 (b). Rather, there is a cusp there whose form is determined by a certain exponent. Finally, this exponent depends on the strength of the interactions in a non-universal manner, and it also governs the power-law fall-offs of the correlation functions at large space-time distances [4].

Let us be more specific about the nature of the low-energy excitations in a one-dimensional system of fermions. Assume that we have a system of length L with a boundary condition to be specified later. The translation invariance and the finite length make the one-particle momenta discrete. Suppose that the system has N_0 particles with a ground state energy $E_0(N_0)$ and a ground state momentum $P_0 = 0$; we are assuming that the system conserves parity. We will be interested in the thermodynamic limit $N_0, L \to \infty$ keeping the particle

density $\rho_0 = N_0/L$ fixed. Let us first consider a single species of non-interacting fermions which have two possible directions of motion, right-moving with $d\epsilon_k/dk = v_F$ and left-moving with $d\epsilon_k/dk = -v_F$. Here ϵ_k is the energy of a low-lying one-particle excitation, k is its momentum measured with respect to a right Fermi momentum k_F and a left Fermi momentum $-k_F$ respectively, and v_F is called the Fermi velocity. (See Fig. 2 for a typical picture of the momentum states of a lattice model). The values of k_F and v_F are defined for the non-interacting system; hence they depend on the density ρ_0 but not on the strength of the interaction. Then a low-lying excitation consists of two pieces [2],
(i) a set of bosonic excitations each of which can have either positive momentum q or negative momentum $-q$ with an energy $\epsilon_q = v_F q$, where $0 < q << k_F$, and
(ii) a certain number of particles N_R and N_L added to the right and left Fermi points respectively, where $N_R, N_L << N_0$. (Note that N_R and N_L can be positive, negative or zero. It is convenient to assume that $N_R \pm N_L$ are even integers; then the total number of particles $N_0 + N_R + N_L$ is always even or always odd. We can choose the boundary condition (periodic or antiperiodic) to ensure that the ground state is always non-degenerate).

It turns out that the Hamiltonian and momentum operators for a one-dimensional system (which may have interactions) have the general form

$$\begin{aligned} H &= E_0(N_0) + \sum_{q>0} vq \, [\, \tilde{b}^\dagger_{R,q}\tilde{b}_{R,q} + \tilde{b}^\dagger_{L,q}\tilde{b}_{L,q} \,] \\ &\quad + \mu(N_R + N_L) + \frac{\pi v}{2LK}(N_R + N_L)^2 + \frac{\pi v K}{2L}\,(N_R - N_L)^2 \,, \\ P &= \sum_{q>0} q \, [\, \tilde{b}^\dagger_{R,q}\tilde{b}_{R,q} - \tilde{b}^\dagger_{L,q}\tilde{b}_{L,q} \,] + [\, k_F + \frac{\pi}{L}(N_R + N_L) \,]\,(N_R - N_L) \,, \end{aligned} \tag{4}$$

where v is the sound velocity, q is the momentum of the low-energy bosonic excitations created and annihilated by $\tilde{b}^\dagger_q$ and $\tilde{b}_q$, K is a positive dimensionless number, and μ is the chemical potential of the system. We will see later that v and K are the two important parameters which determine all the low-energy properties of a system. Their values generally depend on both the strength of the interactions and the density. If the fermions are non-interacting, we have

$$v = v_F \quad \text{and} \quad K = 1 \,. \tag{5}$$

Note that one can numerically find the values of v and K by studying the $1/L$ dependence of the low-energy excitations of finite size systems.

It is interesting that the expression for the momentum operator in Eq. (4) is independent of the interaction strength. We can understand the last term in the momentum as follows. For a continuum system, the Fermi momentum $k_F(N)$ is related to the density by the relation

$$L \int_{-k_F(N)}^{k_F(N)} \frac{dk}{2\pi} = N \,. \tag{6}$$

Thus a system of N_0 particles has a Fermi momentum

$$k_F = \frac{\pi N_0}{L} = \pi\rho_0 \,, \tag{7}$$

while a system of $N = N_0 + N_R + N_L$ particles has a Fermi momentum equal to $k_F + (\pi/L)(N_R + N_L)$. If the N particles occupy the momenta states symmetrically about zero momentum, the total momentum of that state is zero; in this state, both the right and left Fermi points have $(N_R + N_L)/2$ particles more than the original ground state. Now let us shift $(N_R - N_L)/2$ particles from the left Fermi point to the right Fermi point, so that the right Fermi point has N_R particles more and the left Fermi point has N_L particles more than the original system. We then see that the total momentum has changed from zero to $[k_F + (\pi/L)(N_R + N_L)](N_R - N_L)$; this is the last term in the expression for the momentum operator.

The form of the parameterization of the last two terms in the Hamiltonian in Eq. (4) can be understood as follows. (Note that these two terms vanish in the thermodynamic limit and do not contribute to the specific heat. However they are required for the completeness of the theory up to terms of order $1/L$, and for a comparison with conformal field theory). Specifically, we will prove that if the coefficients of $(\pi/2L)(N_R + N_L)^2$ and of $(\pi/2L)(N_R - N_L)^2$ in Eq. (4) are denoted by A and B respectively, then

$$AB \;=\; v^2 \;. \tag{8}$$

It will then follow that if A is equal to v/K, B must be equal to vK. Although the expressions in Eq. (4) are valid for lattice models also, let us for simplicity consider a continuum model which is invariant under Galilean transformations. First, let us set $N_R = N_L$, so that we have added $\Delta N = 2N_R$ particles to the system. The sound velocity v of a one-dimensional system is related to the density of particles $\rho = N/L$ (where $N = N_0 + N_R + N_L$), the particle mass m, and the pressure $\mathcal{P}$ as

$$m\rho v^2 \;=\; -\; L \left(\frac{\partial \mathcal{P}}{\partial L} \right)_N \;. \tag{9}$$

The pressure is related to the ground state energy by $\mathcal{P} = -(\partial E_0/\partial L)_N$. Hence

$$m\rho v^2 \;=\; -\; L \left(\frac{\partial^2 E_0}{\partial L^2} \right)_N \;=\; \frac{N^2}{L} \left(\frac{\partial^2 E_0}{\partial N^2} \right)_L \;, \tag{10}$$

where the second equality follows from the first because E_0 depends on N and L only through the combination N/L. Comparing Eqs. (4) and (10), we see that the coefficient of $(\pi/2L)(\Delta N)^2$ is given by

$$A \;=\; \frac{mv^2}{\pi\rho_0} \;. \tag{11}$$

(In certain expressions such as Eq. (11), we have ignored the difference between ρ and ρ_0 since $\Delta N \ll N_0$). Next, let us take $N_L = -N_R$; this corresponds to moving N_R particles from the left Fermi point $-k_F$ to the right Fermi point k_F keeping the total number of particles equal to N_0. The change in momentum is therefore given by $\Delta P = 2\pi\rho_0 N_R$. Since we can also view such an excitation as a center of mass excitation with momentum ΔP, the

change in energy is given by $\Delta E = (\Delta P)^2/(2mN)$ since the total mass of the system is mN. It follows from this that the coefficient of $(\pi/2L)(N_R - N_L)^2$ satisfies

$$B \;=\; \frac{\pi\rho_0}{m} \;. \tag{12}$$

We thus see that $AB = v^2$ independently of the nature of the interactions between the particles.

We now consider the other important property of a Luttinger liquid, namely, the absence of a discontinuity in $n(k)$ at the Fermi momenta or, equivalently, the absence of a pole in the one-particle propagator. Thus the effect of interactions is so drastic in one dimension that the self-energy Σ in Eq. (2) becomes non-analytic at the Fermi points. As a result, $n(k)$ becomes continuous at $k = \pm k_F$ with the form

$$n(k) \;=\; n(k_F) \;+\; \text{constant} \;\cdot\; \text{sign}(k - k_F)\; |k - k_F|^\beta \;, \tag{13}$$

where $\text{sign}(z) \equiv 1$ if $z > 0$, -1 if $z < 0$ and 0 if $z = 0$. The exponent β is a positive number whose value depends on the strength of the interactions; for a non-interacting system, $\beta = 0$ and we recover the discontinuity in $n(k)$. Similarly, the density of states (DOS) is obtained by integrating Eq. (2) over all momenta; near zero energy it vanishes with a power-law form

$$\tilde{n}(\omega) \;\sim\; |\omega|^\beta \;, \tag{14}$$

which signals the absence of one-particle states in the low-energy spectrum. We will see later how the exponent β can be calculated in an interacting system called the Tomonaga-Luttinger model.

2 Bosonization

The basic idea of bosonization is that there are certain objects which can be calculated either in a fermionic theory or in a bosonic theory, and the two calculations give the same answer [2, 1, 3, 5, 7, 8]. Further, a particular quantity may seem very difficult to compute in one theory and may be easily calculable in the other theory. Bosonization works best in two space-time dimensions although there have been some attempts to extend it to higher dimensions.

In two dimensions, bosonization can be studied in either real time (Minkowski space) or in imaginary time (Euclidean space). In both cases, there is a one-to-one correspondence between the correlation functions of some fermionic and bosonic operators. We will work in real time here because bosonization has an added advantage in that case, namely, that there is a direct relationship between the creation and annihilation operators for a boson in terms of the corresponding operators for a fermion [8]. To show this, we just need to consider a bosonic and a fermionic Fock space. A Hamiltonian is *not* needed at this stage; we need to introduce a Hamiltonian only when discussing interactions and time-dependent correlation functions.

2.1 Bosonization of a fermion with one chirality

Let us begin by considering just one component, say, right-moving, of a single species of fermions on a circle of length L with the following boundary condition on the one-particle wave functions $\tilde{\psi}(x)$,

$$\tilde{\psi}(L) \;=\; e^{-i\pi\sigma}\;\tilde{\psi}(0)\;. \tag{15}$$

Thus $\sigma = 0$ and 1 correspond to periodic and antiperiodic boundary conditions, but any value of σ lying in the range $0 \leq \sigma < 2$ is allowed in principle. (If we assume that the particles are charged, then $\pi\sigma$ can be identified with an Aharonov-Bohm phase and can be varied by changing the magnetic flux through the circle). The normalized one-particle wave functions are then given by

$$\begin{aligned} \tilde{\psi}_{n_k} &= \frac{1}{\sqrt{L}}\; e^{ikx}\;, \\ k &= \frac{2\pi}{L}\;(n_k \;-\; \frac{\sigma}{2})\;, \end{aligned} \tag{16}$$

where $n_k = 0, \pm 1, \pm 2, ...$ is an integer. We now introduce a second quantized Fermi field

$$\psi_R(x) \;=\; \frac{1}{\sqrt{L}}\;\sum_{k=-\infty}^{\infty}\;c_{R,k}\;e^{ikx}\;, \tag{17}$$

where the subscript R stands for right-moving, and

$$\{c_{R,k}, c_{R,k'}\} \;=\; 0\;, \quad \text{and} \quad \{c_{R,k}, c^\dagger_{R,k'}\} \;=\; \delta_{kk'}\;. \tag{18}$$

Using the identity

$$\sum_{n=-\infty}^{\infty}\;e^{iny} \;=\; 2\pi\;\sum_{m=-\infty}^{\infty}\;\delta(y - 2\pi m)\;, \tag{19}$$

we obtain

$$\begin{aligned} \{\psi_R(x), \psi_R(x')\} &= 0\;, \\ \text{and} \quad \{\psi_R(x), \psi^\dagger_R(x')\} &= \delta(x - x') \quad \text{for} \quad 0 \leq x, x' \leq L\;. \end{aligned} \tag{20}$$

We define the vacuum or Fermi sea of the system to be the state $|0\rangle$ satisfying

$$\begin{aligned} c_{R,k}\;|0\rangle &= 0 \quad \text{for} \quad k > 0\;, \\ c^\dagger_{R,k}\;|0\rangle &= 0 \quad \text{for} \quad k \leq 0\;, \end{aligned} \tag{21}$$

as shown in Fig. 3. (Following this definition of the vacuum state, some people prefer to write the particle annihilation operator $c_{R,k}$ as a hole creation operator $d^\dagger_{R,-k}$ for $k \leq 0$). Given any operator A which can be written as a product of a string of c's and $c^\dagger$'s, we denote its normal ordered form by the symbol $:A:$. This new operator is defined by moving all the c_k with $k > 0$ and $c^\dagger_k$ with $k \leq 0$ to the right of all the c_k with $k \leq 0$ and $c^\dagger_k$ with $k > 0$. This

is achieved by transposing as many pairs of creation and annihilation operators as necessary, remembering to multiply by a factor of -1 for each transposition. (It is sometimes claimed that $:A:=A-\langle 0|A|0\rangle$. This is true if A is quadratic in the c's and $c^\dagger$'s, but it is not true in general).

Next we define the fermion number operator

$$\hat{N}_R \;=\; \sum_{k=-\infty}^{\infty} \;:c^\dagger_{R,k}c_{R,k}: \;=\; \sum_{k>0} c^\dagger_{R,k}c_{R,k} \;-\; \sum_{k\leq 0} c_{R,k}c^\dagger_{R,k}\;. \tag{22}$$

Thus $\hat{N}_R|0\rangle = 0$. Now consider all possible states $|\Psi\rangle$ satisfying $\hat{N}_R|\Psi\rangle = 0$. Clearly, any such state can only differ from $|0\rangle$ by a certain number of particle-hole excitations, *i.e.*, it must be of the form

$$|\Psi\rangle \;=\; c^\dagger_{R,k_1}c_{R,k_2}c^\dagger_{R,k_3}c_{R,k_4}c^\dagger_{R,k_5}c_{R,k_6}...|0\rangle\;, \tag{23}$$

where the k_i are all different from each other, $k_1, k_3, ... > 0$, and $k_2, k_4, ... \leq 0$. Two such excitations are shown in Fig. 4. We will now see that all such states can be written in terms of certain bosonic creation operators acting on the vacuum. Let us define the operators

$$\begin{aligned} b^\dagger_{R,q} &= \frac{1}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c^\dagger_{R,k+q}c_{R,k}\;, \\ b_{R,q} &= \frac{1}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c^\dagger_{R,k-q}c_{R,k}\;, \\ q &= \frac{2\pi}{L}\, n_q\;, \end{aligned} \tag{24}$$

where $n_q = 1, 2, 3, ...$ Note that we have defined the boson momentum label q to be positive. Also, the fermion boundary condition parameter σ does not appear in the definitions in Eq. (24). We can check that

$$\begin{aligned} [\hat{N}_R, b_{R,q}] &= [\hat{N}_R, b^\dagger_{R,q}] \;=\; 0\;, \\ [b_{R,q}, b_{R,q'}] &= 0\;, \\ [b_{R,q}, b^\dagger_{R,q'}] &= \delta_{qq'}\;. \end{aligned} \tag{25}$$

Checking the last identity for $q = q'$ is slightly tricky due to the presence of an infinite number of fermion momenta k. One way to derive the commutators is to multiply each c_k and $c^\dagger_k$ by a factor of $\exp[-\alpha|k|/2]$ in Eq. (24), and to let $\alpha \to 0$ at the end of the calculation. We should emphasize that the length scale α is not to be thought of as a short-distance cut-off like a lattice spacing; if we had introduced a lattice, the number of fermion modes would have been finite, and the bosonization formulas in Eq. (24) would not have given the correct commutation relations.

We see that the vacuum defined above satisfies $b_{R,q}|0\rangle = 0$ for all q. If we consider any operator A consisting of a string of b's and $b^\dagger$'s, we can define its bosonic normal ordered form $:A:$ by taking all the b_q's to the right of all the $b^\dagger_q$'s by suitable transpositions. Given

an operator A which can be written in terms of either fermionic or bosonic operators, normal ordering it in the fermionic and bosonic ways do not always give the same result. However, it will always be clear from the context which normal ordering we mean.

We can now begin to understand why bosonization works. First of all, note that there is a one-to-one correspondence between the particle-hole excitations described in Eq. (23) and the bosonic excitations created by the $b^\dagger$'s [9]. For instance, consider a bosonic excitation in which states with the momenta labeled by the integers $n_1 \geq n_2 \geq ... \geq n_j > 0$ (following the convention in Eq. (24)) are excited. Some of these integers may be equal to each other; that would mean that particular momenta has an occupation number greater than 1. Now we can map this excitation to a fermionic excitation in which j fermions occupying the states labeled by the momenta integers $0, -1, -2, ..., -j+1$ (following the convention in Eq. (16)) are excited to momenta labeled by $n_1, n_2 - 1, n_3 - 2, ..., n_j - j + 1$ respectively. This is clearly a one-to-one map, and we can reverse it to uniquely obtain a bosonic excitation from a given fermionic excitation. This mapping allows us to show, once an appropriate Hamiltonian is defined, that thermodynamic quantities like the specific heat are identical in the fermionic and bosonic models.

The above mapping makes it plausible, although it requires more effort to prove, that *all* particle-hole excitations can be produced by combinations of $b^\dagger$'s acting on the vacuum. For instance, the state in Fig. 4 (a) is given by $b^\dagger_{R,1}|0\rangle$. However the state in Fig. 4 (b) has a more lengthy expression in terms of bosonic operators, although it is also a single particle-hole excitation just like (a); to be explicit, it is given by the linear combination $(1/6)[2b^\dagger_{R,3} + 3b^\dagger_{R,2}b^\dagger_{R,1} + (b^\dagger_{R,1})^3]|0\rangle$.

Next, we define bosonic field operators and show that some bilinears in fermionic fields, such as the density $\rho_R(x)$, have simple expressions in terms of bosonic fields. Define the fields

$$\begin{aligned}
\chi_R(x) &= \frac{i}{2\sqrt{\pi}} \sum_{q>0} \frac{1}{\sqrt{n_q}}\, b_{R,q}\, e^{iqx - \alpha q/2} \,, \\
\chi^\dagger_R(x) &= -\frac{i}{2\sqrt{\pi}} \sum_{q>0} \frac{1}{\sqrt{n_q}}\, b^\dagger_{R,q}\, e^{-iqx - \alpha q/2} \,, \\
\phi_R(x) &= \chi_R(x) + \chi^\dagger_R(x) - \frac{\sqrt{\pi}x}{L}\, \hat{N}_R \,. \qquad (26)
\end{aligned}$$

The last term in the definition of $\phi_R(x)$ has been put in for later convenience; it simplifies the expressions for the Hamiltonian and the fermion density in terms of ϕ_R. (Some authors prefer not to include that term in the definition of ϕ_R but add it separately in the Hamiltonian and density). Note that $\hat{N}_R$ commutes with both χ_R and $\chi^\dagger_R$. From the commutation relations in Eq. (25), we see that

$$\begin{aligned}
[\chi_R(x), \chi_R(x')] &= 0 \,, \\
[\chi_R(x), \chi^\dagger_R(x')] &= -\frac{1}{4\pi} \ln \left[1 - \exp\left(-\frac{2\pi}{L}(\alpha + i(x - x')) \right) \right] , \\
&= -\frac{1}{4\pi} \ln \left[\frac{2\pi}{L}(\alpha + i(x - x')) \right] \quad \text{in the limit} \quad L \to \infty \,. \qquad (27)
\end{aligned}$$

Henceforth, the limit $L \to \infty$ will be assumed wherever convenient. We find that

$$\begin{aligned} [\phi_R(x), \phi_R(x')] &= \frac{1}{4\pi} \ln\Big[\frac{\alpha - i(x-x')}{\alpha + i(x-x')} \Big] , \\ &= -\frac{i}{4} \text{ sign } (x-x') \quad \text{in the limit} \quad \alpha \to 0 . \end{aligned} \tag{28}$$

Thus the commutator of two ϕ's looks like a step function which is smeared over a region of length α.

Now we use the operator identity

$$\exp A \ \exp B \ = \ \exp \ (A + B + \frac{1}{2}[A,B]) , \tag{29}$$

if $[A, B]$ commutes with both A and B. It follows that

$$\exp \ [i2\sqrt{\pi}\chi_R^\dagger(x)] \ \exp \ [i2\sqrt{\pi}\chi_R(x)] \ \exp \ [i\frac{2\pi x}{L}\hat{N}_R] = \Big(\frac{L}{2\pi\alpha}\Big)^{1/2} \ \exp \ [i2\sqrt{\pi}\phi_R(x)] . \tag{30}$$

We observe that the left hand side of this equation is normal ordered while the right hand side is not; that is why the two sides are related through a divergent factor involving L/α.

We can show that the fermion density operator is linear in the bosonic field, namely,

$$\begin{aligned} \rho_R(x) &= : \psi_R^\dagger(x)\psi_R(x) : \\ &= \frac{1}{L} \sum_{q>0} \sqrt{n}_q \ (\ b_{R,q}e^{iqx} \ + \ b_{R,q}^\dagger e^{-iqx} \ ) \ + \ \frac{1}{L} \sum_k \ : c_{R,k}^\dagger c_{R,k} : \\ &= -\frac{1}{\sqrt{\pi}} \frac{\partial \phi_R}{\partial x} . \end{aligned} \tag{31}$$

We now go in the opposite direction and construct fermionic field operators from bosonic ones. To do this, we first define the Klein factors η_R and $\eta_R^\dagger$ which are unitary operators satisfying

$$\begin{aligned} [\hat{N}_R, \eta_R^\dagger] &= \eta_R^\dagger , \quad [\hat{N}_R, \eta_R] \ = \ - \ \eta_R , \\ [\eta_R, b_{R,q}] &= [\eta_R, b_{R,q}^\dagger] \ = \ 0 . \end{aligned} \tag{32}$$

Pictorially, in terms of Figs. 3 and 4, the action of $\eta_R^\dagger$ is to raise all the occupied fermion states by one unit of momentum, while the action of η_R is to lower all the fermion occupied states by one unit of momentum. Although these actions are easy to describe in words, the explicit expressions for η_R and $\eta_R^\dagger = {\eta_R}^{-1}$ in terms of the c's and $c^\dagger$'s are rather complicated[8]. The Klein factors will be needed to ensure the correct anticommutation relations between the fermionic operators constructed below.

We observe that

$$\begin{aligned} [b_{R,q}, \psi_R(x)] &= -\frac{e^{-iqx}}{\sqrt{n}_q} \ \psi_R(x) , \\ [b_{R,q}^\dagger, \psi_R(x)] &= -\frac{e^{iqx}}{\sqrt{n}_q} \ \psi_R(x) . \end{aligned} \tag{33}$$

Since $b_{R,q}$ annihilates the vacuum, we have

$$b_{R,q}\ \psi_R(x)\ |0\rangle\ =\ -\ \frac{e^{-iqx}}{\sqrt{n}_q}\ \psi_R(x)\ |0\rangle\ . \tag{34}$$

Thus $\psi_R(x)|0\rangle$ is an eigenstate of $b_{R,q}$ for every value of q, namely, it is a coherent state. We therefore make the ansatz

$$\begin{aligned}\psi_R(x)\ |0\rangle\ &=\ Q(x)\ \exp\ [\ -\ \sum_{q>0}\ \frac{e^{-iqx}}{\sqrt{n}_q}\ b^\dagger_{R,q}\ |0\rangle\ ]\ |0\rangle\ ,\\ &=\ Q(x)\ \exp\ [\ -i2\sqrt{\pi}\ \chi^\dagger_R(x)\ ]\ |0\rangle\ ,\end{aligned} \tag{35}$$

where $Q(x)$ is some operator which commutes with all the b's and $b^\dagger$'s. Since ψ_R reduces the fermion number by 1, Q must contain a factor of η_R. Let us try the form $Q(x) = F(x)\eta_R$, where $F(x)$ is a c-number function of x. The form of F is determined by computing

$$\begin{aligned}F(x)\ &=\ \langle 0|\ \eta^\dagger_R\eta_R F(x)\ |0\rangle\\ &=\ \langle 0|\ \eta^\dagger_R\psi_R(x)\ |0\rangle\\ &=\ \frac{e^{-i\pi\sigma x/L}}{\sqrt{L}}\ .\end{aligned} \tag{36}$$

(The last line in Eq. (36) has been derived by using the actions of $\eta^\dagger_R$ above and of ψ_R in Eq. (17). To see this explicitly, note that $\langle 0|\eta^\dagger_R$ is the conjugate of the state in which the top most fermion has been removed from the vacuum. Hence, in $\psi_R|0\rangle$, we only have to consider the state in which the top most fermion has been removed; so we require the wave function of the state with $n_k = 0$ in Eq. (16)). We now obtain

$$\psi_R(x)\ |0\rangle\ =\ \frac{e^{-i\pi\sigma x/L}}{\sqrt{2\pi\alpha}}\ \eta_R\ e^{-i2\sqrt{\pi}\phi_R(x)}\ |0\rangle\ , \tag{37}$$

where we have used Eq. (30) and the fact that $\chi_R(x)$ and $\hat{N}_R$ annihilate the vacuum. We are thus led to the plausible conjecture

$$\psi_R(x)\ =\ \frac{e^{-i\pi\sigma x/L}}{\sqrt{2\pi\alpha}}\ \eta_R\ e^{-i2\sqrt{\pi}\phi_R(x)}\ . \tag{38}$$

To prove this, we need to show that the two sides of this equation have the same action on *all* states, not just the vacuum. Such a proof is given in Ref. 7. Eq. (38) is one of the most important identities in bosonization.

We next introduce a non-interacting Hamiltonian by defining the energy of the fermion mode with momentum k to be

$$\epsilon_k\ =\ v_F k \tag{39}$$

for all values of k. The Hamiltonian is

$$\begin{aligned} H_0 &= v_F \sum_{k=-\infty}^{\infty} k \ : c_{R,k}^\dagger c_{R,k} : \ + \ \frac{\pi v_F}{L} \ \hat{N}_R^2 \\ &= - v_F \int_0^L dx \ : \psi_R^\dagger i \partial_x \psi_R : \ + \ \frac{\pi v_F}{L} \ \hat{N}_R^2 \ . \end{aligned} \tag{40}$$

This defines the chiral Luttinger model. (The term proportional to $\hat{N}_R^2$ has been introduced in Eq. (40) so as to reproduce similar terms in Eq. (4) after we introduce left-moving fields in the next section). We can check that $H_0|0\rangle = 0$ and

$$\begin{aligned} [H_0, b_{R,q}] &= - v_F q \ b_{R,q} \ , \\ [H_0, b_{R,q}^\dagger] &= v_F q \ b_{R,q}^\dagger \ , \end{aligned} \tag{41}$$

To reproduce these relations in the bosonic language, we must have

$$\begin{aligned} H_0 &= v_F \sum_{q>0} q b_{R,q}^\dagger b_{R,q} \ + \ \frac{\pi v_F}{L} \ \hat{N}_R^2 \\ &= v_F \int_0^L dx \ : (\partial_x \phi)^2 : \ . \end{aligned} \tag{42}$$

We can introduce an interaction in this model which is quadratic in the fermion density. Let us consider the interaction

$$V \ = \ \frac{1}{2} \int_0^L g_4 \ {\rho_R}^2(x) \ = \ \frac{g_4}{2\pi} \sum_{q>0} q \ b_{R,q}^\dagger b_{R,q} \ . \tag{43}$$

Physically, such a term could arise if there is a short-range (*i.e.*, screened) Coulomb repulsion or a phonon mediated attraction between two fermions. We will therefore not make any assumptions about the sign of the interaction parameter g_4. If we add Eq. (43) to Eq. (42), we see that the only effect of the interaction in this model is to renormalize the velocity from v_F to $v_F + (g_4/2\pi)$.

In the next section, we will consider a model containing fermions with opposite chiralities; we will then see that a density-density interaction can have more interesting effects than just renormalizing the velocity.

2.2 Bosonization of a fermion with two chiralities

Let us consider a fermion with both right- and left-moving components as depicted in Figs. 3 and 5 respectively. For the left-moving fermions in Fig. 5, we define the momentum label k as increasing towards the left; the advantage of this choice is that the vacuum has the negative k states occupied and the positive k states unoccupied for both chiralities. We introduce a chirality label ν, such that $\nu = R$ and L refer to right- and left-moving particles

respectively. Sometimes we will use the numerical values $\nu = 1$ and -1 for R and L; this will be clear from the context. Let us choose periodic boundary conditions on the circle so that $\sigma = 0$. Then the Fermi fields are given by

$$\begin{aligned} \psi_\nu(x) &= \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} c_{\nu,k}\, e^{i\nu kx} \ , \\ k &= \frac{2\pi}{L}\, n_k \ , \end{aligned} \tag{44}$$

where $n_k = 0, \pm 1, \pm 2, ...$, and

$$\begin{aligned} \{c_{\nu,k}, c_{\nu',k'}\} &= 0 \ , \\ \{c_{\nu,k}, c^\dagger_{\nu',k'}\} &= \delta_{\nu\nu'}\ \delta_{kk'} \ . \end{aligned} \tag{45}$$

The vacuum is defined as the state satisfying

$$\begin{aligned} c_{\nu,k}\ |0\rangle &= 0 \quad \text{for} \quad k > 0 \ , \\ c^\dagger_{\nu,k}\ |0\rangle &= 0 \quad \text{for} \quad k \leq 0 \ . \end{aligned} \tag{46}$$

We can then define normal ordered fermion number operators $\hat{N}_\nu$ in the usual way.

Next we define bosonic operators

$$\begin{aligned} b^\dagger_{\nu,q} &= \frac{1}{\sqrt{n}_q} \sum_{k=-\infty}^{\infty} c^\dagger_{\nu,k+q} c_{\nu,k} \ , \\ b_{\nu,q} &= \frac{1}{\sqrt{n}_q} \sum_{k=-\infty}^{\infty} c^\dagger_{\nu,k-q} c_{\nu,k} \ . \end{aligned} \tag{47}$$

Note that $b^\dagger_{R,q}$ and $b^\dagger_{L,q}$ create excitations with momenta q and $-q$ respectively, where the label q is always taken to be positive. We can show as before that

$$[b_{\nu,q}, b_{\nu',q'}] \ = \ 0 \ , \quad \text{and} \quad [b_{\nu,q}, b^\dagger_{\nu',q'}] \ = \ \delta_{\nu\nu'}\ \delta_{qq'} \ . \tag{48}$$

The unitary Klein operators η_ν ($\eta^\dagger_\nu$) are defined to be operators which raise (lower) the momentum label k of all the occupied states for fermions of type ν. We then have

$$\begin{aligned} \{\eta_R, \eta_L\} &= \{\eta_R, \eta^\dagger_L\} \ = \ 0 \ , \\ [\hat{N}_\nu, \eta^\dagger_{\nu'}] &= \delta_{\nu\nu'}\ \eta^\dagger_{\nu'} \ , \quad [\hat{N}_\nu, \eta_{\nu'}] \ = \ - \ \delta_{\nu\nu'}\ \eta_{\nu'} \ , \\ [\eta_\nu, b_{\nu',q}] &= [\eta_\nu, b^\dagger_{\nu',q}] \ = \ 0 \ . \end{aligned} \tag{49}$$

We now define the chiral creation and annihilation fields

$$\begin{aligned} \chi_\nu(x) &= \frac{i\nu}{2\sqrt{\pi}} \sum_{q>0} \frac{1}{\sqrt{n}_q}\, b_{\nu,q}\ e^{i\nu qx - \alpha q/2} \ , \\ \chi^\dagger_\nu(x) &= - \frac{i\nu}{2\sqrt{\pi}} \sum_{q>0} \frac{1}{\sqrt{n}_q}\, b^\dagger_{\nu,q}\ e^{-i\nu qx - \alpha q/2} \ . \end{aligned} \tag{50}$$

Then

$$\begin{aligned}
[\chi_\nu(x), \chi'_\nu(x')] &= 0 \ , \\
[\chi_\nu(x), \chi^\dagger_{\nu'}(x')] &= -\frac{1}{4\pi}\,\delta_{\nu\nu'}\ \ln\,[\ \frac{2\pi}{L}(\ \alpha - i\nu(x-x')\ )\ ] \quad \text{in the limit} \quad L\to\infty \ . \qquad (51)
\end{aligned}$$

The chiral fields

$$\phi_\nu(x) \ = \ \chi_\nu(x) \ + \ \chi^\dagger_\nu(x) \ - \ \frac{\sqrt{\pi}x}{L}\ \hat{N}_\nu \ , \qquad (52)$$

satisfy

$$[\phi_\nu(x), \phi_{\nu'}(x)] \ = \ -\frac{i\nu}{4}\ \delta_{\nu\nu'}\ \text{sign}\ (x-x') \qquad (53)$$

in the limit $\alpha \to 0$. Finally, we can define two fields dual to each other

$$\begin{aligned}
\phi(x) &= \phi_R(x) \ + \ \phi_L(x) \ , \\
\theta(x) &= -\ \phi_R(x) \ + \ \phi_L(x) \ , \qquad (54)
\end{aligned}$$

such that $[\phi(x), \phi(x')] = [\theta(x), \theta(x')] = 0$, while

$$[\phi(x), \theta(x')] \ = \ \frac{i}{2}\ \text{sign}\ (x-x') \ . \qquad (55)$$

The fermion density operators $\rho_\nu(x) =: \psi^\dagger_\nu(x)\psi_\nu(x) :$ satisfy $\rho_\nu = \partial_x\phi_\nu/\sqrt{\pi}$. Hence the total density and current operators are given by

$$\begin{aligned}
\rho(x) &= \rho_R + \rho_L \ = \ -\ \frac{1}{\sqrt{\pi}}\ \partial_x\phi \ , \\
j(x) &= v_F(\rho_R - \rho_L) \ = \ \frac{v_F}{\sqrt{\pi}}\ \partial_x\theta \ , \qquad (56)
\end{aligned}$$

where v_F is a velocity to be introduced shortly.

We can again show that the fermionic fields are given in terms of the bosonic ones as

$$\begin{aligned}
\psi_R(x) &= \frac{1}{\sqrt{2\pi\alpha}}\ \eta_R\ e^{-i2\sqrt{\pi}\phi_R} \ , \\
\psi_L(x) &= \frac{1}{\sqrt{2\pi\alpha}}\ \eta_L\ e^{i2\sqrt{\pi}\phi_L} \ . \qquad (57)
\end{aligned}$$

As before, we introduce a linear dispersion relation $\epsilon_{\nu,k} = v_F k$ for the fermions. The non-interacting Hamiltonian then takes the form

$$\begin{aligned}
H_0 &= v_F \sum_{k=-\infty}^{\infty}\ k\ [\ :c^\dagger_{R,k}c_{R,k} \ + \ c^\dagger_{L,k}c_{L,k}\ :\ ] \ + \ \frac{\pi v_F}{L}(\hat{N}^2_R + \hat{N}^2_L) \\
&= -v_F\int_0^L dx\ [:\psi^\dagger_R(x) i\partial_x\psi_R(x) \ - \ \psi^\dagger_L(x) i\partial_x\psi_L(x):] \ + \ \frac{\pi v_F}{L}(\hat{N}^2_R + \hat{N}^2_L) \qquad (58)
\end{aligned}$$

in the fermionic language, and

$$\begin{aligned} H_0 &= v_F \sum_{q>0} q \, (\, b^\dagger_{R,q} b_{R,q} \, + \, b^\dagger_{L,q} b_{L,q} \,) \, + \, \frac{\pi v_F}{L} (\hat{N}_R^2 + \hat{N}_L^2) \\ &= v_F \int_0^L dx \, [\, : (\partial_x \phi_R)^2 \, + \, (\partial_x \phi_L)^2 : \,] \\ &= \frac{v_F}{2} \int_0^L dx \, [\, : (\partial_x \phi)^2 \, + \, (\partial_x \theta)^2 : \,] \end{aligned} \tag{59}$$

in the bosonic language. If we use this Hamiltonian to transform all the fields to time-dependent Heisenberg fields, we find that ψ_R, ϕ_R become functions of $x_R = x - v_F t$ while ψ_L, ϕ_L become functions of $x_L = x + v_F t$.

From Eq. (55), we see that the field canonically conjugate to ϕ is given by

$$\Pi \, = \, \partial_x \theta \, . \tag{60}$$

Thus

$$[\phi(x), \Pi(x')] \, = \, i\delta(x - x') \, , \tag{61}$$

and

$$H_0 \, = \, \frac{v_F}{2} \int_0^L dx \, [\, \Pi^2 \, + \, (\partial_x \phi)^2 \,] \, . \tag{62}$$

We now study the effects of four-fermi interactions. In the beginning it is simpler to work in the Schrödinger representation in which the fields are time-independent; we will transform to the Heisenberg representation later when we compute the correlation functions. Let us consider an interaction of the form

$$V \, = \, \frac{1}{2} \int_0^L dx \, [\, 2g_2 \, \rho_R(x) \rho_L(x) \, + \, g_4 \, (\, \rho_R^2(x) \, + \, \rho_L^2(x) \,) \,] \, . \tag{63}$$

Physically, we may expect an interaction such as $g : \rho^2(x) :$, so that $g_2 = g_4 = g$. However, it is instructive to allow g_2 to differ from g_4 to see what happens. For reasons explained before, we will again not assume anything about the signs of g_2 and g_4. In the fermionic language, the interaction takes the form

$$\begin{aligned} V = \frac{1}{2L} \sum_{k_1,k_2,k_3=-\infty}^{\infty} [\quad & 2g_2 c^\dagger_{R,k_1+k_3} c_{R,k_1} c^\dagger_{L,k_2+k_3} c_{L,k_2} \\ & + g_4 (c^\dagger_{R,k_1+k_3} c_{R,k_1} c^\dagger_{R,k_2-k_3} c_{R,k_2} \, + \, c^\dagger_{L,k_1+k_3} c_{L,k_1} c^\dagger_{L,k_2-k_3} c_{L,k_2})]. \end{aligned} \tag{64}$$

From this expression we see that g_2 corresponds to a two-particle scattering involving both chiralities; in this model, we can call it either forward scattering or backward scattering since there is no way to distinguish between the two processes in the absence of some other quantum number such as spin. The g_4 term corresponds to a scattering between two fermions with the same chirality, and therefore describes a forward scattering process.

The quartic interaction in Eq. (64) seems very difficult to analyze. However we will now see that it is easily solvable in the bosonic language; indeed this is one of the main motivations

behind bosonization. The bosonic expression for the total Hamiltonian $H = H_0 + V$ is found to be

$$\begin{aligned} H = \quad & \sum_{q>0} q \, [v_F \, (b^\dagger_{R,q} b_{R,q} + b^\dagger_{L,q} b_{L,q}) \; + \; \frac{g_2}{2\pi} (b^\dagger_{R,q} b^\dagger_{L,q} + b_{R,q} b_{L,q}) \; + \; \frac{g_4}{2\pi} (b^\dagger_{R,q} b_{R,q} + b^\dagger_{L,q} b_{L,q})] \\ & + \frac{\pi v_F}{L} \, (\, \hat{N}_R^2 \; + \; \hat{N}_L^2 \,) \; + \; \frac{g_2}{L} \, \hat{N}_R \hat{N}_L \; + \; \frac{g_4}{2L} \, (\, \hat{N}_R^2 \; + \; \hat{N}_L^2 \,) \; . \end{aligned} \tag{65}$$

The g_4 term again renormalizes the velocity. The g_2 term can then be rediagonalized by a Bogoliubov transformation. We first define two parameters

$$\begin{aligned} v \quad &= \quad \Big[\, (\, v_F + \frac{g_4}{2\pi} - \frac{g_2}{2\pi} \,) \, (\, v_F + \frac{g_4}{2\pi} + \frac{g_2}{2\pi} \,) \, \Big]^{1/2} \, , \\ K \quad &= \quad \Big[\, (\, v_F + \frac{g_4}{2\pi} - \frac{g_2}{2\pi} \,) \, / \, (\, v_F + \frac{g_4}{2\pi} + \frac{g_2}{2\pi} \,) \, \Big]^{1/2} \, . \end{aligned} \tag{66}$$

Note that $K < 1$ if g_2 is positive (repulsive interaction), and > 1 if g_2 is negative (attractive interaction). (If g_2 is so large that $v_F + g_4/(2\pi) - g_2/(2\pi) < 0$, then our analysis breaks down. The system does not remain a Luttinger liquid in that case, and is likely to go into a different phase such as a state with charge density order). The Bogoliubov transformation now takes the form

$$\begin{aligned} \tilde{b}_{R,q} \quad &= \quad \frac{b_{R,q} \; + \; \gamma \; b^\dagger_{L,q}}{\sqrt{1-\gamma^2}} \, , \\ \tilde{b}_{L,q} \quad &= \quad \frac{b_{L,q} \; + \; \gamma \; b^\dagger_{R,q}}{\sqrt{1-\gamma^2}} \, , \\ \gamma \quad &= \quad \frac{1-K}{1+K} \, , \end{aligned} \tag{67}$$

for each value of the momentum q. The Hamiltonian is then given by the quadratic expression

$$\begin{aligned} H \; = \quad & \sum_{q>0} v q \, [\, \tilde{b}^\dagger_{R,q} \tilde{b}_{R,q} \; + \; \tilde{b}^\dagger_{L,q} \tilde{b}_{L,q} \,] \\ & + \frac{\pi v}{2L} \, [\, \frac{1}{K} \, (\hat{N}_R + \hat{N}_L)^2 \; + \; K \, (\hat{N}_R - \hat{N}_L)^2 \,] \, . \end{aligned} \tag{68}$$

Equivalently,

$$H \; = \; \frac{1}{2} \, \int_0^L dx \, [\, v K \Pi^2 \; + \; \frac{v}{K} \, (\partial_x \phi)^2 \,] \, . \tag{69}$$

The old and new fields are related as

$$\begin{aligned} \phi_R \quad &= \quad \frac{(1+K) \, \tilde{\phi}_R \; - \; (1-K) \, \tilde{\phi}_L}{2\sqrt{K}} \, , \\ \phi_L \quad &= \quad \frac{(1+K) \, \tilde{\phi}_L \; - \; (1-K) \, \tilde{\phi}_R}{2\sqrt{K}} \, , \\ \phi \quad &= \quad \sqrt{K} \, \tilde{\phi} \quad \text{and} \quad \theta \; = \; \frac{1}{\sqrt{K}} \, \tilde{\theta} \, . \end{aligned} \tag{70}$$

Note the important fact that the vacuum changes as a result of the interaction; the new vacuum $|\tilde{0}\rangle$ is the state annihilated by the operators $\tilde{b}_{\nu,q}$. Since the various correlation functions must be calculated in this new vacuum, they will depend on the interaction through the parameters v and K. In particular, we will see in the next section that the power-laws of the correlation functions are governed by K.

Given the various Hamiltonians, it is easy to guess the forms of the corresponding Lagrangians. For the non-interacting theory ($g_2 = g_4 = 0$), the Lagrangian density describes a massless Dirac fermion,

$$\mathcal{L} \;=\; i\psi_R^\dagger(\partial_t + v_F\partial_x)\psi_R \;+\; i\psi_L^\dagger(\partial_t - v_F\partial_x)\psi_L \tag{71}$$

in the fermionic language, and a massless real scalar field,

$$\mathcal{L} \;=\; \frac{1}{2v_F}\,(\partial_t\phi)^2 \;-\; \frac{v_F}{2}\,(\partial_x\phi)^2 \tag{72}$$

in the bosonic language. For the interacting theory in Eq. (69), we find from Eq. (70) that

$$\mathcal{L} \;=\; \frac{1}{2vK}\,(\partial_t\phi)^2 \;-\; \frac{v}{2K}\,(\partial_x\phi)^2 \;=\; \frac{1}{2v}\,(\partial_t\tilde{\phi})^2 \;-\; \frac{v}{2}\,(\partial_x\tilde{\phi})^2\;. \tag{73}$$

The momentum operator in Eq. (4) has the same expression in terms of the old and new fields, namely,

$$P \;=\; k_F(\hat{N}_R - \hat{N}_L) \;+\; \int_0^L dx\;\partial_x\phi\partial_x\theta\;. \tag{74}$$

We can check that $[P,\phi] = -i\partial_x\phi$ and $[P,\theta] = -i\partial_x\theta$.

Let us now write down the fields $\tilde{\phi}$ and $\tilde{\theta}$ in the Heisenberg representation. This is simple to do once we realize that the right- and left-moving fields must be functions of $x_R = x - vt$ and $x_L = x + vt$ respectively. We find that

$$\begin{aligned}
\tilde{\phi}(x,t) = \frac{i}{2\sqrt{\pi}}\;\sum_{q>0}\frac{1}{\sqrt{n}_q}[\;\; & \tilde{b}_{R,q}\;e^{iq(x_R+i\alpha/2)} \;-\; \tilde{b}_{R,q}^\dagger\;e^{-iq(x_R-i\alpha/2)} \;-\; \tilde{b}_{L,q}\;e^{-iq(x_L-i\alpha/2)} \\
& + \tilde{b}_{L,q}^\dagger\;e^{iq(x_L+i\alpha/2)}\;] \\
-\frac{\sqrt{\pi}}{L}[\;\; & \frac{x}{\sqrt{K}}\;(\hat{N}_R \;+\; \hat{N}_L) \;-\; \sqrt{K}vt(\hat{N}_R \;-\; \hat{N}_L)\;]\;, \\
\tilde{\theta}(x,t) = \frac{i}{2\sqrt{\pi}}\;\sum_{q>0}\frac{1}{\sqrt{n}_q}[\;\; & -\;\tilde{b}_{R,q}\;e^{iq(x_R+i\alpha/2)} \;+\; \tilde{b}_{R,q}^\dagger\;e^{-iq(x_R-i\alpha/2)} \;-\; \tilde{b}_{L,q}\;e^{-iq(x_L-i\alpha/2)} \\
& + \tilde{b}_{L,q}^\dagger\;e^{iq(x_L+i\alpha/2)}\;] \\
+\frac{\sqrt{\pi}}{L}[\;\; & \sqrt{K}x\;(\hat{N}_R \;-\; \hat{N}_L) \;-\; \frac{vt}{\sqrt{K}}\;(\hat{N}_R \;+\; \hat{N}_L)\;]\;.
\end{aligned} \tag{75}$$

We observe that the coefficients of $\hat{N}_R$ and $\hat{N}_L$ have terms which are linear in t. This is necessary because we want the conjugate momentum field to satisfy

$$\tilde{\Pi} \;=\; \frac{1}{v}\,\partial_t\tilde{\phi} \;=\; \partial_x\tilde{\theta}\;. \tag{76}$$

We note that a dual equation holds, namely,

$$\frac{1}{v}\,\partial_t\tilde{\theta} \;=\; \partial_x\tilde{\phi}\;. \tag{77}$$

(One can check from Eq. (75) that $\tilde{\phi}_R$ and $\tilde{\phi}_L$ are functions of x_R and x_L alone). In terms of θ, the Lagrangian density is

$$\mathcal{L} \;=\; \frac{K}{2v}\,(\partial_t\theta)^2 \;-\; \frac{Kv}{2}\,(\partial_x\theta)^2 \;=\; \frac{1}{2v}\,(\partial_t\tilde{\theta})^2 \;-\; \frac{v}{2}\,(\partial_x\tilde{\theta})^2\;. \tag{78}$$

Although the Lagrangians in Eq. (73) and Eq. (78) have opposite signs, the Hamiltonians derived from the two are identical.

Before ending this section, let us comment on a global symmetry of all these models. It is known that fermionic systems with a conserved charge are invariant under a global phase rotation

$$\psi_R \;\rightarrow\; e^{i\lambda}\,\psi_R\;, \quad \text{and} \quad \psi_L \;\rightarrow\; e^{i\lambda}\,\psi_L\;, \tag{79}$$

where λ is independent of (x,t). Eq. (57) then implies that the corresponding bosonic theories must remain invariant under

$$\phi \;\rightarrow\; \phi\;, \quad \text{and} \quad \theta \;\rightarrow \theta + \;\frac{\lambda}{\sqrt{\pi}}\;. \tag{80}$$

This provides a constraint on the kinds of terms which can appear in the Lagrangians of such theories.

2.3 Field theory of modes near the Fermi momenta

In the last section, we discussed bosonization for a model of fermions which has the following properties.
(i) There are an infinite number of right- and left-moving modes with the momenta going from $-\infty$ to ∞, and
(ii) the relation between energy and momentum is linear for all values of the momentum.
Neither of these properties is true in condensed matter systems which typically are non-relativistic and have a finite (though possibly very large) number of states. The question is the following: can bosonization give useful results even if these two properties do not hold? We will see that the answer is yes, provided that we are only interested in the long-wavelength, low-frequency and low-temperature properties of such systems.

In an experimental system, the fermions may be able to move either on a discrete lattice of points such as in a crystal, or in a continuum such as the conduction electrons in a metal. For instance, non-interacting fermions moving in a continuum have a dispersion $\epsilon_k = k^2/2m$, while fermions hopping on a lattice have a dispersion such as $\epsilon_k = -t\cos(ka)$ if a is the lattice spacing and t is the nearest neighbor hopping amplitude. In either case, a non-interacting system in one-dimension will, at zero temperature, have a Fermi surface consisting of two

points in momentum space given by $k = \pm k_F$ (see Fig. 2). As stated before, we define the one-particle energy to be zero at the Fermi points. At low temperatures T or low frequencies ω, the only modes which can contribute are the ones lying close to those points, *i.e.*, with excitation energies of the order of or smaller than $k_B T$ or ω. Near the Fermi points, we can approximate the dispersion relation by a linear one, with the velocity being defined to be $v_F = (d\epsilon_k/dk)_{k=k_F}$. We thus restrict our attention to the right-moving modes with momenta lying between $k_F - \Lambda$ and $k_F + \Lambda$, and the left-moving modes with momenta lying between $-k_F - \Lambda$ and $-k_F + \Lambda$. Here Λ is taken to be much smaller than the full range of the momentum (which is $2\pi/a$ on a lattice if the lattice spacing is a), but $v_F \Lambda$ is much larger than the temperatures or frequencies of interest. If we include only these regions of momenta, then the second quantized Fermi field can be written in the approximate form

$$\psi(x,t) \; = \; \psi_R(x,t) \; e^{ik_F x} \; + \; \psi_L(x,t) \; e^{-ik_F x} \; , \tag{81}$$

where ψ_R and ψ_L vary slowly over spatial regions which are large compared to the distance scale $1/\Lambda$. The momentum components of these slowly varying fields are related to those of ψ as

$$\begin{aligned} \psi_{R,k}(t) &= \psi_{k+k_F}(t) \; , \\ \psi_{L,k}(t) &= \psi_{-k-k_F}(t) \; , \end{aligned} \tag{82}$$

where $-\Lambda \leq k \leq \Lambda$. These long-wavelength fields are the ones to which the technique of bosonization can be applied.

The definitions in Eqs. (81-82) tell us the forms of the various terms in a microscopic model and also tell us which of them survive in the long-wavelength limit. For instance, the density is given by

$$\begin{aligned} \rho &= \; : \psi^\dagger \psi : \; = \; : \psi_R^\dagger \psi_R \; + \; \psi_L^\dagger \psi_L \; + \; e^{-i2k_F x} \; \psi_R^\dagger \psi_L \; + \; e^{i2k_F x} \; \psi_L^\dagger \psi_R : \\ &= \; - \; \frac{1}{\sqrt{\pi}} \; \frac{\partial \phi}{\partial x} \; + \; \frac{1}{2\pi\alpha} \; [\eta_R^\dagger \eta_L e^{i(2\sqrt{\pi}\phi - 2k_F x)} \; + \; \eta_L^\dagger \eta_R e^{-i(2\sqrt{\pi}\phi - 2k_F x)} \;] \; . \end{aligned} \tag{83}$$

The terms containing $\exp(\pm i 2k_F x)$ in Eq. (83) vary on a distance scale k_F^{-1} which is typically of the same order as the inverse particle density ρ^{-1}. These terms can therefore be ignored if we are only interested in the asymptotic behavior of correlation functions at distances much larger than k_F^{-1}. In a lattice model, we have to be more careful about this argument since the lattice momentum only needs to be conserved modulo $2\pi/a$ in any process. However, since $0 < k_F < \pi/a$ in general, and x/a is an integer, we see that the last two terms in Eq. (83) vary on the scale of the lattice unit a; we can therefore ignore those terms if we are only interested in phenomena at distance scales which are much larger than a.

On the other hand, there are situations when a density term like $\rho \cos(2k_F x)$ is generated in the model; for instance, this happens below a Peierls transition if the fermions are coupled to lattice phonons. We then find that the slowly varying terms in the continuum field theory

are given by

$$\begin{aligned} \cos(2k_F x)\ \rho &= \frac{1}{2}\ [\ \psi_R^\dagger \psi_L\ +\ \psi_L^\dagger \psi_R\ ] \quad \text{in general}\ , \\ &= \psi_R^\dagger \psi_L\ +\ \psi_L^\dagger \psi_R \quad \text{if} \quad e^{i4k_F x}\ =\ 1\ . \end{aligned} \tag{84}$$

The second possibility can arise in a lattice model if $4k_F a = 2\pi$, *i.e.*, at half-filling; we then call it a dimerized system. We will call the term on the right hand sides of Eq. (84) the mass operator. We will see below that for any value of $K < 2$, this term produces a gap in the low-energy spectrum. This is called the dimerization gap if it occurs in a lattice system.

We should emphasize an important difference between models defined in the continuum and those defined on a lattice. In the continuum, $\psi_R^2(x) = \psi_L^2(x) = 0$ due to the anticommutation relations. Therefore a term like $\psi_R^{\dagger 2}(x)\psi_L^2(x)$ is equal to zero in the continuum. However such a term need not vanish on a lattice, if we take the two factors of $\psi_R^\dagger$ (or ψ_L) as coming from two neighboring sites separated by a distance a. In fact, this term is allowed by momentum conservation on a lattice if $4k_F a = 2\pi$, and it leads to umklapp scattering.

3 Correlation functions and dimensions of operators

We will now use bosonization to compute the correlation functions of some fermionic operators in the interacting theory discussed above. The power-law fall-offs of the correlation functions will tell us the dimensions of those operators.

The bosonic correlation function can be found from the commutation relations in Eq. (51), remembering that all normal-orderings have to be done with respect to the new vacuum $|\tilde{0}\rangle$. (Henceforth we will omit the tilde denoting the new vacuum, but we will continue to use the tilde for the new ϕ fields). For instance,

$$\langle 0|\ T\tilde{\phi}(x,t)\tilde{\phi}^\dagger(0,0)\ |0\rangle\ =\ \frac{1}{4\pi}\ \ln\ \Big[\ \Big(\ \frac{2\pi}{L}\ \Big)^2\ \Big(\ x^2\ -\ (vt - i\alpha\ \mathrm{sign}(t))^2\ \Big)\ \Big]\ . \tag{85}$$

We can use the expressions in Eq. (57) and identities like Eq. (30) to obtain the correlation functions of various operators. For instance,

$$\begin{aligned} \langle 0|\ T e^{i2\sqrt{\pi}\beta\tilde{\phi}_R(x,t)} e^{-i2\sqrt{\pi}\beta\tilde{\phi}_R(0,0)}\ |0\rangle &\sim \Big(\ \frac{\alpha}{vt - x - i\alpha\ \mathrm{sign}(t)}\Big)^{\beta^2}\ , \\ \langle 0|\ T e^{i2\sqrt{\pi}\beta\tilde{\phi}_L(x,t)} e^{-i2\sqrt{\pi}\beta\tilde{\phi}_L(0,0)}\ |0\rangle &\sim \Big(\ \frac{\alpha}{vt + x - i\alpha\ \mathrm{sign}(t)}\Big)^{\beta^2}\ , \end{aligned} \tag{86}$$

in the limit $L \to \infty$; we will assume henceforth that this limit is taken in the calculation of all correlation functions. Consider now the positive-chirality fermion field; according to Eq. (57),

$$\psi_R\ =\ \frac{1}{\sqrt{2\pi\alpha}}\ \eta_R\ e^{-i2\sqrt{\pi}\phi_R}\ , \tag{87}$$

where ϕ_R is given in Eq. (70) in terms of $\tilde{\phi}_R$ and $\tilde{\phi}_L$. Hence its time-ordered correlation function takes the form

$$\langle 0|T\psi_R(x,t)\psi_R^\dagger(0,0)|0\rangle \sim \frac{\alpha^{(1-K)^2/2K}}{2\pi(vt-x-i\alpha\mathrm{sign}(t))^{(1+K)^2/4K}(vt+x-i\alpha\mathrm{sign}(t))^{(1-K)^2/4K}}. \tag{88}$$

We see that the correlation function falls off at large space-time distances (*i.e.*, large compared to α) with the power $(1+K^2)/(2K)$. This means that the scaling dimension of the operator ψ_R or $\psi_R^\dagger$ is $(1+K^2)/4K$; this agrees with the familiar value of $1/2$ for non-interacting fermions.

If we set $x=0$ in Eq. (88), and Fourier transform over time, we find that the one-particle density of states (DOS) has a power-law form near zero frequency,

$$\tilde{n}(\omega) \;\sim\; |\omega|^\beta \;, \tag{89}$$

where

$$\beta \;=\; \frac{(1-K)^2}{2K} \;. \tag{90}$$

The same result holds for the DOS of the negative-chirality fermions. We therefore see that for any non-zero interaction, either repulsive or attractive, the one-particle DOS vanishes as a power. (This result is not to be confused with the *bosonic* DOS which, from Eq. (68), is a constant near zero energy since the energy is linearly related to the momentum which has a constant density. That leads to a specific heat which is linear in the temperature at low temperatures). Alternatively, we may set $t=0$ in Eq. (88) and Fourier transform over space, with a factor of $\exp(ik_Fx)$ since the momentum of the right-chirality fermions is measured with respect to the Fermi momentum k_F. We then see that the momentum distribution function is continuous at k_F with a power-law form,

$$n(k) \;=\; n(k_F) \;+\; \text{constant} \;\cdot\; \mathrm{sign}(k-k_F)\;|k-k_F|^\beta \;, \tag{91}$$

as we have sketched in Fig. 1 (b). These expressions for $\tilde{n}(\omega)$ and $n(k)$ are characteristic features of a Luttinger liquid.

Next let us compute the correlation function of an operator which is bilinear in the fermion fields, namely, the mass operator

$$M \;=\; \psi_R^\dagger\psi_L \;+\; \psi_L^\dagger\psi_R \;=\; \frac{1}{2\pi\alpha}\left[\;\eta_R^\dagger\eta_L\; e^{i2\sqrt{\pi}\phi} \;+\; \eta_L^\dagger\eta_R\; e^{-i2\sqrt{\pi}\phi}\;\right]. \tag{92}$$

Using the same technique as before, we find that

$$\langle 0|\; TM(x,t)M(0,0)\;|0\rangle \;\sim\; \frac{\alpha^{2(K-1)}}{4\pi^2\;((vt-i\alpha\;\mathrm{sign}(t))^2 \;-\; x^2)^K} \;. \tag{93}$$

This shows that the scaling dimension of the mass operator is K. For the non-interacting case $K=1$, we see that the addition of such a term to the Lagrangian density in Eq. (71)

makes the Dirac fermion massive; this is why we have called it the mass operator. (For convenience, we will sometimes omit the Klein factors when writing fermionic operators in the bosonic language. We will of course need to restore those factors when calculating the correlation functions; clearly, correlation functions will vanish if the numbers of η_R and $\eta_R^\dagger$ (or η_L and $\eta_L^\dagger$) are not equal).

An important operator to consider is the density ρ. From Eqs. (83), (85) and (93), we see that the density-density equal-time correlation function is asymptotically given by

$$\langle 0|\ \rho(x,0)\rho(0,0)\ |0\rangle \ = \ - \ \frac{K}{2\pi^2 x^2} \ + \ \text{const} \ \cdot \ \frac{\cos(4k_F x)}{x^{2K}} \ . \tag{94}$$

We should emphasize that this is only the asymptotic expression; the complete expression generally contains oscillatory terms like $\cos(4nk_F x)/x^{2n^2K}$ for all positive integers n. However the form of the denominator shows that these terms decay rapidly with x as n increases.

In general, we can consider an operator of the form

$$O_{m,n} \ = \ e^{i2\sqrt{\pi}(m\phi+n\theta)} \ . \tag{95}$$

(Such an operator can arise from a product of several ψ's and $\psi^\dagger$'s if we ignore the Klein factors; then Eq. (57) implies that $m \pm n$ must take integer values). We then find the following result for the two-point correlation function

$$\begin{aligned} &\langle 0|\ T O_{m,n}(x,t) O^\dagger_{m',n'}(0,0)\ |0\rangle \\ &\sim \delta_{mm'}\delta_{nn'} \frac{\alpha^{2(m^2K+n^2/K)}}{(vt-x-i\alpha\text{sign}(t))^{(m\sqrt{K}-n/\sqrt{K})^2}(vt+x-i\alpha\text{sign}(t))^{(m\sqrt{K}+n/\sqrt{K})^2}} \ , \end{aligned} \tag{96}$$

where we have taken the limit $L \to \infty$ as usual. (If L had been kept finite, the correlation function in Eq. (96) would have been non-zero even if $m \neq m'$ or $n \neq n'$. This may seem surprising since the global phase invariance in Eqs. (79 - 80) should lead to the Kronecker δ's in Eq. (96) even for finite values of L. The resolution of this puzzle is that we need to include the appropriate Klein factors in the definition in (95) to show that the correlation function of a product of fermionic operators is zero if it is not phase invariant). We conclude that the scaling dimension of $O_{m,n}$ is given by

$$d_O \ = \ m^2 K \ + \ \frac{n^2}{K} \ . \tag{97}$$

The appearance of the cut-off α in the expressions for the various correlation functions may seem bothersome. This may be eliminated by redefining the operators $O_{m,n}$ in Eq. (95) by multiplying them with appropriate K-dependent powers of α; then the two-point correlation function has a well-defined limit as $\alpha \to 0$. The important point to note is that all the correlation functions fall off as power-laws asymptotically, and that the exponents give the scaling dimensions of those operators. The significance of the scaling dimension will be discussed in the next section.

For certain applications of bosonization, it is useful to know the forms of the correlation functions in imaginary time. From the various expressions above, it is clear that if x is held fixed at some non-zero value, then the poles in the complex t plane are either in the first or in the third quadrant. We may therefore rotate t by $\pi/2$ without crossing any poles. After doing this, we write $t = i\tau$ where τ is a real variable. Eq. (96) then takes the form

$$\langle\ O_{m,n}(x,t)O^{\dagger}_{m',n'}(0,0)\rangle\ \sim\ \delta_{mm'}\delta_{nn'}\ e^{i4mn\zeta}\ \Big(\ \frac{\alpha^2}{x^2+v^2\tau^2}\ \Big)^{m^2K+n^2/K}\ , \tag{98}$$

where $\zeta = \tan^{-1}(v\tau/x)$, and we have dropped the α-dependent terms in the denominator since there are no longer any poles for non-zero values of x.

4 Renormalization group analysis of perturbed models

We will now study the effects of some perturbations on the low-energy properties of Luttinger liquids. A standard way to do this is to use the renormalization group (RG) idea. Suppose that we are given an action at a microscopic length scale which may be a lattice spacing a; the action contains some small perturbations proportional to certain dimensionless parameters λ_i, such that for $\lambda_i = 0$, we have a gapless system with an infinite correlation length ξ, *i.e.*, all correlations fall off as power laws. Then the RG procedure typically consists of the following steps.
(i) First, a small range of high momentum modes of the various fields are integrated out. Specifically, we will assume that the momenta lie in the range $[-\Lambda, \Lambda]$ while the frequencies go all the way from $-\infty$ to ∞. Then we will integrate out the modes with momenta lying in the two intervals $[-\Lambda, -\Lambda/s]$ and $[\Lambda/s, \Lambda]$ and with all frequencies from $-\infty$ to ∞. Here $s = e^{dl}$ where dl is a small positive number. The asymmetry between the momentum and frequency integrations is necessary to ensure that the action remains local in time at all stages. (Note that we are using sharp momentum cut-offs in this section, whereas we used a smooth momentum cut-off with the parameter α in the previous sections).
(ii) Secondly, the space-time coordinates, the fields and the various parameters are rescaled by appropriate powers of s so that the new action looks exactly like the old action. This new action is effectively at a larger length scale equal to ae^{dl}. Clearly, the changes in the parameters λ_i must be proportional to the small number dl. Since we are going to repeat the process of integrating out high momentum modes, we introduce the idea of an effective length scale $a(l) = ae^l$; we also define length scale dependent parameters $\lambda_i(l)$, where $\lambda_i(0)$ denote the values of λ_i in the original action. We then define the β-functions

$$\beta(\lambda_i)\ =\ \frac{d\lambda_i(l)}{dl}\ . \tag{99}$$

These are functions of all the $\lambda_i(l)$'s so that we get a set of coupled non-linear equations in general. In principle, the β-functions are given by infinite power series in the λ_i, but

in practice, we can only obtain the first few terms depending on the number of loops of the various Feynman diagrams that we can compute. The RG analysis is therefore usually limited to small values of λ_i.
(iii) Finally, we integrate the RG equations, *i.e.*, the differential equations described by the β-functions, in order to obtain the functions $\lambda_i(l)$. For simplicity, let us consider the case of a single perturbation with a coefficient λ. Then one of three things can happen as l increases from 0. Either $\lambda(l)$ goes to zero in which case we recover the unperturbed theory at long distances; or λ does not change with l; or $\lambda(l)$ grows with l till its value becomes of order 1. In the last case, the RG equation cannot be trusted beyond that length scale since the β-functions are generally only known up to some low order in the λ's. All that we can say is that beyond the length scale ae^l where $\lambda(l)$ becomes of order 1, a completely new kind of action is likely to be required to describe the system. Large perturbations of a gapless system often (but not always) correspond to a gapped system whose correlation length ξ (which governs the exponential decay of various correlation functions) is of the same order as that length scale ae^l. Thus, although the blowing up of a parameter λ at some scale does not tell us what the new action must be beyond that scale, it can give us an idea of the correlation length of that new theory. This is the main use that is made of RG equations. To complete the picture and find the new theory beyond the scale ξ, one usually has to do some other kind of analysis.

Let us now examine in a little more detail the various kinds of RG equations which can arise at low orders. Suppose that to first order, the RG equation for a single perturbing term is given by

$$\frac{d\lambda}{dl} \;=\; b_1\lambda \;, \tag{100}$$

where b_1 is some constant. If $b_1 < 0$, any non-zero value of λ at $l = 0$ flows to 0 as l increases. Such a perturbation is called irrelevant. If $b_1 > 0$, it is called a relevant perturbation. A small perturbation then grows exponentially with l and reaches a number of order 1 at a distance scale given by $e^{b_1 l} \sim 1/\lambda(0)$. In many models, this gives an estimate of the correlation length ξ and of the energy gap ΔE of the system, namely,

$$\begin{aligned} \xi \;&=\; ae^l \;=\; \frac{a}{\lambda(0)^{1/b_1}} \;, \\ \text{so} \quad \Delta E \;&=\; \frac{v}{\xi} \;=\; \frac{v\lambda(0)^{1/b_1}}{a} \;. \end{aligned} \tag{101}$$

Finally, if $b_1 = 0$, the perturbation is called marginal. One then has to go to second order in λ. If the RG equation takes the form

$$\frac{d\lambda}{dl} \;=\; b_2\lambda^2 \;, \tag{102}$$

then a small perturbation of one particular sign flows to zero and is called marginally irrelevant, while a small perturbation of the opposite sign grows and is called marginally relevant.

For instance, suppose that $b_2 > 0$. Then the above equation gives

$$\lambda(l) \;=\; \frac{\lambda(0)}{1 \;-\; b_2\lambda(0)l} \;. \tag{103}$$

If we start with a negative value of $\lambda(0)$, $\lambda(l)$ flows to 0. For large l, $\lambda(l)$ goes to zero logarithmically in the distance scale as $-1/(b_2 l)$ independently of the starting value. (It turns out that this produces logarithmic corrections to the power-law fall-offs of the correlation functions at large distances and the various excitation energies [10]). On the other hand, if we start with a small positive value of $\lambda(0)$, then $\lambda(l)$ grows and becomes of order 1 at a distance scale which we identify with a correlation length

$$\xi = a e^{1/(b_2\lambda(0))} \;. \tag{104}$$

The corresponding energy gap $\Delta E = v/\xi$ is extremely small for small values of $\lambda(0)$; it may be very hard to distinguish this kind of a system from a gapless system by numerical studies. This is in sharp contrast to the situation with a relevant perturbation where the gap scales as a power of $\lambda(0)$.

There is a simple relation between the scaling dimension of an operator O (assumed to be hermitian for simplicity), and the first-order coefficient b_1 in its β-function. We recall that the scaling dimension d_O is defined as half of the exponent appearing in the two-point correlation function at large distances, namely,

$$\langle\; O(x,0)O(0,0) \;\rangle \;=\; |x|^{-2d_{\mathcal{O}}} \;. \tag{105}$$

It is convenient to define the normalization of O in such a way that the right hand side of Eq. (105) has a prefactor equal to 1. Consequently, O has the engineering dimensions of $a^{-d_{\mathcal{O}}}$. Let us now add a perturbation to the Hamiltonian (or to the Lagrangian with a negative sign) of the form

$$\delta H \;=\; \lambda a^{d_{\mathcal{O}}-2} v \int dx\; O \;, \tag{106}$$

where the factors of a and v (the velocity of the unperturbed Luttinger liquid) are put in to make λ dimensionless; note that v/a has the dimensions of energy. Then the first-order RG equation for λ must take the form given in Eq. (100) with

$$b_1 \;=\; 2 \;-\; d_O \;. \tag{107}$$

This important statement will be proved below for the class of operators $O_{m,n}$ introduced in Eq. (95). If $d_O = 2$, the perturbation is marginal and we have to proceed to Eq. (102). It turns out that b_2 can be obtained from a *three-point* correlation function, but we will not pursue that here [10].

It will not come as a surprise that the RG equations for interacting quantum systems in one dimension can often be derived in two different ways, namely, using the fermionic theory or the bosonic one. Although both the derivations are limited in practice to small values of

the perturbations λ_i, we will see that the bosonic derivation is superior because it can handle the interactions in Eq. (63) exactly. In the fermionic derivation, we have to assume that not only the λ's but also the interaction parameters g_2 and g_4 are small. We will now discuss some simple examples of β-function calculations to first order in the two kinds of theories.

As a particularly simple exercise, consider a non-interacting massive Dirac theory, where the mass term is to be treated as a perturbation. We define the Fourier components of ψ_ν as

$$\begin{aligned} \psi_R(x,t) &= \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i(kx-\omega t)}\, \psi_R(k,\omega)\ , \\ \psi_L(x,t) &= \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i(kx+\omega t)}\, \psi_L(k,\omega)\ . \end{aligned} \tag{108}$$

Then the action takes the form

$$\begin{aligned} S[\psi_\nu,\psi_\nu^\dagger] = \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{\infty}^{\infty} \frac{d\omega}{2\pi} \Big[\quad & \psi_R^\dagger(k,\omega)(\omega - vk)\psi_R(k,\omega) + \psi_L^\dagger(k,\omega)(\omega - vk)\psi_L(k,\omega) \\ & -\mu\Big(\psi_R^\dagger(-k,\omega)\psi_L(k,\omega) + \psi_L^\dagger(-k,\omega)\psi_R(k,\omega)\Big)\ \Big]\ . \end{aligned} \tag{109}$$

Since μ has the dimensions of energy, the dimensionless parameter must be taken to be

$$\lambda \ = \ \frac{a\mu}{v}\ . \tag{110}$$

(The value of a is completely arbitrary here and it will not appear in any physical quantity as we will see). We consider the partition function in the functional integral representation,

$$Z \ = \ \int \mathcal{D}\psi_\nu \mathcal{D}\psi_\nu^\dagger\ e^{iS}\ . \tag{111}$$

We integrate out the modes in the momentum and frequency ranges specified in step (i) of the RG procedure outlined above. Since Eq. (109) describes non-interacting fermions, the mode integration produces an action which looks exactly the same, except that the momentum integrations go from $-\Lambda/s$ to Λ/s. To restore this to the original range of $[-\Lambda,\Lambda]$, we define the new (primed) quantities

$$\begin{aligned} k' &= sk\ , \\ \omega' &= s\omega\ , \\ \psi'_\nu(k',\omega') &= s^{-3/2}\ \psi_\nu(k,\omega)\ , \\ \lambda' &= s\lambda\ . \end{aligned} \tag{112}$$

The resultant action in terms of the new variables and fields looks exactly the same as the original action in terms of the old variables. Note that we had to rescale the mass parameter also in order to achieve this. Since $s = e^{dl}$, we obtain the RG equation

$$\frac{d\lambda(l)}{dl} \ = \ \lambda(l)\ . \tag{113}$$

Clearly, this describes a relevant perturbation, and $\lambda(l)$ grows to 1 at a length scale

$$\xi \;=\; ae^{l} \;=\; \frac{v}{\mu}\;. \tag{114}$$

The energy gap is $\Delta E = v/\xi = \mu$ as expected.

Now let us add density-density interactions as in Eq. (63) to the above massive theory. The question is the following: do ξ and ΔE scale in the same way with μ as they do in the non-interacting theory? Clearly, it is not easy to answer this in the fermionic language since the interactions themselves are not easy to handle in that language, and the mass perturbation is an additional complication. But bosonization comes to our rescue here since the bosonic theory remains quadratic even after including the four-fermi interactions; hence the mass perturbation is the only thing that needs to be studied.

Let us consider a more general perturbing operator of the form

$$O \;=\; O_{m,0} \;+\; O^{\dagger}_{m,0}\;, \tag{115}$$

where $O_{m,n}$ is defined in Eq. (95); the reason for setting $n = 0$ will be explained later. From Eq. (97), the scaling dimension of O is given by $d_O = m^2 K$; note that this contains the effects of the four-fermion interaction in a non-trivial way through the parameter K. In the bosonic language, the perturbed action has the sine-Gordon form,

$$S[\tilde{\phi}] \;=\; \int dxdt\, \Big[\, \frac{1}{2v}\,(\partial_t\tilde{\phi})^2 \;-\; \frac{v}{2}\,(\partial_x\tilde{\phi})^2 \;-\; \frac{v\lambda}{a^2}\,\cos(2m\sqrt{\pi K}\tilde{\phi})\,\Big]\;, \tag{116}$$

where we have changed variables from ϕ to $\tilde{\phi}$ using Eq. (70). We now have to apply the RG procedure to this action. We introduce the Fourier components of $\tilde{\phi}$ as

$$\tilde{\phi}(x,t) \;=\; \int_{-\Lambda}^{\Lambda}\frac{dk}{2\pi}\int_{\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{i(kx-\omega t)}\,\tilde{\phi}(k,\omega)\;. \tag{117}$$

(In principle, the momentum cut-offs for fermion and boson fields need not be equal, but we will use the same symbol Λ for convenience). Next we consider the partition function

$$Z \;=\; \int \mathcal{D}\tilde{\phi}\; e^{iS}\;, \tag{118}$$

and expand e^{iS} in powers of λ to obtain an infinite series. Let us write the field $\tilde{\phi}$ as the sum

$$\tilde{\phi} \;=\; \tilde{\phi}_{<} \;+\; \tilde{\phi}_{>}\;, \tag{119}$$

where both $\tilde{\phi}_{<}$ and $\tilde{\phi}_{>}$ contain all frequencies, but $\tilde{\phi}_{<}$ only contains momenta lying in the range $[-\Lambda/s, \Lambda/s]$, whereas $\tilde{\phi}_{>}$ only contains momenta lying in the ranges $[-\Lambda, -\Lambda/s]$ and $[\Lambda/s, \Lambda]$. Following step (i) of the RG procedure, we have to perform the functional integration over $\tilde{\phi}_{>}$, and then re-exponentiate the infinite series to obtain the new action in terms of $\tilde{\phi}_{<}$. We will do this calculation only to first order in λ. This is not difficult since

$e^{\pm i2m\sqrt{\pi K}\tilde{\phi}}$ can be written as the product of exponentials of $\tilde{\phi}_<$ and $\tilde{\phi}_>$, while the quadratic part of the action decouples as $S_0[\tilde{\phi}] = S_0[\tilde{\phi}_<] + S_0[\tilde{\phi}_>]$. Let us denote the expectation value of a functional $F[\tilde{\phi}_>]$ as

$$\langle\ F[\tilde{\phi}_>]\ \rangle\ =\ \int\ \mathcal{D}\tilde{\phi}_>\ e^{iS_0[\tilde{\phi}_>]}\ F[\tilde{\phi}_>]\ . \tag{120}$$

Now we have to compute

$$\langle\ e^{\pm i2m\sqrt{\pi K}\tilde{\phi}_>(x,t)}\ \rangle\ . \tag{121}$$

By translation invariance, the value of this is independent of the coordinates (x,t), so we can evaluate it at the point $(0,0)$. We then use the fact that $\langle\tilde{\phi}^n_>(0,0)\rangle = 0$ if n is odd, while

$$\langle\ \tilde{\phi}^n_>(0,0)\ \rangle\ =\ (n-1)(n-3)\cdots 1\ \langle\ \tilde{\phi}^2_>(0,0)\ \rangle^{n/2} \tag{122}$$

if n is even. Thus the expectation value in (121) is given by

$$\begin{aligned}\langle\ e^{\pm i2m\sqrt{\pi K}\tilde{\phi}_>(0,0)}\ \rangle\ &=\ \sum_{n=0}^{\infty}\ \frac{1}{n!}\ (\pm i2m\sqrt{\pi}K)^n\ \langle\ \tilde{\phi}^n_>(0,0)\ \rangle \\ &=\ e^{-2m^2\pi K\langle\tilde{\phi}^2_>(0,0)\rangle}\ .\end{aligned} \tag{123}$$

Now we use the fact that

$$\langle\ \tilde{\phi}^2_>(0,0)\ \rangle\ =\ 2\int_{\Lambda/s}^{\Lambda}\ \frac{dk}{2\pi}\ \int_{-\infty}^{\infty}\ \frac{d\omega}{2\pi}\ \frac{i}{\omega^2/v\ -\ vk^2+i\epsilon}\ =\ \frac{\ln s}{2\pi} \tag{124}$$

to show that the left hand side of Eq. (123) is equal to s^{-m^2K}. Putting everything together, we find the new action to be

$$S[\tilde{\phi}_<]\ =\ \int\ dxdt\ \Big[\ \frac{1}{2v}\ (\partial_t\tilde{\phi}_<)^2\ -\ \frac{v}{2}\ (\partial_x\tilde{\phi}_<)^2\ -\ \frac{v\lambda s^{-m^2K}}{a^2}\ \cos(2m\sqrt{\pi K}\tilde{\phi}_<)\ \Big]\ , \tag{125}$$

where the momentum integrals only go from $-\Lambda/s$ to Λ/s. To restore the range of the momentum to $[-\Lambda,\Lambda]$ and to recover the form of the action in Eq. (116), we have to define

$$\begin{aligned} k' &= sk\ , \quad \text{and} \quad x'\ =\ s^{-1}\ x\ , \\ \omega' &= s\omega\ , \quad \text{and} \quad t'\ =\ s^{-1}\ t\ , \\ \tilde{\phi}'(k',\omega') &= \tilde{\phi}_<(k,\omega)\ , \\ \lambda' &= s^{2-m^2K}\ \lambda\ , \end{aligned} \tag{126}$$

and write the action in terms of primed variables. Since $s = e^{dl}$, we see that $d\lambda = \lambda' - \lambda$ satisfies the RG equation

$$\frac{d\lambda}{dl}\ =\ (2\ -\ m^2K)\lambda\ . \tag{127}$$

This proves the relation between the first-order β-function coefficient b_1 and the scaling dimension d_O. Note that the β-functions of the parameters v and K remain zero up to this

order in the perturbation. However they do get a contribution to second order in λ as shown in Ref. 3.

The mass perturbation corresponds to the special case of Eq. (115) with $m = 1$. We now see that it is marginal for $K = 2$ and is relevant if $K < 2$. In the latter case, $\lambda(l)$ grows till we reach a length scale $\xi = a/\lambda(0)^{1/(2-K)}$ where the length scale of the coefficient of the cosine term in the Lagrangian becomes of the same order as a; that is the appropriate point to stop the RG flow of λ. The expression for ξ implies that the energy gap of the system is given by

$$\Delta E \;=\; \frac{v}{a}\,\lambda(0)^{1/(2-K)}\;. \tag{128}$$

Thus the effect of the renormalization is to produce a sine-Gordon theory with the Lagrangian density

$$\mathcal{L} \;=\; \frac{1}{2v}\,(\partial_t\tilde{\phi})^2 \;-\; \frac{v}{2}\,(\partial_x\tilde{\phi})^2 \;-\; \text{const}\;\cdot\;\frac{(\Delta E)^2}{v}\;\cos(2\sqrt{\pi K}\tilde{\phi})\;, \tag{129}$$

where x and t in this expression denote the *original* coordinates, and it is understood that this Lagrangian is *not* to be renormalized any further. This theory is exactly solvable and its spectrum is known in detail [11]. It has both bosonic and fermionic (soliton) excitations, and both of them have energy gaps of the order of ΔE given in Eq. (128).

Finally, let us briefly consider some other relevant and marginal perturbations that can appear in a system which, at the microscopic model, involves fermions on a lattice. If the model has the global phase invariance discussed in Eqs. (79 - 80), then the operators $O_{m,n}$ appearing in the bosonized theory must necessarily have $n = 0$. The scaling dimension is then $d_O = m^2K$. Since $m \geq 1$, there is only a finite number of relevant operators possible depending on the value of K. For $K > 2$, there are no relevant operators at all. For $1/2 < K < 2$, the mass operator is the only relevant term, and so on.

Turning to the possible marginal operators, we see that the umklapp operator $O_{2,0} = \psi_R^{\dagger 2}\psi_L^2$ is marginal for $K = 1/2$. This is a particularly important case to consider because a Luttinger liquid at $K = 1/2$ is known to have a global $SU(2)$ symmetry; it therefore describes a large number of gapless systems involving spins. From conformal field theory, the value of b_2 in the RG equation Eq. (102) for the umklapp operator O is exactly known to be $4\pi/\sqrt{3}$ for the normalization given in Eq. (105). The coefficient of O in the Hamiltonian, namely λ, depends on the microscopic parameters of the model. In general, a system will have a non-zero value of λ. As discussed above, for one sign of λ, the system remains gapless but with logarithmic corrections to various physical quantities; for instance, a $1/\ln T$ term appears in the magnetic susceptibility of a spin system at low temperatures. For the other sign of λ, the system spontaneously dimerizes producing a finite correlation length and an energy gap; this leads to an exponentially vanishing susceptibility at low temperatures.

5 Applications of Bosonization

We will now study various applications of the method of bosonization. The method, as you have learned, can only be applied in one dimension, so we restrict ourselves to one-dimensional models. As you have also seen, the main advantage of the method of bosonization is that many interacting fermion theories can often be recast (within some approximations) as non-interacting boson theories. This enables the explicit calculation of correlation functions. This is an advantage, even in Bethe ansatz solvable one-dimensional models, because it is often not possible to compute correlation functions using the Bethe ansatz.

We will concentrate on the applications of the bosonization technique in the following problems - (i) the quantum antiferromagnetic spin 1/2 chain, (ii) the Hubbard model in one dimension, (iii) transport in clean quantum wires and (iv) transport through isolated impurities. Since the physics of each of these applications is a huge subject by itself, here we will only concentrate on explaining the model and the quantities that we can obtain through the use of bosonization, rather than go into details of its phenomenology.

6 Quantum antiferromagnetic spin 1/2 chain

The model

The first problem that we shall study is the model of a spin 1/2 antiferromagnetic chain. We are picking this model, since you have already learned a lot about the model from the course on quantum spin chains and spin ladders. Here, we will restrict ourselves to just the study of the spin 1/2 anisotropic Heisenberg model with the Hamiltonian given by

$$H = \sum_{i=1}^{N} \left[\frac{J}{2} \left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ \right) + J_z \, S_i^z S_{i+1}^z \right], \tag{130}$$

where the interactions are only between nearest neighbor spins, and $J > 0$. $S_i^+ = S_i^x + iS_i^y$ and $S_i^- = S_i^x - iS_i^y$ are the spin raising and lowering operators. Although this model can be exactly solved using the Bethe ansatz and one has the explicit result that the model is gapless for $-J \leq J_z \leq J$ and gapped for $J_z > J$, (there is a phase transition exactly at the isotropic point $J_z = J$), it is not easy to compute explicit correlation functions in that approach. Hence, it is more profitable to use field theory methods.

Symmetries of the model

Note that this spin model has a global $U(1)$ invariance, which is rotations about the S^z axis. Precisely when $J_z = J$, the $U(1)$ invariance is enhanced to an $SU(2)$ invariance, because at this point the model can simply be written as $H = J \sum_i \mathbf{S_i} \cdot \mathbf{S_{i+1}}$. The model also has discrete symmetries under $S^x \to -S^x$, $S^y \to S^y$, and under $S^z \to -S^z$. Note also that one can change the sign of the XY part of the Hamiltonian by making a rotation by π about the S^z axis on alternate sites, without affecting the J_z term, although this is not an extra symmetry of the model.

Aside on non-linear sigma models

Even using field theory methods, there are two distinct approaches to the problem. In the large-S limit, there exists a semiclassical field theory approach to this model, which leads to an $O(3)$ non-linear sigma model ($NL\sigma M$), with integer and half-integer spins being distinguished by the absence or presence of a Hopf term in the action. In this approach, it is easy to see that integer spin models have a gap in the spectrum. However, it is less easy to study the effect of the Hopf term and show that 1/2-integer spin models are gapless. In fact, in this case, it was the spin model which gave information about field theories with the Hopf term!

Jordan-Wigner transformation

For spin 1/2 models, it is possible to fermionize and then bosonize the spin model and study its spectrum. That is the approach we will follow in the rest of this lecture. First, we will try to convince you that it is possible to rewrite the spin model in terms of spinless fermions. The spin 1/2 model has two states possible at every site - spin $\uparrow$ or spin $\downarrow$. Hence,it can be mapped to another two state model which we can construct in terms of fermions. We shall assume that an $\uparrow$ spin or $\downarrow$ can be denoted by the presence or absence of a fermion at that site. Since no more than a single spinless fermion can sit at a site, the degrees of freedom in both the models are the same. This mapping is implemented by introducing a fermion annihilation operator ψ_i at each site and writing the spin at the site as

$$\begin{aligned} S_i^z &= \psi_i^\dagger \psi_i - 1/2 = n_i - 1/2 \\ S_i^- &= (-1)^i \ \psi_i e^{i\pi \sum_j n_j} \ , \end{aligned} \tag{131}$$

where the sum runs from one boundary of the chain up to the $(i-1)^{\text{th}}$ site and S_i^+ is the hermitian conjugate of S_i^-. So an $\uparrow$-spin is denoted by $n_i = 1$ and the $\downarrow$-spin by $n_i = 0$ at the site i. One might have naively guessed that the spin-lowering operator should be expressed by ψ_i which denotes annihilation of a fermion (with the spin raising operator being given by the hermitian conjugate). One can explicitly check that this gives the correct commutation relations of the spin operators at a site because $[S_i^+, S_i^-] = 2S_i^z$ just reproduces the correct anticommutation relations for the fermions $\{\psi_i, \psi_i^\dagger\} = 1$. The extra string factor has to be added in order to correct for different site statistics - the fermions at different sites anticommute, whereas the spin operators commute. In fact, it is instructive to check explicitly that the string operator changes the commutation relation on different sites.
(H.W. Exercise 1. Check the above explicitly).

The Hamiltonian

Now, we rewrite the spin model in terms of the fermions. We find that

$$\begin{aligned} H &= -\frac{J}{2} \sum_i \ [\psi_i^\dagger e^{i\pi n_i} \psi_{i+1} + h.c.] \ + \ J_z \sum_i \ [(n_i - 1/2)(n_{i+1} - 1/2)] \\ &= -\frac{J}{2} \sum_i \ [\psi_i^\dagger \psi_{i+1} + h.c.] \ + \ J_z \sum_i \ [(n_i - 1/2)(n_{i+1} - 1/2)] \ . \end{aligned} \tag{132}$$

The point to notice is that the string operator has cancelled out in the nearest neighbor interaction, except for a phase term, which also can be explicitly shown to be just 1 because $e^{i\pi n_i}$ precedes a creation operator $\psi_i^\dagger$ which can only act if $n_i = 0$. The spin-flip terms are like the hopping terms in the fermion Hamiltonian and give rise to motion of fermions whereas the S^z-S^z interaction term leads to a four fermion interaction between fermions on adjacent sites (the analog of the on-site Hubbard interaction for spinless fermions). So for non-zero J_z, the fermionic model is non-trivial. There exists a competition between the hopping term or kinetic energy term, which gains in energy when the electrons are free to hop from site to site, and the potential energy which costs J_z if there are electrons present on adjacent sites. So naively, for large J_z, one expects the potential energy to win and electrons to be localized on non-adjacent sites, and for small J_z, one expects the kinetic energy to win and to have delocalized fermions. Let us see whether this expectation is true and how it comes about.

Set $J_z = 0$

To make the problem simpler, we first consider the case where $J_z = 0$ or where there are no interaction terms. Then this is just the model of free spinless fermions. By Fourier transforming the fermions, - $\psi_j = \sum_k \psi_k e^{ikja}/\sqrt{N}$, (a is the lattice spacing) where the k sum is over momentum values in the first Brillouin zone, - we find that the Hamiltonian is given by

$$H \;=\; -\; J \sum_k \; \cos(ka)\; \psi_k^\dagger \psi_k \; . \tag{133}$$

(H.W. Exercise 2. Obtain the above Hamiltonian explicitly).
The discrete symmetry of the model under $S_i^- \leftrightarrow -S_i^+$ and $S_i^z \rightarrow -S_i^z$ implies a particle-hole symmetry $\psi_i \leftrightarrow \psi_i^\dagger$ in the fermion language. Thus, the ground state has to have total spin $M \equiv \sum_i S_i^z = 0$ or equivalently in the fermionic language, the ground state is precisely half-filled. This symmetry can be broken by the addition of a magnetic field term that couples linearly to S^z. In the fermionic language, this is equivalent to adding a chemical potential term (which couples to n_i which is the S^z term) in which case, the ground state no longer has $M = 0$ and the fermion model is no longer half-filled. Thus, for $M = 0$, the band is precisely half-filled and the Fermi surface ($E = 0$) occurs exactly at $ka = \pm\pi/2 \equiv k_F a$ (because the density of states is symmetric about $E = 0$.) Low energy excitations are particle-hole excitations about the Fermi surface, which can occur either at a single Fermi point *i.e.*, $k \sim 0$ modes, or across Fermi points, which are the $k \sim 2k_F = \pi/a$ modes.

Effective field theory

The next step is to write down an effective field theory for the low energy modes. Now comes the approximation. Let us make the assumption that it is only the modes near the Fermi surface (or here, the Fermi points), which are relevant at low energies. Hence, we are only interested in ka values near $ka = \pm\pi/2$ and we may approximate the dispersion relation around the Fermi points to be linear - *i.e.*, $\cos(ka) = \cos(\pm k_F a + k'a) = \cos(\pm\pi/2 + k'a) = \mp\sin(k'a) = \mp(k'a)$. We introduce the labels left and right to denote fermion modes near $ka = -\pi/2$ and $ka = \pi/2$ respectively and henceforth drop the primes on the momenta and assume that they are always measured from the Fermi points; as before, we take k as

increasing towards the right near the right Fermi point, and increasing towards the left near the left Fermi point. If we want to solve the problem without any approximations, we have to allow for excitations about the Fermi points with arbitrary k. The approximation that is made is that we only allow small values of k compared to k_F. This is why the excitations around the left and right Fermi points can be thought of as independent excitations. In this approximation, the Hamiltonian breaks up into

$$H = Ja \sum_k \; k \; (\psi^\dagger_{R,k}\psi_{R,k} + \psi^\dagger_{L,k}\psi_{L,k}) \; . \tag{134}$$

(Note that we have incorporated the change in sign mentioned below Eq. (133)). The fermions around the Fermi points are Dirac fermions since we have linearized the dispersion. These fields do not contain any high momentum modes. In real space, the original non-relativistic fermion field, which has high energy modes (rapidly oscillating factors), has been split up as exponential prefactors times smoothly varying fields -

$$\begin{aligned} \psi_j &\sim e^{-ik_F ja} \int_{-k_F a-\Lambda}^{-k_F a+\Lambda} \frac{d(ka)}{2\pi} e^{ikja}\psi_k \; + e^{ik_F ja} \int_{k_F a-\Lambda}^{k_F a+\Lambda} \frac{d(ka)}{2\pi} e^{ikja}\psi_k \\ &\equiv e^{-ik_F ja}\psi_{Lj} + e^{ik_F ja}\psi_{Rj} \; . \end{aligned} \tag{135}$$

We assume that the $\Lambda << k_F a$ and that it is sufficient to keep just these modes, if we are interested in physics at length scales much greater than $1/\Lambda$, which is of course much greater than the lattice spacing. (The real physical cutoff is the lattice length or in momentum space, the Fermi momentum. As a low energy approximation, we are introducing the larger length cutoff $1/\Lambda$ or the smaller momentum cutoff Λ). For both R and L fermions, states with $k > 0$ are empty and correspond to electron operators (c_k), while states with $k < 0$ are filled and correspond to hole operators ($d^\dagger_k$). (See Fig. 2). In terms of these operators, the Hamiltonian can be rewritten as

$$H = Ja \sum_{k>0} \; k \; (c^\dagger_{L,k}c_{L,k} + d^\dagger_{L,k}d_{L,k} + c^\dagger_{R,k}c_{R,k} + d^\dagger_{R,k}d_{R,k}) \; . \tag{136}$$

We now introduce continuum fermion fields made up of particle (electron) and anti-particle (hole) operators at the left and right Fermi points as

$$\begin{aligned} \psi_R(x,t) &= \frac{1}{\sqrt{Na}} \sum_{k>0} \; [c_{R,k}e^{-ik(vt-x)} + d^\dagger_{R,k}e^{ik(vt-x)}] \\ \psi_L(x,t) &= \frac{1}{\sqrt{Na}} \sum_{k>0} \; [c_{L,k}e^{-ik(vt+x)} + d^\dagger_{L,k}e^{ik(vt+x)}] \; , \end{aligned} \tag{137}$$

where $v = Ja$ is defined to be the velocity. Note that the factor of $1/\sqrt{a}$ is needed to relate continuum fermions to lattice fermions. (The factor of $\sqrt{a}$ is needed to get the dimensions right. The lattice fermions satisfy $\{\psi_i, \psi^\dagger_j\} = \delta_{ij}$, whereas continuum fermions satisfy $\{\psi(x), \psi^\dagger(y)\} = \delta(x-y)$ where the Dirac δ-function has the dimension of 1/length. Also

$Na = L$ gives the conventional box normalization of the continuum fermions). Note also that the standard inclusion of the e^{-ikvt} for the particle fields and the e^{ikvt} for the anti-particle fields, show that the right-movers are a function only of $x_R = x - vt$ and the left-movers are a function only of $x_L = x + vt$. This observation will come in useful when we compute correlation functions. In general, we only need to compute equal time correlation functions. The time-dependent correlations are then obtained by replacing x by x_R for right-movers and by x_L for left-movers.

In terms of the continuum fields, the Hamiltonian is obtained as

$$H = iv \int dx \; [-\psi_R^\dagger \frac{d}{dx}\psi_R + \psi_L^\dagger \frac{d}{dx}\psi_L] \; . \tag{138}$$

(H.W. Exercise 3. Check that this Hamiltonian reduces to the one in Eq. (136) using Eqs. (137)).
We see that the corresponding Lagrangian density is just the standard one for free fermions given by

$$L = i\psi_R^\dagger(\partial_t + v\partial_x)\psi_R + i\psi_L^\dagger(\partial_t - v\partial_x)\psi_L \; . \tag{139}$$

Using the standard rules of bosonization, this Lagrangian can also be rewritten as

$$L = \; \frac{1}{2v}(\partial_t\phi)^2 - \frac{v}{2}(\partial_x\phi)^2 = \frac{1}{2} \; \partial_\mu\phi\partial^\mu\phi \; , \tag{140}$$

where the last equality requires setting $v = 1$. **Note**: It is also worth checking to see that the same Hamiltonian in Eq. (138) is obtained by directly starting with the real space lattice model given in Eq. (132), rewriting the lattice fermions in terms of the continuum fermions remembering the $\sqrt{a}$ conversion factor, using $\sum_i a = \int dx$ and using $\psi_{i+1} = \psi_i + a\partial_x\psi_i$.

Correlation functions

Thus, we have a Lorentz-invariant massless Dirac fermion field theory in the low energy approximation. All low energy properties can be obtained from the field theory, which in fact are trivially computed, since this is a free massless field theory. As far as fermionic correlation functions are concerned, one does not even require bosonization. However, for the spin correlations, it depends on how the spins can be expressed in terms of fermions. For instance, we can explicitly obtain the following spin-spin correlation function

$$G^{zz}(x,t) \equiv < S^z(x,t)S^z(0,0) > \tag{141}$$

simply using Wick theorem. We start by writing S_j^z in term of the fermions as $S_j^z = n_j - 1/2 = \psi_j^\dagger\psi_j - 1/2 =: \psi_j^\dagger\psi_j :$, since the expectation value of n_j is half. Since the lattice fermion can be written in terms of the continuum fermions as

$$\psi_j = \sqrt{a} \; [e^{ik_F j}\psi_R(x = ja) + e^{-ik_F j}\psi_L(x = ja)] \; , \tag{142}$$

and since $e^{i2k_F j} = e^{(i\pi)x/a} = (-1)^{x/a}$, we find that the spin operator can be written as

$$S_j^z/a = S^z(x = ja, t) =: \psi_L^\dagger\psi_L : + : \psi_R^\dagger\psi_R : +(-1)^{x/a}[\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R] \; . \tag{143}$$

Directly using the Wick theorem and the fermion correlators

$$\begin{aligned} < T\psi_L(x,t)\psi_L^\dagger(0,0) > &= \frac{-i}{2\pi(x_L - i\alpha \text{ sign}(t))} \\ \text{and} \quad < T\psi_R(x,t)\psi_R^\dagger(0,0) > &= \frac{i}{2\pi(x_R + i\alpha \text{ sign}(t))} \ , \end{aligned} \tag{144}$$

we see that

$$G^{zz}(x,t) = -\frac{a^2}{4\pi^2} \ [(\frac{1}{x_R^2} + \frac{1}{x_L^2}) - (-1)^{x/a}\frac{1}{x_R x_L}] \ , \tag{145}$$

where $x_R = x - t$ and $x_L = x + t$.
(H.W. Exercise 4. Obtain this explicitly).
This can also be computed using bosonization. Note that even without doing the calculation, one could have guessed that the four-point correlation of the fermions must go as $1/l^2$, where l is a distance, because in 1+1 dimensions, the fermion field has a mass dimension of 1/2 or distance dimension of $-1/2$. So, in the absence of any other scale in the problem (the fermion field is massless and there are no interactions to cause divergences or introduce any anomalous mass scale), as long as the spin correlations can be expressed purely in terms of local fermion fields, no calculations are needed to see that correlations go as $1/l^2$. But we do need to calculate to get the explicit coefficients of $1/x_R^2$, etc, because they could be multiplied by dimensionless quantities like $f(x_R/x_L)$, *etc.*

However, to obtain the correlation function $< S^+(x,t)S^-(0,0) >$ in the fermionic language is more difficult because of the non-local string operator. Here, simple dimensional analysis is not sufficient to give the answer and one actually needs bosonization. The correlation function can be written as

$$\begin{aligned} G^{+-}(x,t) &= < S^+(x,t)S^-(0,0) > \\ &= (-1)^{x/a} \ [e^{-ik_F x/a}\psi_R^\dagger(x,t) + e^{ik_F x/a}\psi_L^\dagger(x,t)] \times \\ &\quad [e^{i\pi\int_0^x (:\psi^\dagger(x',t)\psi(x',t):+1/2a)dx'} + h.c.] \times \\ &\quad [\psi_R(0,0) + \psi_L(0,0)] \ , \end{aligned} \tag{146}$$

where the string operator stretches between the two positions of the spin operator. (The other terms cancel out between S^- and S^+). Also, we have explicitly made the string operator hermitian, since it is hermitian in the lattice model. The reason bosonization comes in handy here is because the non-local operator when written in terms of bosons, turns out to be perfectly simple. We just use the bosonization identity

$$\begin{aligned} \int_0^x dx' : \psi^\dagger(x',t)\psi(x',t) : &= -\frac{1}{\sqrt{\pi}}\int_0^x dx' \partial_{x'}\phi = -\frac{1}{\sqrt{\pi}} \ [\phi(x,t) - \phi(0,t)] \\ &= -\frac{1}{\sqrt{\pi}} \ [\phi_R(x,t) + \phi_L(x,t) - \phi_R(0,t) - \phi_L(0,t)] \ . \end{aligned} \tag{147}$$

Substituting this in Eq. (146), and substituting for the other fermion operators in terms of bosons, we get

$$\begin{aligned} G^{+-}(x,t) &= (-1)^{x/a}\ \frac{a}{2\pi\alpha}[\eta_R^\dagger e^{-ik_Fx/a}e^{i2\sqrt{\pi}\phi_R(x,t)} + \eta_L^\dagger e^{ik_Fx/a}e^{-i2\sqrt{\pi}\phi_L(x,t)}\ ]\times \\ &\quad [e^{ik_Fx/a - i\sqrt{\pi}(\phi_R(x)+\phi_L(x)-\phi_R(0)-\phi_L(0))} + e^{-ik_Fx/a + i\sqrt{\pi}(\phi_R(x)+\phi_L(x)-\phi_R(0)-\phi_L(0))}\ ]\times \\ &\quad [\eta_R e^{-i2\sqrt{\pi}\phi_R(0,0)} + \eta_L e^{i2\sqrt{\pi}\phi_L(0,0)}\ ] \end{aligned} \tag{148}$$

fully in terms of bosons. Now we use the operator identity $e^{A+B} = e^Ae^Be^{-[A,B]/2}$ to write each of the 8 terms that appear in the above equation in terms of products of exponential factors. Just for illustration, we explicitly write the first term which appears by multiplying the first term in each of the square brackets in the above equation.

$$G^{+-}(x,t) = \frac{a}{2\pi\alpha}\ [\eta_R^\dagger\eta_R e^{i\sqrt{\pi}\phi_R(x,t)}e^{-i\sqrt{\pi}\phi_L(x,t)}e^{-i\sqrt{\pi}\phi_R(0,0)}e^{i\sqrt{\pi}\phi_L(0,0)} + 7\ \text{ other terms}]\ . \tag{149}$$

Now, we use the standard commutators
$[\phi_{R/L}(x), \phi_{R/L}(y)] = (-/+)\ i\ \text{sign}(x-y)/4$, and $[\phi_{R/L}(x), \phi_{L/R}(y)] = 0$ (since we are using Klein factors), and the standard algorithm for computing the correlation function

$$\begin{aligned} &< e^{i2\sqrt{\pi}m_1\phi_L(x)}e^{-i2\sqrt{\pi}m_2\phi_L(0)} > \ \sim\ Lim_{\alpha\to 0}\ \Big(\frac{\alpha}{x_L - i\alpha\ \text{sign}(t)}\Big)^{m_1m_2} \\ \text{and}\quad &< e^{i2\sqrt{\pi}m_1\phi_R(x)}e^{-i2\sqrt{\pi}m_2\phi_R(0)} > \ \sim\ Lim_{\alpha\to 0}\ \Big(\frac{\alpha}{x_R + i\alpha\ \text{sign}(t)}\Big)^{m_1m_2}\ , \end{aligned} \tag{150}$$

when m_1 and m_2 have the same sign and vanish when they have opposite signs [6]. This implies that of the 8 terms above, four of them give zero contribution. Adding up the contributions of the remaining four, we obtain

$$G^{+-}(x,t) \sim \frac{1}{(x_Rx_L)^{1/4}}\ [(-1)^{x/a}\ +\ \text{const}\ (\frac{1}{x_R^2} + \frac{1}{x_L^2})]\ . \tag{151}$$

Note that the Klein factors always come as $\eta_i^\dagger\eta_i = 1$ in this correlation function. Also note that one cannot fix the arbitrary constant that can appear between the uniform and the alternating parts of the correlation function because of the normal ordering ambiguities. It is only the exponents which can be found.
(H.W. Exercise 5. Obtain the above explicitly).
Thus even for the non-interacting theory or purely the XY model, bosonization comes in handy to compute the correlation functions. As we have already said, the reason the correlation functions are not obtainable just by naive scaling arguments is because the expression for the 'off-diagonal' spin correlations in terms of the fermion operators is non-trivial, because of the presence of the string term. These are the only non-zero correlators in the theory. The other correlators such as G^{z+} or G^{++} are zero by symmetry - *i.e* because of $U(1)$ invariance in the spin model or because of charge conservation in the fermion model.

Case when $J^z \neq 0$

We now consider the Hamiltonian in Eq. (132). In the fermionic language, the last term is given by

$$\delta H = J_z \sum_j : \psi_j^\dagger \psi_j :: \psi_{j+1}^\dagger \psi_{j+1} : \ . \tag{152}$$

At very large J_z, we would expect electrons to be localized on every alternate site so that adjacent sites are not occupied. However, this will not be true for small J_z, so the point of the exercise is to see when this happens and what the ground state looks like, for both small J_z and large J_z. In the low energy limit, we can rewrite this term in terms of the continuum Dirac fermions at the Fermi points (use Eq. (142)) as

$$\begin{aligned} \delta H = aJ_z \int dx \quad & [\quad : \psi_R^\dagger(x)\psi_R(x) + \psi_L^\dagger(x)\psi_L(x) + (-1)^{x/a} M(x) :] \times \\ & [\quad : \psi_R^\dagger(x+a)\psi_R(x+a) + \psi_L^\dagger(x+a)\psi_L(x+a) + (-1)^{x/a+1} M(x+a) :], \\ \text{where} \quad M(x) \ & = \ \psi_R^\dagger(x)\psi_L(x) + \psi_L^\dagger(x)\psi_R(x) \ . \end{aligned} \tag{153}$$

Using the notation $\rho_L(x) = \psi_L^\dagger(x)\psi_L(x)$ and $\rho_R(x) = \psi_R^\dagger(x)\psi_R(x)$, ($\rho_R + \rho_L$ is the charge density, and $\rho_R - \rho_L$ is the current density; ρ_L and ρ_R are also called the left and right moving currents respectively), we can rewrite Eq. (153) as

$$\begin{aligned} \delta H = aJ_z \int dx \quad & [\quad \rho_R(x)\rho_R(x+a) + \rho_L(x)\rho_L(x+a) + \rho_R(x)\rho_L(x+a) + \rho_L(x)\rho_R(x+a) \\ & \quad -M(x)M(x+a)] \ . \end{aligned} \tag{154}$$

Here we have used the fact that oscillatory factors integrate to zero. (More precisely, they give rise to higher dimension operators, which, however, are irrelevant and ignored in this analysis). In the current-current terms, we can use the expansions $\rho_L(x+a) = \rho_L(x) + a\partial_x\rho_L(x), \psi_L(x+a) = \psi_L(x) + a\partial_x\psi_L(x)$, *etc.*, and the fact that square of a Fermi field vanishes, *e.g.*, $\psi_L^2(x) = 0$, to deduce that terms of the form $\rho_L(x)\rho_L(x+a)$ are higher dimension operators (they have four fermion operators and at least one derivative term) and renormalize to zero in the $a \to 0$ limit. So among the current-current terms, we are only left with ρ_L-ρ_R cross terms of the form $\rho_L(x)\rho_R(x)$ as the lowest dimension operators. For the four fermion terms in the second line also, we apply the same expansion. Dropping higher derivative terms, we see that the only term which survives in the product of the curly brackets is of the form $-\rho_R(x)\rho_L(x) - \rho_L(x)\rho_R(x)$. The extra negative sign is because we need to anticommute one of the fields. This adds to the $\rho_R\rho_L$ term coming from the first line and we are finally left with

$$\delta H = 4J_z a \int dx \ \rho_L(x)\rho_R(x) \ . \tag{155}$$

This is a four fermion term which in continuum quantum field theory is called the Thirring term. In the fermionic language, this is an interacting quantum field theory. However, it is easy to solve by bosonization.

By the standard rules of bosonization for non-interacting fermions, we can write

$$\rho_L = \frac{1}{2\sqrt{\pi}} \, (\frac{1}{v}\partial_t + \partial_x) \, \phi \ , \quad \text{and} \quad \rho_R = \frac{1}{2\sqrt{\pi}} \, (-\frac{1}{v}\partial_t + \partial_x) \, \phi \ . \tag{156}$$

In that case, using the units that $Ja = v = 1$, we get

$$\delta L = -\delta H = \frac{J_z}{J\pi} \, \partial_\mu \phi \partial^\mu \phi \ , \tag{157}$$

where δL denotes the change in the Lagrangian. This is precisely of the same form as the bosonization of the free fermion Hamiltonian. So the new Lagrangian is given by

$$L = \frac{1}{2K} \, \partial_\mu \phi \partial^\mu \phi \ , \tag{158}$$

where

$$\frac{1}{K} = 1 + \frac{2J_z}{J\pi} \ . \tag{159}$$

This can be made to look like the free term by redefining the field ϕ - *i.e.*, we define a new field $\tilde{\phi} = \phi/\sqrt{K}$, so that in terms of $\tilde{\phi}$, the Lagrangian is just $\frac{1}{2}(\partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi})$. However, the canonical momentum obtained from the rescaled Lagrangian is just $\tilde{\Pi} = \partial_0 \tilde{\phi}$ whereas the momentum obtained from the Lagrangian in Eq. (158) $\Pi = \partial_0 \phi / K$. Hence, the momentum gets rescaled compared to the original momentum as $\tilde{\Pi} = \sqrt{K}\Pi$. Clearly, the new coordinate and momenta satisfy the canonical commutation relations, since they are rescaled in opposite ways. (Remember that ϕ takes values on a compact circle, since the original spin operators are defined in terms of exponential of the boson fields and are invariant under periodic changes of ϕ). But since the right and left mover fields are defined by taking both the field and the canonical momentum, and they scale in different ways, one can no longer write the right and left moving fields in the tilde representation as just scaled versions of the right and left moving fields of the original theory - in fact, they mix up the left and right moving fields. Explicitly,

$$\begin{aligned} \tilde{\phi}_R(t,x) &= \frac{1}{2} \, [\tilde{\phi}(t,x) - \int_{-\infty}^{x} dx' \ \tilde{\Pi}(t,x')] = \frac{1}{2} \, [\phi(t,x)/\sqrt{K} - \int_{-\infty}^{x} dx' \ \sqrt{K}\Pi(t,x')] \\ &= \frac{(\phi_R + \phi_L)}{2\sqrt{K}} - \frac{\sqrt{K}(\phi_L - \phi_R)}{2} \\ &= \frac{1}{2}(\frac{1}{\sqrt{K}} + \sqrt{K}) \ \phi_R + \frac{1}{2}(\frac{1}{\sqrt{K}} - \sqrt{K}) \ \phi_L = \cosh\beta \ \phi_R + \sinh\beta \ \phi_L \ , \end{aligned} \tag{160}$$

where $e^{-\beta} = \sqrt{K}$. Similarly

$$\tilde{\phi}_L(t,x) = \cosh\beta \ \phi_L + \sinh\beta \ \phi_R \ . \tag{161}$$

One can now express the spin fields in terms of the $\tilde{\phi}$ fields. They are given by

$$\begin{aligned} S^z(x,t) &\simeq \sqrt{\frac{K}{\pi}}\partial_x\tilde{\phi} + (-1)^{x/a}\ \text{const}\ e^{i2\sqrt{\pi K}\tilde{\phi}} \\ S^-(x,t) &\simeq (-1)^{x/a}e^{i\sqrt{\pi/K}(\tilde{\phi}_R-\tilde{\phi}_L)} + \text{const}\ \times \\ &\quad [e^{i(2\sqrt{\pi K}(\tilde{\phi}_R+\tilde{\phi}_L)+\sqrt{\pi/K}(\tilde{\phi}_R-\tilde{\phi}_L))} + e^{i(-2\sqrt{\pi K}(\tilde{\phi}_R+\tilde{\phi}_L)+\sqrt{\pi/K}(\tilde{\phi}_R-\tilde{\phi}_L))}] \ . \end{aligned} \quad (162)$$

With these substitutions, it is trivial (albeit algebraically more tedious!) to recalculate the spin-spin correlators $G^{zz}(x,t)$ and $G^{+-}(x,t)$. Since the method is exactly the same as for the free case, we just quote the answers here.

$$\begin{aligned} G^{zz}(x,t) &\simeq -\frac{K}{4\pi}(\frac{1}{x_L^2}+\frac{1}{x_R^2}) + (-1)^{x/a}\ \text{const}\ (x_Rx_L)^{-K} \\ G^{+-}(x,t) &\simeq (-1)^{x/a}(x_Rx_L)^{-1/4K} + \ \text{const}\ (x_Rx_L)^{-(\frac{1}{2\sqrt{K}}-\sqrt{K})^2}(\frac{1}{x_L^2}+\frac{1}{x_R^2}) \ . \end{aligned} \quad (163)$$

(H.W. Exercise 6: Obtain the above expressions).
Note that at $K = 1/2$, the two correlations above are the same.

Limitations of this calculation

So the end result is that we have now obtained spin-spin correlation functions even including J_z. But since we have made a low energy continuum approximation and included only a few low-lying modes around the Fermi point, this derivation of the correlation functions is not true for arbitrary J_z. For instance, we left out terms that were irrelevant by naive power counting, which only works in the non-interacting case. Once we have interactions, some of those operators could acquire anomalous dimensions and hence become relevant. In other words, we have seen that interactions change the dimensions of operators. However, we have only studied operators of the form $\rho_L(x)\rho_R(x)$, which were marginal to start with and seen how they evolved. But we did not keep all the irrelevant operators and see how they evolved. Sometimes, they will also become relevant with sufficiently strong interactions.

A more general effective action approach

However, one can try to understand what can possibly change if we include other corrections that we left out in our approximation. One way of doing this is to look at all possible relevant terms that can appear consistent with the symmetries of the problem. The idea is not to try and derive these terms but to write them down in the effective Lagrangian assuming that if they are not explicitly prohibited by a symmetry, then they will appear. This is the philosophy behind what are called effective field theories.

Aside on how to 'read off' dimensions of operators

We know that to see whether an operator is relevant or irrelevant, we have to compute its correlation function and find out its scaling dimension. Then, we have to check whether the scaling dimension is such that the coefficient of the operator grows or becomes smaller as the energy scale is reduced. So given any operator O_i in terms of bosons, we first compute

the correlation function $< O_i(x,t)O_i(x',0) >$ which goes as $1/(x-x')^{2d_i}$. For a free fermion theory with no interactions, (equivalently a boson theory with the interaction parameter $K=1$) we know that $< O_i(x,t)O_i(x',0) >= 1/(x-x')^{2\tilde{d}_i}$ where $\tilde{d}_i$ is just the naive scaling dimension or the engineering dimension of the operator O_i. The difference between $\tilde{d}_i$ and the d_i that appears when we actually compute the correlation function is because of the interactions and is called the anomalous dimension of the operator. As was explained in the other courses in this school [13], the extra dimensional parameter comes from the cutoff scale. We shall use the term scaling dimension to mean d_i itself. For an operator of the form $O_i \sim e^{i2\sqrt{\pi}\beta(\phi_L+\phi_R)}$, the scaling dimension is given by $d_i = \beta^2$, for the standard (non-interacting) form of the Hamiltonian. Since the space-time dimension is two, it is clear that $d_i > 2$ implies that the coefficient λ_i of the operator has to have dimension $2-d_i < 0$. So each time the cutoff is scaled down by a factor Λ, $\lambda_i \to \lambda_i \Lambda^{2-d_i}$. Hence, after successive rescalings, this term in the action is irrelevant and scales to zero. On the other hand, $d_i < 2$ denotes relevant operators, whose coefficients grow under scaling downs of the cutoff. $d_i = 2$ is a marginal operator, whose coefficient remains unchanged under rescalings. (We will come back to this when we study impurity scattering and scaling dimensions of 'boundary operators').

Back to the effective action

The only possible Lorentz-invariant relevant terms that can be added to the Lagrangian is either $\cos 2\sqrt{\pi}\beta(\tilde{\phi}_L + \tilde{\phi}_R)$ or $\cos 2\sqrt{\pi}\beta(\tilde{\phi}_L - \tilde{\phi}_R)$; both of these have dimension β^2 and are thus relevant for $\beta < \sqrt{2}$. (The real problem on a lattice, of course, does not have Lorentz invariance. However, in the long distance or low energy limit, all such Lorentz non-invariant interactions will probably be irrelevant). Of these, the $U(1)$ symmetry under rotations about the z-axis in fermion language implies that ψ_L and ψ_R have to be multiplied by the same phase (because S_z which has terms of the form $\psi_L^\dagger \psi_R + h.c.$ should not change). This in turn means that $\tilde{\phi}_L \to \tilde{\phi}_L + c$ and $\tilde{\phi}_R \to \tilde{\phi}_R - c$ so that $\tilde{\phi}_L + \tilde{\phi}_R \to \tilde{\phi}_L + \tilde{\phi}_R$ and $\tilde{\phi}_L - \tilde{\phi}_R \to \tilde{\phi}_L - \tilde{\phi}_R$+constant. Thus, to be consistent with this symmetry, we can only allow $\cos 2\sqrt{\pi}\beta(\tilde{\phi}_L + \tilde{\phi}_R)$.

Furthermore, since the spin operators are all expressed in terms of exponentials of the boson fields, (see Eq. (162)) the boson fields need to be 'compactified on a circle'. This only means that the boson fields are periodic -

$$\tilde{\phi} \leftrightarrow \tilde{\phi} + \sqrt{\frac{\pi}{K}} \tag{164}$$

since the spin fields cannot distinguish between $\tilde{\phi}$ and $\tilde{\phi} + \sqrt{\pi/K}$. This restricts β in $\cos 2\sqrt{\pi}\beta(\tilde{\phi}_L + \tilde{\phi}_R)$ to be of the form $n\sqrt{K}$ where n is an integer.

Finally, we use an unusual feature which occurs in the continuum field theories of many lattice spin models. The translational symmetry of the lattice spin model by one site (or more sites for more general models) maps to a discrete symmetry in the continuum model, which is distinct from translational symmetry. This can be seen from the continuum definition of the spin in terms of the Dirac fermions - Eq. (143). When we change j to $j+1$, the oscillatory

factor $(-1)^j \to -(-1)^j$. This is a drastic change from site to site. So if in the continuum version, we want to define smooth fields without having this rapid oscillations, we need to define one field for every pair of sites. Thus invariance under translation by $2a$ on the lattice becomes translational invariance in the continuum model. But from Eq. (143), we see that translational symmetry through a single site corresponds to the discrete symmetry

$$\psi_L \to i\psi_L, \quad \psi_R \to -i\psi_R \ . \tag{165}$$

In the bosonic language, this corresponds to

$$\tilde{\phi}_L \to \tilde{\phi}_L + \frac{1}{2}\sqrt{\frac{\pi}{K}}, \quad \text{and} \quad \tilde{\phi}_R \to \tilde{\phi}_R + \frac{1}{2}\sqrt{\frac{\pi}{K}} \ . \tag{166}$$

This symmetry implies that the only terms that can be added to the Lagrangian are of the form $\cos 2\sqrt{\pi}2n\sqrt{K}(\tilde{\phi}_L + \tilde{\phi}_R)$, so that $\beta = 2n\sqrt{K}$. This is relevant when $K < 1/2$ when $n = 1$. So the system is in a massless phase till K reaches $1/2$ below which it develops a relevant interaction, and a mass gap.

However, we cannot use our low energy approximate result to estimate the point at which the spin model develops a relevant interaction. Besides adding the possible relevant term mentioned above, the most general thing the other terms that we have neglected can do is to change the relation between K and J_z in an unpredictable way. In fact, the low energy result relating K to the perturbation J_z is only true to lowest order in J_z/J. This particular spin-chain model is, in fact, solvable by Bethe ansatz and the exact answer is

$$\frac{1}{K} = 1 + \frac{2}{\pi}\sin^{-1}(\frac{J_z}{J}) \tag{167}$$

which, to lowest order in J_z/J, gives us the relation in Eq. (159). From this, we see that $1/K \to 2$ at $J_z = J$. This is precisely the K value for which the cosine interaction term becomes relevant. To prove that a relevant interaction necessarily leads to a mass gap is non-trivial, but it is certainly plausible. Once, there is a relevant interaction, its coefficient grows under renormalization. It becomes divergent as we make the energy scale lower and lower, so we have to cut it off at some scale, which is the mass scale associated with the theory. However, it could also lead us to a new fixed point, which may not have a mass gap.

To get higher orders in J_z/J in this effective field theory approach is not easy, because one needs to go beyond the region of linear dispersion. Also, once one starts including modes with $k \simeq k_F$, we need to be careful to put in the restriction that $-k_F < k < k_F$. Also, since the Bethe ansatz already gives the exact answer to all orders, there may not be much point in trying to do this for this problem.

So what does all this formalism gain us? How does the ground state evolve as J_z changes? For small values of J_z all that happens is that the spin-spin correlations have a slightly different power law fall-off with anomalous non-integer exponents. Does this continue for all values of J_z? No. Once, J_z reaches $J_z = J$, the isotropic point, there exists a phase transition to a massive phase where spin-spin correlations fall off exponentially fast at large separations. In this particular problem, of course, one knew this answer from the Bethe ansatz, but the

point is that the bosonization method can be used even for other models, which are not exactly solvable by the Bethe ansatz. But without the Bethe ansatz, one cannot analytically find the value of the parameter where the phase transition into a massive phase occurs. The other important gain that we have in this method is that it allows the computation of correlation functions, which is not possible using the Bethe ansatz. Finally, since it is a symmetry analysis, it tells us that for any Hamiltonian of this type, the model is likely to be massless only when the theory has $U(1)$ symmetry and the Z_2 symmetry of translation by a single site and even then, only for some restricted values of the parameter space.

The best reference for this application is Affleck's lectures [5] on field theories and critical phenomena, which we have followed fairly faithfully.

7 Hubbard model

The Hubbard model is one of the simplest realistic models that one can study which has a competition between the kinetic energy and the potential energy. The kinetic energy or the hopping term gains, or rather the energy gets lowered, if the fermions are delocalized - free to move throughout the sample. In this model, the potential energy represents screened Coulomb interactions between electrons and the model is constructed so that it costs energy to put two electrons at the same place. So the potential energy prefers each electron to sit at its own site. The model is given by

$$H = -\frac{t}{2}\sum_{j\alpha}\ (\psi^\dagger_{j\alpha}\psi_{j+1\alpha} + h.c.) + U\sum_{j}\ n_{j\uparrow}n_{j\downarrow} + \mu\sum_{j,\alpha}\ \psi^\dagger_{j\alpha}\psi_{j\alpha}\ , \tag{168}$$

where t is the hopping parameter, U is a positive constant denoting the repulsion between two electrons at a site, μ is the chemical potential and α is the spin index which can be $\uparrow$ or $\downarrow$. This model is very similar to the fermion model we studied for the spin chain except that the electrons have spin and the chemical potential term allows for arbitrary fillings. The U term or Hubbard term is analogous to the nearest neighbor J_z interaction term for spinless electrons.

At half-filling (one electron/site, since a filled band implies two electrons/site), for large U, the model is expected to describe an insulator. One can easily understand this, because at infinite U, the ground state will have one electron at every site. Any excitation will cost an energy of U. So there is a gap to excitations and the model behaves as an insulator. It is called a Mott-Hubbard insulator (as opposed to other band insulators) because here the insulating gap is created by interactions.

The question that one would like to ask is, at what value of U does the Mott-Hubbard gap open, because naively one may think that at very small values of U, the model allows free propagation of electrons and describes a metal. Using bosonization, we will show that in one dimension, this expectation is wrong. The Mott-Hubbard gap opens up for any finite U if the filling is half and not otherwise. For any other filling, the model at low energies is an example of a Luttinger liquid with separate spin and charge excitations. The spin

modes are always gapless whereas the charge modes are gapless at any filling other than half-filling; precisely at half-filling a charge gap opens up. The model for arbitrary filling (other than half-filling) and positive U is said to be in the Luttinger liquid phase. Spin and charge correlations fall-off as power laws and we expect power law transport. At half-filling, the model is in a charge-gapped phase called the charge-density wave phase.
Aside: For negative U, it is found that the spin excitations are always gapped. Here, the model is said to be in the Luther-Emery phase or spin gapped phase.

Bosonization of the model without interactions

How do we go about seeing all that we have described above? In higher dimensions, we would do a mean field theory, but in one space dimension, we know that a mean field analysis is not very useful because of the infrared divergences of the low energy fluctuations. (In other words, if we write down a mean field theory and then try to do systematic corrections about the mean field theory, then order by order in perturbation theory, we find that the integrals which appear in the corrections are divergent). So it seems like a good idea to try and use bosonization. In fact, the way this model is analyzed is very similar to the way we analyzed the spinless fermion model in the previous section. We first switch off the interactions and start with the Fourier decomposition

$$\psi_{j\alpha} = \frac{1}{\sqrt{N}} \sum_k \ \psi_{k\alpha} e^{ikja} \ . \tag{169}$$

We rewrite the Hamiltonian as

$$H_0 = \sum_{k\alpha} \ (\mu - t\cos ka) \ \psi^\dagger_{k\alpha}\psi_{k\alpha} \ , \tag{170}$$

where the k values go from $-\pi/a$ to π/a. In the ground state, all states with $|k| < k_F$ are filled, where k_F is determined by the chemical potential from the equation $\mu = \cos k_F a$. Just as in the spinless case, we will look only at the low energy modes near the Fermi surface, so that each fermion is written as

$$\begin{aligned} \psi_{j\alpha} \ &\sim \ e^{-ik_F ja} \int_{-k_F a-\Lambda}^{-k_F a+\Lambda} \frac{d(ka)}{2\pi} \ e^{ikja}\psi_{k\alpha} + e^{ik_F ja} \int_{k_F a-\Lambda}^{k_F a+\Lambda} \frac{d(ka)}{2\pi} \ e^{ikja}\psi_{k\alpha} \\ &\equiv \ e^{-ik_F ja}\psi_{Lj\alpha} + e^{ik_F ja}\psi_{Rj\alpha} \ , \end{aligned} \tag{171}$$

so that the $\psi_{Lj\alpha}$ and $\psi_{Rj\alpha}$ do not contain high energy modes. Substituting this expression in Eq. (168), we get

$$\begin{aligned} H_0 = -\frac{t}{2}\sum_{j\alpha} \ \ [\ \ & (e^{-ik_F a}\psi^\dagger_{Lj\alpha}\psi_{Lj+1\alpha} + e^{ik_F a}\psi^\dagger_{Rj\alpha}\psi_{Rj+1\alpha} + \\ & e^{-i2k_F ja - ik_F a}\psi^\dagger_{Lj\alpha}\psi_{Rj+1\alpha} + e^{i2k_F ja + ik_F a}\psi^\dagger_{Rj\alpha}\psi_{Lj+1\alpha}) + h.c.] \\ +\mu\sum_{j\alpha} \ \ [\ \ & \psi^\dagger_{Lj\alpha}\psi_{Lj\alpha} + \psi^\dagger_{Rj\alpha}\psi_{Rj\alpha} + e^{-i2k_F ja}\psi^\dagger_{Lj\alpha}\psi_{Rj\alpha} + e^{i2k_F ja}\psi^\dagger_{Rj\alpha}\psi_{Lj\alpha}] \ . \end{aligned} \tag{172}$$

The oscillatory terms do not contribute because they have the form

$$\sum_j e^{i2k_F ja}\psi^\dagger_{Rj\alpha}\psi_{Lj\alpha} \sim \sum_j e^{i2k_F ja}\sum_{k,k'} e^{-ikja}e^{ik'ja}\psi^\dagger_{k\alpha}\psi_{k'\alpha} \sim \sum_{k,k'}\psi^\dagger_{k\alpha}\psi_{k'\alpha}\sum_j e^{i2k_F ja-ikja+ik'ja} \ , \tag{173}$$

and the sum over j in the last expression produces $\delta_{2k_F,k-k'}$ which cannot be satisfied for small values of k, k'. (Note that even for the half-filled case where $k_F a = \pi/2$, this cannot be satisfied). Hence, we may drop these terms and we are left with only

$$\begin{aligned} H_0 = & - \frac{t}{2}\sum_{j\alpha} [(e^{-ik_F a}\psi^\dagger_{Lj\alpha}\psi_{Lj+1\alpha} + e^{ik_F a}\psi^\dagger_{Rj\alpha}\psi_{Rj+1\alpha}) + h.c.] \\ & + \mu\sum_{j\alpha} (\psi^\dagger_{Lj\alpha}\psi_{Lj\alpha} + \psi^\dagger_{Rj\alpha}\psi_{Rj\alpha}) \ . \end{aligned} \tag{174}$$

Now, we expand $\psi_{j+1\alpha} \equiv \psi_\alpha(j+1) = \psi_\alpha(j) + a\partial_x\psi_\alpha(j) +$ higher order irrelevant terms and use the fact that $\cos k_F a = \mu$ (which means that part of the hopping term cancels with the μ term) to get

$$\begin{aligned} H &= -\frac{at}{2}\sum_{j\alpha} (e^{-ik_F a}\psi^\dagger_{Lj\alpha}\partial_x\psi_{Lj\alpha} + e^{ik_F a}\psi^\dagger_{Rj\alpha}\partial_x\psi_{Rj\alpha} + h.c.) \\ &= iat\sin(k_F a)\sum_{j\alpha} (\psi^\dagger_{Lj\alpha}\partial_x\psi_{Lj\alpha} - \psi^\dagger_{Rj\alpha}\partial_x\psi_{Rj\alpha}) \ , \end{aligned} \tag{175}$$

where we have also integrated the hermitian conjugate terms by part to get it in the form of the second equation above. Finally, we can rewrite this Hamiltonian as a continuum Hamiltonian in terms of continuum fields (defined with the usual factor of $\sqrt{a}$ as $\psi_\alpha(j)/\sqrt{a} = \psi_\alpha(x)$) and using $\sum_j a = \int dx$

$$H_0 = it\sin(k_F a)\sum_\alpha \int dx\ [\psi^\dagger_{L\alpha}(x)\partial_x\psi_{L\alpha}(x) - \psi^\dagger_{R\alpha}(x)\partial_x\psi_{R\alpha}(x)\ ] \ , \tag{176}$$

where we call $t\sin(k_F a) = v_F a$, the Fermi velocity times a. The derivation here is very similar to the one for spinless fermions, except that here we have carried it out in real space instead of momentum space. This Hamiltonian can be bosonized using the usual rules of bosonization and we get

$$H_0 = \frac{v_F a}{2}\sum_\alpha \int dx\ [\Pi^2_\alpha + (\partial_x\phi_\alpha)^2] \ . \tag{177}$$

(H.W. Exercise 7. Derive the Hamiltonian in Eq. (176) through a momentum space derivation).

Bosonization of the interaction term

The next step is to figure out the low energy part of the on-site Hubbard interaction. Here, again, the principle is the same. We rewrite the four-fermion term written in terms

of the original fermions in terms of the low energy Dirac fermion modes. Just as in the spin model, the $S^z - S^z$ term or the four fermion term corresponded to a product of normal ordered bilinears, here also the four fermion term in Eq. (168) can be written in terms of the product of normal ordered bilinears if we subtract the average charge densities of the $\uparrow$ and $\downarrow$ fields. So we may write

$$H_{\rm int} = U \sum_j \; n_{j\uparrow} n_{j\downarrow} = U \sum_i \; : n_{j\uparrow} :: n_{j\downarrow} : \; . \tag{178}$$

In terms of the Dirac fields, this becomes

$$H_{\rm int} = U \sum_j [(: \psi^\dagger_{jL\uparrow}\psi_{jL\uparrow} : + : \psi^\dagger_{jR\uparrow}\psi_{jR\uparrow} : + \psi^\dagger_{jR\uparrow}\psi_{jL\uparrow}e^{-i2k_F ja} + \psi^\dagger_{jL\uparrow}\psi_{jR\uparrow}e^{i2k_F ja}) \times (\uparrow \rightarrow \downarrow)]. \tag{179}$$

We now expand the products and keep only the terms with no oscillatory factor, to get

$$H_{\rm int} = U \sum_j \; (J_{jR\uparrow} + J_{jL\uparrow})(J_{jR\downarrow} + J_{jL\downarrow}) + U \sum_j \; (\psi^\dagger_{jR\uparrow}\psi_{jL\uparrow}\psi^\dagger_{jL\downarrow}\psi_{jR\downarrow} + h.c.) \; . \tag{180}$$

The remaining terms have the oscillatory factors of either $e^{i2k_F ja}$ or $e^{i4k_F ja}$ and can be set to zero for arbitrary filling. Notice however, that $e^{i4k_F ja} = 1$ and is not oscillatory at half-filling since $k_F a = \pi/2$. We will come back to this point later. Now we first express these fields in terms of the continuum fields and just use the standard bosonization formulae to get

$$H_{\rm int} = Ua \int dx \; [\frac{1}{\pi}\partial_x\phi_\uparrow \partial_x \phi_\downarrow + \eta^\dagger_{R\uparrow}\eta_{L\downarrow}\eta^\dagger_{L\downarrow}\eta_{R\uparrow}\frac{Ua}{2\pi^2\epsilon^2}\cos\sqrt{4\pi}(\phi_{R\uparrow} + \phi_{L\uparrow} - \phi_{R\downarrow} - \phi_{L\downarrow})] \; . \tag{181}$$

(H.W. Exercise 8. Derive the above).
The interesting point to note here is that the cosine term only depends on $\phi_\uparrow - \phi_\downarrow$. (We use the earlier defined notation that $\phi = \phi_L + \phi_R$ and $\theta = -\phi_R + \phi_L$). So if we define the charge and spin fields

$$\phi_c = \frac{\phi_\uparrow + \phi_\downarrow}{\sqrt{2}} \; , \quad \text{and} \quad \phi_s = \frac{\phi_\uparrow - \phi_\downarrow}{\sqrt{2}} \; , \tag{182}$$

the Hamiltonian is completely separable in terms of these two fields and we may write $H = H_0 + H_{\rm int} = H_c + H_s$ with

$$\begin{aligned} H_c &= \frac{v_F}{2}\int dx \; [\Pi_c^2 + (1 + \frac{U}{\pi v_F})(\partial_x\phi_c)^2] \\ H_s &= \frac{v_F}{2}\int dx \; [\Pi_s^2 + (1 - \frac{U}{\pi v_F})(\partial_x\phi_s)^2 + \frac{Ua}{2\pi\epsilon^2}\cos\sqrt{8\pi}\phi_s] \; , \end{aligned} \tag{183}$$

where the bosonized form of the kinetic energy term given by H_0 in Eq. (177) along with the first term in Eq. (181) ($U\partial_x\phi_\uparrow\partial_x\phi_\downarrow/\pi$) can also be written in terms of the charge and spin fields as above. The charge sector is massless, but for the spin sector, one has a cosine term in the Hamiltonian. From our earlier experience of spinless models, we know that a cosine term can lead to a mass gap, when it becomes relevant. So we need to compute the dimension of

the operator and see when it becomes relevant. Note that we have chosen the product of the Klein factors to be unity[3]. But we only know how to compute correlation functions when the quadratic Hamiltonian is in the standard form. To get that, we need to rescale the ϕ fields and their conjugate momenta (in the opposite way so that the commutation relations are preserved) as

$$\bar{\phi}_c = (1 + \frac{U}{\pi v_F})^{1/4} \phi_c \ , \quad \text{and} \quad \bar{\Pi}_c = (1 + \frac{U}{\pi v_F})^{-1/4} \Pi_c \ , \tag{184}$$

and similarly for the spin fields to get the Hamiltonian in the standard form, from which we can directly read out the dimensions of the operators. In terms of the bar fields, we see that

$$\begin{aligned} H_c &= (1 + \frac{U}{\pi v_F})^{1/2} \frac{v_F a}{2} \int dx [\bar{\Pi}_c^2 + (\partial_x \bar{\phi}_c)^2] \\ H_s &= (1 - \frac{U}{\pi v_F})^{1/2} \frac{v_F a}{2} \int dx [\bar{\Pi}_s^2 + (\partial_x \bar{\phi}_s)^2 + \frac{Ua}{2\pi\epsilon^2} \frac{1}{(1 - \frac{U}{\pi v_F})^{1/2}} \cos \sqrt{\frac{8\pi}{(1 - \frac{U}{\pi v_F})^{1/4}}} \bar{\phi}_s]. \end{aligned} \tag{185}$$

The charge sector is purely quadratic (both before and after rescaling!) and remains massless, whereas for the spin sector, the rescaling was necessary to 'read off' the dimension of the cosine operator. Since its scaling dimension is given by $d = 2/(1 - \frac{U}{\pi v_F})^{1/4}$, it is irrelevant ($d < 2$) for any weak positive U and the spin sector is also massless. On the other hand, for any negative U, this term has dimension $d > 2$ and is relevant. As we explained in the spinless case, this means that the spin sector acquires a mass gap for all negative U.

Also note that the velocities of the charge and the spin modes have got renormalized in different ways. $v_c = (1 + \frac{U}{\pi v_F})^{1/2} v_F$ is the velocity of the charge mode and $v_s = (1 - \frac{U}{\pi v_F})^{1/2} v_F$ is the velocity of the spin mode. Thus, spin and charge move independently. This is one of the hallmarks of Luttinger liquid behavior in one-dimensional fermion models. It is only for $U = 0$, that the spin and the charge modes move together.

How does one look for such spin-charge separation in one-dimensional models? Experimentally, one has to look at different susceptibilities and measure the Wilson ratio, which is the ratio of the spin susceptibility to the specific heat coefficient. The specific heat coefficient depends both on spin and charge modes and is given by

$$\frac{\gamma}{\gamma_0} = \frac{1}{2} \left(\frac{v_F}{v_c} + \frac{v_F}{v_s} \right) \ , \tag{186}$$

where γ_0 is the specific heat of non-interacting electrons with velocity v_F. However, spin susceptibility only depends on the spin mode and is given by

$$\frac{\chi}{\chi_0} = \frac{v_F}{v_s} \ . \tag{187}$$

[3]For single chain problems, the Klein factors usually cause no problems and can be set to be unity, in most cases. The only care that we need to take is to remember the negative sign that one gets when two of them are exchanged. But for multi-chain models, when more than four explicit Klein factors exist, one needs to be more careful.

Thus, the Wilson ratio is given by

$$R_W = \frac{\chi/\chi_0}{\gamma/\gamma_0} = \frac{2v_c}{v_c + v_s}. \tag{188}$$

Clearly, when there is no spin-charge separation, this is given by one. So deviations of the Wilson ratio from unity are a sign of spin-charge separation in real systems.

Finally, let us consider the case exactly at half-filling, $k_F a = \pi/2$. In this case, the $e^{i4k_F ja}$ term we neglected in Eq. (179) as oscillatory, is no longer oscillatory, since $e^{i4k_F a} = 1$. In this case, there exists a term in the Hamiltonian of the form

$$H_{\text{umklapp}} = U \sum_j \left(\psi^\dagger_{jR\uparrow}\psi_{jL\uparrow}\psi^\dagger_{jR\downarrow}\psi_{jL\downarrow} + h.c.\right). \tag{189}$$

Note that this term destroys two right movers and creates two left movers or vice-versa. So there is an overall change in momentum by $4k_F a = 2\pi$, which has to be absorbed by the lattice. It is an umklapp process unlike the earlier interaction term for arbitrary filling which created and destroyed a particle at the left Fermi point and also created and destroyed a particle at the right Fermi point and did not change any momentum. It is easy to see that this term also gives rise to a cosine term by bosonizing, which, after rescaling gives $\frac{Ua}{2\pi\epsilon^2(1+U/\pi v_F)^{1/2}} \cos\sqrt{\frac{8\pi}{(1+U/\pi v_F)^{1/4}}}\bar{\phi}_c$ neglecting Klein factors. Thus it appears in the Hamiltonian of the charge sector. This term is irrelevant for any negative U, but relevant for any positive U. Thus, precisely at half-filling, the charge sector has a gap. This is similar to the case for spinless fermions where the spin model actually corresponded to a half-filled spinless fermion model. But unlike the case for the spinless fermions where the gap only opens up at $J = J_z$, here the gap opens up for any positive U, however small.

Is there any way one can understand these results in a physical way? For negative U, we found that the spin sector has a gap. This can be understood by saying that since there is an attractive interaction between the spin $\uparrow$ and the spin $\downarrow$ densities of fermions, they will like to form singlets and sit on a single site. So to make a spin excitation, one needs to break a pair and this costs energy. But charge excitations can move around as bound spin singlet pairs with no cost in energy. On the other hand, for positive U, there exists a repulsion between two electrons at a site. So each electron will tend to sit on a different site. At half-filling, hence, there is no way for an electron to move, without trying to sit at a site, at which an electron is already present. And this costs a repulsive energy U. Hence, there is a gap to charge excitations. But one can flip spins at a site and hence have spin excitations with no cost in energy.

So what results has bosonization given us here? We started with electrons with spin and charge moving together via a hopping term, but with a strong on-site Coulomb repulsion. This term could not be treated perturbatively. However, when we rewrote the theory in terms of bosons, with one boson for the $\uparrow$ spin and one for the $\downarrow$ spin, we found that the theory decoupled in terms of new spin and charge bosons. For a generic filling, the charge boson was just a massless free boson excitation, whereas the spin boson Hamiltonian had a cosine term, which was relevant when U was negative, but irrelevant for positive U. But at

half-filling, the charge excitations develop a gap, for any positive U. The most important thing to note here is that the charge and spin degrees of freedom have completely decoupled. Since the two fields are scaled differently, they move with different velocities in the system. This is a result that one could never have obtained perturbatively. Thus, at any filling other than half-filling, the low energy limit of the Hubbard model is a Luttinger liquid with massless spin and charge excitations moving with different velocities. A good reference for this part is Shankar's article [6] which also explains in great detail how to compute correlation functions.

8 Transport in a Luttinger liquid - Clean Wire

The last two applications involved the study of correlation functions, with the aim of finding out the different phases possible in a one-dimensional system of interacting fermions. In this part of the course, we will study another application of bosonization, which is to study transport, in particular the DC (or zero frequency) conductivity in one-dimensional wires of interacting fermions.

Firstly, are one-dimensional wires experimentally feasible? The general idea to make narrow wires is to 'gate' 2D electron gases. In recent times, technology has developed enough to make these wires so narrow, that they contain only one transverse channel. So these are good enough approximations to one-dimensional wires. Another good approximation to coupled chains of one-dimensional models are carbon nanotubes, though those are not the kind of models we will study here.

The next point to note is that even at a qualitative level, transport in low dimensional systems is extremely different from transport in higher dimensions. To understand this point, we will first make qualitative statements about transport and conductivity before we explicitly start computing it using bosonization. The usual aim is to compute the conductance as a function of the voltage, temperature, presence of impurities or disorder and so on. Normally, when currents are measured in wires, one does not worry about quantum effects, because wires are still macroscopic objects, but that is clearly not the case here, since we are interested in one-dimensional wires. In fact, whenever the physical dimensions of the conductor becomes small, (it need not be really one-dimensional), the usual Ohmic picture of conductance where the conductance is given by

$$G = \sigma \ \frac{W}{L} = \sigma \ \frac{\text{width of conductor}}{\text{length of conductor}} \ , \tag{190}$$

where σ is a material dependent quantity called conductivity, breaks down. A whole new field called 'mesoscopic physics' has now been created to deal with electronic transport in such systems. The term 'mesoscopic' in between microscopic and macroscopic is used for systems, where the sizes of the devices are such that it is comparable with a) the de Broglie wavelength (or kinetic energy) of the electron, b) the mean free path of the electron and c) the phase relaxation length (the length over which the particle loses memory of its phase)

of the electron. Ohmic behavior is guaranteed only when all these length scales are small compared to the size which happens for any macroscopic object. These lengths actually vary greatly depending on the material and also on the temperature. Typically, at low temperatures, they vary between a nanometer for metals to a micrometer for quantum Hall systems.

For mesoscopic wires, in general, quantum effects need to be taken into account. One way of computing these conductances is by using the quantum mechanical formulation of transmission and reflection through impurities and barriers. This formulation is called the Landauer-Buttiker formulation and works for Fermi liquids. However, it does not include interactions. But for one dimensional wires, interactions change the picture dramatically, since the quasi-particles are no longer fermion-like. Hence the Landauer-Buttiker formalism cannot be directly applied and one needs to compute conductances in Luttinger wires taking interactions into account right from the beginning. One way of doing this is by using bosonization and this is the method that we will follow here.

The aim is to compute the conductance of a one-dimensional wire. First, we will compute the conductance through a clean wire (no impurities or barriers) and argue why the conductance is not renormalized by the interaction. Then we will study the conductance again after introducing a single impurity. Here, we will see that the interactions change the picture dramatically. For a non-interacting one-dimensional wire, from just solving usual one-dimensional quantum mechanics problems, we know that we can get both transmission and reflection depending on the strength of the scattering potential. But for an interacting wire, we shall find that for any scattering potential, however small, for repulsive interactions between the electrons, there is zero transmission and full reflection (implies conductance is zero, or that the wire is 'cut') and for attractive interactions between electrons (which is of course possible only for some renormalized 'effective' electrons), there is full transmission and zero reflection (implying perfect conductance or 'healing' of the wire).

Ballistic conductor

Let us first define the conductance of a mesoscopic ballistic conductor (*i.e.*, a conductor with no scattering) without taking interactions into account. We said earlier that the usual definition of conductance as $G = \sigma \frac{W}{L}$ breaks down for mesoscopic systems. For instance, it is seen that instead of the conductance smoothly going down as a function of the area or width of the wire W, it starts going down discretely in steps, each of height $2e^2/h$. Also as L decreases, instead of increasing indefinitely, G saturates at some limiting value G_c. The general understanding now, is that as the wire becomes thinner and thinner, the current is carried in a very few channels, each of them carrying a current of $2e^2/h$ (two for spin degeneracy) until we reach the lowest value which is just a single channel (which we interpret as the lowest eigenstate of the transverse Hamiltonian) carrying this current. Moreover, as the length decreases, the resistance does not decrease indefinitely but instead reaches a limiting value. One way of understanding this is to simply consider this to be a contact resistance, independent of the length of the wire, which arises simply because the conductor and the contacts are different. One cannot make the contacts the same as the

conductor, because then our assumption that the voltage drop is across the conductor alone does not make sense. That makes sense only if we assume that the contacts are infinitely more conducting than the conductor. So we are finally left with a non-zero resistance and the wire does not become infinitely conducting. In fact, in this limit, the conductance or resistance of the wire is purely a 'boundary' property and the 'conductivity' of the wire has no real significance. In fact, whether we get a finite conductivity or infinite conductivity depends on how one defines it.

However, for a single channel wire, clearly, the wire is one-dimensional and we know that interactions can change the picture drastically. The question that we want to answer here is precisely that. What is the conductance of a clean one-dimensional interacting wire or Luttinger wire?

Computing conductance of a clean one-dimensional (mesoscopic) wire

(a) Without leads

First, we shall perform a calculation to compute the conductance of a Luttinger liquid without any consideration of contacts or leads. (We shall restrict ourselves to spinless fermions since spin only increases the degrees of freedom and gives an overall multiplicative factor of two in the conductance). The conductance of a wire is calculated by applying an electric field to a finite region L of an infinitely long wire and the current I is related to the field as

$$I(x) = \int_0^L dx' \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\sigma_\omega(x,x')E_\omega(x')\ , \tag{191}$$

where $E_\omega(x')$ is the frequency ω component of the time Fourier transform of the electric field. The conductivity $\sigma_\omega(x,x')$, in turn, is related to the (imaginary time) current-current correlation function by the usual Kubo formula as

$$\sigma_\omega(x,x') = -\frac{e^2}{\bar{\omega}}\int_0^\beta\ d\tau\ \ < T_\tau j(x,\tau)j(x',0) > e^{-i\bar{\omega}\tau}\ , \tag{192}$$

where $\tau = it$, $\omega = i\bar{\omega} + \epsilon$, T_τ is the (imaginary) time ordering operator and $j(x,\tau)$ is the current operator. Both these formulae are standard in many books [14] on many body techniques, so here we will confine ourselves to just describing what they mean. The first equation describes the current as a response to an electric field (externally applied plus induced) of frequency ω. The proportionality function is the conductivity. To get the usual Ohmic formula, all we need to do is replace $\sigma = \sigma_0\delta(x-x')$ or remember that the $\sigma(x,x')$ is generally a function which is centered around $x \simeq x'$ and which falls off sufficiently fast elsewhere. The point for mesoscopic systems is that the length of the wire is roughly comparable with the range of $\sigma(x,x')$. Hence, the current gets contributions from the electric field all over the wire, which is different from what happens in the usual case, where the current at a point gets contributions only from the electric field very near that point. The second equation tells us that the conductivity is related to the current-current correlation function. This is derived by computing the current $I(x)$ in a Hamiltonian formulation to first order in the perturbation which is the applied electric field. The Euclidean formulation

is used so that the generalization to finite temperature calculations is straightforward, but we shall only work at zero temperature and hence take the $\beta \to \infty$ limit.

Our aim here will be to compute the current-current correlation function and hence the conductance for a Luttinger wire using bosonization. We shall denote the Euclidean time action of a generic Luttinger liquid as

$$S_E = \frac{1}{2K} \int d\tau \int dx \; [\frac{1}{v}(\partial_\tau \phi)^2 + v(\partial_x \phi)^2] \; . \tag{193}$$

(Note that in the spin model and Hubbard model, τ was replaced by it). The current can directly be expressed in terms of the boson operators as

$$j(x,\tau) \equiv v(\rho_R - \rho_L) = -\frac{i}{\sqrt{\pi}} \partial_\tau \phi \; . \tag{194}$$

(The extra factor of i is because we are now using imaginary time τ). Our first step is to obtain the correlation function $< j(x,\tau) j(x',0) >$ which is similar to the correlation functions for spinless fermions that we computed earlier when we were studying spin models, except that we are now interested in the Euclidean correlation functions. Since we can pull out the ∂_τ outside the correlation function[4], all we have to do is compute the propagator given by

$$G(\tau, x, x') =< T_\tau \phi(x,\tau) \phi(x',0) > \;\; , \tag{195}$$

or equivalently, its Fourier transform

$$G_{\bar{\omega}}(x,x') = \int_0^\beta d\tau < T_\tau \phi(x,\tau) \phi(x',0) > e^{-i\bar{\omega}\tau} \; . \tag{196}$$

The conductivity is then given by

$$\begin{aligned} \sigma_{\bar{\omega}}(x,x') &= \frac{e^2}{\bar{\omega}\pi} \int_0^\beta d\tau < T_\tau \partial_\tau \phi(x,\tau) \partial_\tau \phi(x',0) > e^{-i\bar{\omega}\tau} \\ &= \frac{e^2 \bar{\omega}}{\pi} G_{\bar{\omega}}(x,x') \; . \end{aligned} \tag{197}$$

So now, to compute the conductance, all we have to do is compute the propagator for the boson with a free Euclidean action. The propagator satisfies the equation

$$\frac{1}{K} \; (-v\partial_x^2 + \frac{\bar{\omega}^2}{v}) G_{\bar{\omega}}(x,x') = \delta(x-x') \; , \tag{198}$$

from which upon integrating once, we get

$$\frac{v}{K} \partial_x G(x,x') |_{x=x'-0}^{x=x'+0} = -1 \; . \tag{199}$$

[4]See R. Shankar in [15] for subtleties in pulling out the derivative outside the time ordering operator. One gets an extra term which cancels another term that we have ignored here, a singular c-number term.

The solution to the differential equation is given by

$$\begin{aligned} G(x,x') &= Ae^{|\bar{\omega}|(x-x')/v}, \quad x<x' \\ &= Ae^{-|\bar{\omega}|(x-x')/v}, \quad x>x' \ . \end{aligned} \tag{200}$$

Using this in Eq. (199), we see that

$$(\frac{2\bar{\omega}}{K})A = 1, \tag{201}$$

$$\begin{aligned} \text{so that} \quad G_{\bar{\omega}}(x,x') &= \frac{K}{2\bar{\omega}}e^{|\bar{\omega}|(x-x')/v}, \quad x<x' \\ &= \frac{K}{2\bar{\omega}}e^{-|\bar{\omega}|(x-x')/v}, \quad x>x' \end{aligned} \tag{202}$$

$$\begin{aligned} \text{leading to} \quad \sigma_{\bar{\omega}}(x,x') &= \frac{Ke^2}{2\pi}e^{|\bar{\omega}|(x-x')/v}, \quad x<x' \\ &= \frac{Ke^2}{2\pi}e^{-|\bar{\omega}|(x-x')/v}, \quad x>x' \ . \end{aligned} \tag{203}$$

The point to note is that in the $\bar{\omega} \to 0$ limit or static limit, the conductivity is finite and does not drop down to zero even for large $|x-x'|$. This is the main difference from macroscopic conductivities which always decay to zero as $|x-x'| \to \infty$. Furthermore, for $x=x'$, even for arbitrary $\bar{\omega}$, the Green's function has a finite value, which is responsible for the saturation value of the conductance. This only happens for a one-dimensional Green's function. In any other dimension, the Green's function and hence conductivity will be divergent at $x=x'$. Using this in the equation for the current, Eq. (191) for a static electric field $\bar{E}_{\omega}(x) = 2\pi\delta(\omega)E(x)$, we finally get

$$I(x) = \frac{Ke^2}{2\pi}\int_0^L dx' \ E(x') = \frac{Ke^2}{2\pi}(V_L - V_0) \tag{204}$$

which gives the final result for the conductance as

$$g = \frac{Ke^2}{2\pi} \ . \tag{205}$$

There are several subtle points to note in this calculation. One is that we have taken the $\omega \to 0$ before $|x-x'| \to \infty$, which is opposite to the usual order of limits in the Kubo formula. The physical justification for the usual order of limits in the Kubo formula comes from the fact that if we first take ω to zero, then we have a static electric field, which is periodic in space. This means that the charge will seek an equilibrium distribution after which there will be no flow of current. Setting $|x-x'| \to \infty$, on the other hand, means taking the thermodynamic limit or infinite length limit first, which allows for an unlimited supply of electrons and is probably equivalent to having reservoirs even if we have do not really have infinite length wires. For the mesoscopic systems, however, it is not correct to take the thermodynamic limit first. The physical situation here, is that one applies a static

electric field to a finite length of the wire L, which in fact, is comparable to the range of the conductivity. If we take the $|x - x'| \to \infty$ limit first, then it is as if we are looking at a long length of the wire beyond the range of the conductivity. This is the usual limit and we will get the usual Ohm's law, which however, is wrong in this context. In fact, it is instructive to try out the calculation with the other order of limits -*i.e.* by computing $\sigma_{\bar{\omega}}(q)$ and taking the $q \to 0$ limit first.
(H.W. Exercise 9: Try the above).

The second point is something we have mentioned earlier - *i.e.*, we have not taken contacts or leads into account. This was the initial computation by Kane and Fisher [16] and they obtained the answer in Eq. (205) that the conductance of the clean wire depends on the interaction parameter K.

(b) Including leads

When any experiment is done, however, one does have explicit contacts or leads. In fact, when a measurement was actually done under conditions where one expected to measure the Luttinger parameter K, it was found to $\simeq 1$, instead of 0.7, which was expected from other measurements of the K value of the wire. (We will see how else K can be measured after considering impurity scattering). So we need to understand what happens when we actually try to measure the conductance of a Luttinger wire.

How do we model the leads? The simplest model to consider is that the Luttinger wire is connected to Fermi liquid leads on either side. (See Fig. 6). So the regions A and C can be modelled by the same bosonic model with $K_L = 1$ and the wire in region B can be modelled as before as a Luttinger wire with $K = K$. But now, we have to put appropriate boundary conditions at the points P and P' between A and B and between B and C respectively. Note that we are making the assumption that one has the same ϕ field or same quasiparticle in all the three regions and it is only the LL parameters which are changing. Although, it is interesting to compute the conductance in this case, it is still not clear that this brings the calculation any closer to real experiments, because real experiments will have three dimensional reservoirs.

We start with the action in all the three regions in Euclidean space as

$$S_E = \frac{1}{2} \int_0^\beta d\tau \int_0^L dx \; [\frac{(\partial_\tau \phi)^2}{K(x)v(x)} + \frac{v(x)}{K(x)}(\partial_x \phi)^2] \; , \tag{206}$$

with $K(x) = K_L$, $v(x) = v_L$ in regions A and C and $K(x) = K$, $v(x) = v$ in region B. This is just the free action of a scalar field in all the three regions. Fourier transforming the imaginary time variable with respect to $\bar{\omega}$, we obtain

$$S_E = \frac{1}{2} \int_0^\beta d\tau \int_0^L dx \; [\frac{\bar{\omega}^2 \phi^2}{K(x)v(x)} + \frac{v(x)}{K(x)}(\partial_x \phi)^2] \; , \tag{207}$$

from which we see that the propagator satisfies the equation

$$\{-\partial_x(\frac{v(x)}{K(x)}\partial_x) + \frac{\bar{\omega}^2}{K(x)v(x)}\} \; G_{\bar{\omega}}(x, x') = \delta(x - x') \; . \tag{208}$$

Now let us consider the four regions. We assume that the interaction parameter changes abruptly at P and P', but that the Green's function is continuous and the derivative of the Green's function has the correct discontinuity at all the boundaries. So now, we need to solve the Green's function equation subject to these boundary conditions. Let us choose x' to lie between 0 and L. It is then easy to see that the solution is of the form

$$\begin{aligned} G_{\bar{\omega}}(x,x') = & \quad Ae^{|\bar{\omega}|x/v} \qquad \text{for} \;\; x \leq 0 \\ = & \quad Be^{|\bar{\omega}|x/v} + Ce^{-|\bar{\omega}|x/v} \quad \text{for} \;\; 0 < x \leq x' \\ = & \quad De^{|\bar{\omega}|x/v} + Ee^{-|\bar{\omega}|x/v} \quad \text{for} \;\; x' < x \leq L \\ = & \quad Fe^{-|\bar{\omega}|x/v} \qquad \text{for} \;\; x > L \end{aligned} \tag{209}$$

for semi-infinite leads, because we have assumed that the lengths of the leads are sufficiently long compared to L so that we do not need to put any further boundary conditions on them. Note that here the Green's functions will no longer be functions of $x - x'$ since we have explicitly broken translational invariance. The constants $A, B, ..., F$ are found by matching the boundary conditions. Since we are interested in the DC conductance, we only need the solutions for $\bar{\omega} \to 0$ which are easy to obtain and are given by

$$A = F = \frac{K_L}{2\bar{\omega}}, \quad B = E = \frac{K_L + K}{4\bar{\omega}}, \quad C = D = \frac{K_L - K}{4\bar{\omega}} \; . \tag{210}$$

From this, we see that $\sigma_{\bar{\omega}}(x,x')$ is x and x' independent in the $\bar{\omega} \to 0$ limit and is equal to $K_L e^2/2\pi$ in all regions from which we find the conductance (using Eq. (191)) given by

$$g \equiv \frac{I}{V} = \frac{K_L e^2}{2\pi} \; . \tag{211}$$

is the same in all the regions. Thus, the conductance is determined by the K_L of the leads, which is just $K_L = 1$ for Fermi liquid leads and does not depend on interactions in the wire. This is a highly counter-intuitive answer! It is telling us that whether we measure the conductance in the leads or in the quantum wire, we get the same answer, so long as we take into account the fact that we are attaching leads, which allow for the fermions to enter and leave the quantum wire. At a very naive level, one may understand this by saying that since the wire itself has no impurities, the only source of resistance is the contact effect between the leads and the wire, which has nothing to to do with the interactions in the wire. However, remember that we have taken semi-infinite leads and abrupt contacts and we are only looking for DC conductance. If any of these assumptions are relaxed, certainly, there are differences in the three regions and one could get more interesting answers.

In fact, using a Landauer-Buttiker scattering approach [18], there has been some attempt to understand these results more intuitively.

Physically, the difference between this computation and the earlier one is that any real measurement requires Fermi liquid leads. So the end result is that the measurable conductance of an interacting one-dimensional wire is simply given by $g = e^2/h$ for spinless fermions and $g = 2e^2/h$ for fermions with spin [17].

9 Transport in the presence of isolated impurities

Computing conductance with a single impurity

Now let us consider the case when there is a single impurity at the origin. At first, we will model the impurity as a weak barrier and add a term to the action of the form

$$S_{\rm int} = \int dx d\tau \; V(x)\psi^\dagger(x)\psi(x) \; . \tag{212}$$

We assume that $V(x)$ is weak and is centred around the origin. For instance, we can choose $V(x) = \lambda\delta(x)$, where λ is much less than the Fermi energy.

First, let us think of what happens when we introduce such a perturbation in a non-interacting wire. In that case, all one has is a one-dimensional quantum mechanics problem with a δ-function potential at the origin. We can find the reflection and transmission probabilities for a single particle with momentum k as

$$R = \frac{\lambda^2}{\lambda^2 + k^2} \quad \text{and} \quad T = \frac{k^2}{\lambda^2 + k^2} \; . \tag{213}$$

So for any λ, one gets both reflection and transmission. To get the total current, we just have to sum up the contributions of all the electrons close to the Fermi surface. But it is clear that there will be non-zero conductance for any potential, with the amount of current being transmitted depending on the strength of the potential. However, for the Luttinger wire, since there exists interactions between electrons in the wire, and no convenient quasiparticle picture, one cannot solve the problem this way. We have to use the bosonized field theory and include the impurity potential as a perturbation.

Let us first rewrite the impurity potential in terms of the left- and right- moving low energy Dirac modes. We find that

$$\begin{aligned} \psi^\dagger(x)\psi(x) &= (\psi_R^\dagger e^{-ik_F x} + \psi_L^\dagger e^{ik_F x})(\psi_R e^{ik_F x} + \psi_L e^{-ik_F x}) \\ &= \psi_R^\dagger\psi_R + \psi_L^\dagger\psi_L + e^{-i2k_F x}\psi_R^\dagger\psi_L + e^{i2k_F x}\psi_L^\dagger\psi_R \\ &= -\frac{1}{\sqrt{\pi}}\partial_x\phi + \frac{1}{2\pi\alpha}(\eta_R^\dagger\eta_L e^{i2\sqrt{\pi}(\phi_R+\phi_L+2k_F x)} + \eta_L^\dagger\eta_R e^{-i2\sqrt{\pi}(\phi_R+\phi_L+2k_F x)}), \end{aligned} \tag{214}$$

where the last line is obtained using standard bosonization. So the full action is given by

$$S = S_E + S_{\rm int} = S_E - \frac{\lambda}{\sqrt{\pi}}\partial_x\phi(0) + \frac{\lambda}{2\pi\alpha}\int d\tau \cos 2\sqrt{\pi}[\phi_R(0) + \phi_L(0)] \; , \tag{215}$$

where S_E is given in Eq. (193) and we have incorporated the fact that the potential only acts at the origin. Moreover, we have simply set both $\eta_R^\dagger\eta_L$ and $\eta_L^\dagger\eta_R$ to be one, with the knowledge that in correlation functions, we will compute $O(\tau)O^\dagger(0)$ so that the Klein factors disappear using $\eta_{R/L}\eta_{R/L}^\dagger = \eta_{R/L}^\dagger\eta_{R/L} = 1$. The first term due to the interaction can be taken care of by a simple redefinition of $\partial_x\phi \to \partial_x\phi' = \partial_x\phi + \lambda/2\sqrt{\pi}$, which makes

no difference to the conductance. This could have been seen even from the fermion terms from which it came. The $\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L$ term only causes scattering at the same Fermi point with momentum transfers $q << 2k_F$. This does not change the direction of propagation of the particles and hence does not affect conductance in any appreciable way. But the cosine term, on the other hand, occurs because of backscattering of fermions from the origin. These represent scattering with $q \sim |2k_F|$ - *i.e.*, from the left branch to the right branch and vice-versa and change the direction of propagation of the particles. These scatterings will definitely affect the conductance. The action with this perturbation is no longer quadratic and cannot be exactly solved. However, since λ is a weak perturbation, one can try to use perturbation theory and the renormalization group approach to see the relevance of this perturbation at low energies.

What is the question that we want to answer? We want to compute the conductance through this barrier at low energies. One way to do that is to see whether this barrier coupling strength grows or becomes smaller as we go to lower energy scales. To check that, we need to perform the usual steps of a renormalization group analysis.

Here since the perturbation term is fixed in space, it is more convenient to first integrate out the variables away from the origin and write down the action purely in terms of the $\phi(x = 0, \tau)$ variables. Since integrating out quadratic degrees of freedom is equivalent to using equations of motion for those degrees of freedom, we write down the equations of motion for the action S_0 as

$$\partial_x^2 \phi - \frac{\bar{\omega}^2}{v^2}\phi = 0 \Rightarrow \partial_x^2 \phi - k^2 \phi = 0 \ . \tag{216}$$

The solution to the above equations are given by

$$\begin{aligned} \phi &= A e^{|k|x}, \quad x > 0 \\ &= A e^{-|k|x}, \quad x < 0 \ , \end{aligned} \tag{217}$$

where $A \equiv \phi(x = 0, \tau)$. Using this solution in the action, we get the effective action in terms of $\phi(\bar{\omega}) = \int \phi(x = 0, \tau) e^{i\bar{\omega}\tau} d\tau$ as

$$\begin{aligned} S_{\text{eff}} &= \frac{1}{2K}\int \frac{d\bar{\omega}}{2\pi} \int_{-\infty}^{0} dx \ [v\phi^2 k^2 e^{2kx} + \frac{\bar{\omega}^2}{v} e^{2kx}] + \frac{1}{2K}\int \frac{d\bar{\omega}}{2\pi} \int_{0}^{\infty} dx \ [v\phi^2 k^2 e^{-2kx} + \frac{\bar{\omega}^2}{v} e^{-2kx}] \\ &= \frac{1}{2K}\int \frac{d\bar{\omega}}{2\pi} \ \frac{2\phi^2\bar{\omega}^2}{v} \ [\ \frac{e^{2kx}}{2k}|_{-\infty}^{0} + \frac{e^{-2kx}}{-2k}|_{0}^{\infty} \ ] \\ &= \frac{1}{K}\int \frac{d\bar{\omega}}{2\pi} \ |\bar{\omega}|\phi^2 \end{aligned} \tag{218}$$

using $k = |\bar{\omega}|/v$. Notice the singular dependence on the Matsubara frequency $|\bar{\omega}|$. The reason for its appearance is the following. In real space, even for a quadratic action, all degrees of freedom (dof) are coupled. (It is only in Fourier space that every mode is decoupled). So when we integrate out all dof except the one at the origin, the dispersion of this degree of

freedom can change and has changed. This is why we get the modulus factor in the effective action. When we Fourier transform back to imaginary time, we get

$$\begin{aligned}\sum_{\bar{\omega}} |\bar{\omega}|\phi_n^2 &\rightarrow i\int \frac{d\omega}{2\pi}\int d\tau \int d\tau' \; e^{i\omega(\tau-\tau')}|\omega_n|\phi^*(\tau)\phi(\tau') \\ &= -\int d\tau d\tau' \frac{2}{(\tau-\tau')^2}\phi^*(\tau)\phi(\tau') \; , \end{aligned} \tag{219}$$

i.e., an explicitly non-local interaction in imaginary time.

So now, we have an action solely in terms of the variables at the origin with the action given by

$$S = \frac{1}{2K}\int \frac{d\bar{\omega}}{2\pi} \; |\bar{\omega}|\phi(\bar{\omega})^2 + \lambda \int \frac{d\bar{\omega}}{2\pi} \; \cos[2\sqrt{\pi}\phi(\bar{\omega})] \; . \tag{220}$$

The RG analysis now involves finding out how the coefficient λ behaves as we go to lower and lower energies. Before we perform the RG analysis, we may ask why would we want to go to lower energy scales? The general idea is that in spite of the fact that in different physical problems or models, the parameter λ may be slightly different, qualitatively many such models may have the same behavior. This is because they are all governed by the same fixed point Hamiltonian with the fixed point Hamiltonian being defined as the Hamiltonian one gets when the RG flow stops. So the aim is to keep reducing the energy scale till the RG flow stops so that we can find out the appropriate fixed point Hamiltonian for this model.

In this problem, we want to find out whether the fixed point Hamiltonian has a large barrier or a small barrier. To find that out, let us perform the three steps of the renormalization group transformation. We choose a high frequency cutoff Λ, which is the real physical cutoff of the theory. Then we rescale $\Lambda \rightarrow \Lambda/s$ with $s > 1$ and then divide $\phi(\bar{\omega})$ into $\phi_<(\bar{\omega})$ (slow modes) and $\phi_>(\bar{\omega})$ (fast modes) for the modes with frequencies less than or greater than the cutoff Λ/s respectively. Finally, we integrate out the fast modes, which are the modes between Λ/s and Λ and rescale $\bar{\omega} \rightarrow \bar{\omega}' = s\bar{\omega}$ or $\tau \rightarrow \tau' = \tau/s$ to get back to the original range of integrations. To lowest order, (tree level contribution) we find that

$$\lambda \int d\tau \cos 2\sqrt{\pi}\phi_<(x=0,\tau) \rightarrow \lambda s^{1-d}\int d\tau \cos 2\sqrt{\pi}\phi_<(x=0,\tau) \; , \tag{221}$$

where d is the dimension of the cosine operator. This was explicitly computed earlier and we found that $d = K$. The RG equation is now easily obtained by taking $s = 1 + dl$, for infinitesimal dl. We find that the new λ' after rescaling is given by

$$\begin{aligned} & \lambda' = \lambda(1+dl)^{1-K} \\ \Rightarrow \quad & \lambda' - \lambda = (1-K)\lambda dl \\ \Rightarrow \quad & \frac{d\lambda}{dl} = (1-K)\lambda \; . \end{aligned} \tag{222}$$

Normally, one would have had coupled RG equations for λ and K. But here since $1/K$ is the coefficient of a singular operator, it does not get renormalized to any order.

Notice that in Eq. (221), the coefficient of the operator gets rescaled by a factor s^{1-d} rather than s^{2-d_i} as we had mentioned earlier when we computed the dimension of the cosine operator in the spin model. The difference is that the operator in the spin model, in the action, required integration over both space and time. So we rescaled both the space and time (or equivalently both the momentum and the energy). However, in this case, the operator exists only at a fixed space point. So we only need to integrate over the time coordinate. Hence, the naive scaling dimension or engineering dimension of the operator is 1 and not 2. Such operators are called boundary operators. You will learn more about them in the course on boundary conformal field theory.

The RG equation is now trivial to analyze. For any $K > 1$, (which corresponds to attractive interactions between the electrons), the λ renormalizes to zero and for any $K < 1$, (corresponding to repulsive interactions), it grows stronger and stronger. In other words, for $K > 1$, the fixed point Hamiltonian is just the free boson Hamiltonian with no barrier and for $K < 1$, the fixed point Hamiltonian has two disconnected wires to the left and right of the origin. For $K = 1$, which is the limit of no interactions in the fermionic model, the coupling is marginal. (This was expected, since we know that for free fermions, both transmission and reflection occurs depending on the strength of the barrier potential). Thus, for attractive interactions, the barrier renormalizes to zero and the wire is 'healed', whereas for repulsive interactions, the barrier renormalizes to infinity and the wire is 'cut'. Note that both these answers are completely independent of the strength of the barrier potential [16].

Strong barrier limit

Since, we are doing perturbation theory, we cannot assume that this result holds for arbitrary λ. It is strictly valid only for $\lambda \simeq 0$. Once $\lambda \sim 1$, the perturbative analysis in λ breaks down. So it is worthwhile to try and see what happens in the other limit. Supposing we start with two decoupled wires and then allow a small hopping between the two wires. Will this hopping grow at low energies and heal the wire or will it renormalize to zero?

Here, we start with two semi-infinite Luttinger liquid wires and analyze the effect of adding a small hopping term coupling the two wires at $x = 0$. The models for $x < 0$ and $x > 0$ are given by the action

$$S_E = \frac{1}{2}\int_0^\beta d\tau \int_0^L dx \; [\frac{(\partial_\tau \phi_i)^2}{K(x)v(x)} + \frac{v(x)}{K(x)}(\partial_x \phi_i)^2] \tag{223}$$

for $i =<$ and $i =>$ respectively. We can also write it in terms of the dual variables as

$$S_E = \frac{1}{2}\int_0^\beta d\tau \int_0^L dx \; [\frac{K(x)}{v(x)}(\partial_\tau \theta_i)^2 + K(x)v(x)(\partial_x \theta_i)^2] \; . \tag{224}$$

Note that in terms of the dual variables, the action has $1/K$ in position of K. This is because the roles of the fields and the canonically conjugate momenta have interchanged. The fact that the wire is cut implies that at the point $x = 0$, there is zero density of either $<$ or $>$ particles - $\psi^\dagger_< \psi_<(x = 0) = 0$ and $\psi^\dagger_> \psi_>(x = 0) = 0$. In the bosonic language, this is imposed as $2\sqrt{\pi}\phi_<(x = 0) = 2\sqrt{\pi}\phi_>(x = 0) = \pi/2$ (and also $\partial_x \phi(x = 0) = 0$ as can be seen from Eq.

(214)). Now a term which hops an electron from one wire to another in the Hamiltonian is just

$$\begin{aligned} H &= -t\,[\psi^\dagger_< \psi_> + h.c.] \\ &= -t\,[\psi^\dagger_{R<}\psi_{R>} + \psi^\dagger_{L<}\psi_{L>} + \psi^\dagger_{R>}\psi_{R<} + \psi^\dagger_{L>}\psi_{L<} \\ &\quad + \psi^\dagger_{L<}\psi_{R>} + \psi^\dagger_{L>}\psi_{R<} + \psi^\dagger_{R<}\psi_{L>} + \psi^\dagger_{R>}\psi_{L<}]\ , \end{aligned} \tag{225}$$

where the second equation involves the left and right moving fields and we have already set $x = 0$. Here, again, the terms that involve fields at one Fermi point are low energy forward scattering terms which do not affect the conductance. In terms of the bosonic fields too, they can be taken care of by trivial redefinitions. But the intra-Fermi point scatterings which will affect the conductance can be bosonized and written in the action as

$$\begin{aligned} \delta S = -t \int d\tau\ [&\quad \eta^\dagger_{L<}\eta_{R>} e^{-i(\phi_{L<}+\phi_{R>})} + \eta^\dagger_{L>}\eta_{R<} e^{-i(\phi_{L>}+\phi_{R<})} \\ &+ \eta^\dagger_{R<}\eta_{L>} e^{-i(\phi_{R<}+\phi_{L>})} + \eta^\dagger_{R>}\eta_{L<} e^{-i(\phi_{R>}+\phi_{L<})}]\ . \end{aligned} \tag{226}$$

Now, we impose the boundary condition on the bosonic fields that we mentioned above, which constrains $\phi(0) = \phi_R(0) + \phi_L(0)$ to be equal to $\pi/2$. Using this, we can express the above equation solely in terms of the $\phi_L(0) - \phi_R(0) = \theta(0)$ fields and get

$$\delta S = 4t \int d\tau\ \cos(\theta_> - \theta_<)\ , \tag{227}$$

where once again, we have been able to drop the Klein factors after checking that they do not lead to any extra minus signs in the correlation functions. (Physically, the reason why we only get the θ_i term at the origin is because the constraint has set $\phi_i(x = 0) = \pi/2$). Computing the dimension of this operator, we see that to leading order, the RG equations are given by

$$\frac{dt}{dl} = (1 - \frac{1}{K})t\ . \tag{228}$$

(K has been replaced by $1/K$ because we now have to compute the dimensions in the dual action). Thus, for repulsive interactions ($K < 1$), the hopping term is irrelevant and flows to zero. This confirms the weak barrier calculation that the wire is insulating. On the other hand, for attractive interactions, the hopping strength grows, ultimately healing the wire. This again is in accordance with the weak coupling analysis.

Intermediate fixed points?

We have started from a wire with a weak barrier and shown that under repulsive interactions, the barrier strength grows. We have also started from two decoupled wires and shown that for repulsive interactions, any small hopping term renormalizes to zero. Hence, it seems plausible to conclude that for repulsive interactions in the wire, any barrier will cut the wire and the conductance goes to zero. However, one should keep in mind that our analysis is strictly true only for $\lambda, t \simeq 0$. Hence, it could happen that for intermediate values of the barrier strength, one could have a pair of non-trivial fixed points (see Fig. 7).

Conductance at finite voltage and temperature

The earlier analysis only tells us how the barrier strength or the tunneling amplitude grows or falls as we go to low energies. But instead of allowing the energy scale to become arbitrarily low, we can cut off the energy scale of renormalization at some finite energy scale, which could be the temperature T or the voltage V. Note that the energy scale at which we want to cutoff the integral is related to the initial high energy scale at which we start the RG as $E = E_0 e^{-l}$. So for attractive interactions for which weak barriers are irrelevant and for which one would expect perfect transmission at very low energies will have power law corrections when we put the lower energy cutoff as E. In that case, we have

$$\int_{\Lambda_0}^{\Lambda} \frac{d\lambda}{\lambda} = \int_0^{ln(E_0/E)} dl(1-K) \ , \tag{229}$$

which means that the effective barrier strength Λ is proportional to $\Lambda_0(E/E_0)^{K-1}$. So by choosing $E = T, V$, we see that one can get power law corrections to the naive conductance at $T \to 0, V \to 0$. In other words, if we measure the conductance at a finite temperature T, rather than at $T = 0$, instead of zero conductance for $K < 1$, we will get conductances which go as T^{1-K} (roughly the inverse of the barrier strength). Similarly, if instead of measuring conductances as $V \to 0$, we measure them at finite voltages, we find that the conductances go as V^{1-K}. On the other hand, for repulsive interactions, we need to start at the strong coupling limit with two decoupled Luttinger wires and allow for a small hopping, which is irrelevant in the RG sense. Here, again, if we cutoff the lower energy scale at E, we expect instead of zero transmission, power law corrections of the form $I \sim V^{1-1/K}$ and $I \sim T^{1-1/K}$. The only difference in the analysis at the strong coupling fixed point and the weak coupling fixed point is that K gets replaced by $1/K$ as we saw in the RG equations. This, in fact, is one way in which K can be measured in experiments. They could explicitly make a constriction in the quantum wire and measure conductances through it and extract K.

10 Concluding Remarks

Almost any interacting quantum system in one dimension which is gapless and has a linear dispersion for the low-energy excitations can be described as a Luttinger liquid at low energies and long wavelengths. As we have seen, the properties of a Luttinger liquid are determined by the two parameters v and K. These in turn depend on the various parameters which appear in the microscopic Hamiltonian of the system. Some examples of systems where Luttinger liquid theory and bosonization can be applied are quantum spin chains (including some spin ladders), quasi-one-dimensional organic conductors and quantum wires (with or without impurities), edge states in a fractional quantum Hall system, and the Kondo problem. Some of these examples have been discussed above.

Antiferromagnetic spin-1/2 chains have a long history going back to their exact solution by the Bethe ansatz. In recent years, many experimental systems have been studied which are

well-described by quasi-one-dimensional half-odd-integer spin models with isotropic (Heisenberg) interactions. Such systems behave at low energies as a $K = 1/2$ Luttinger liquid with an $SU(2)$ symmetry. It seems to be difficult to vary K experimentally in spin systems. In contrast, a single-channel quantum wire (which is basically a system of interacting electrons which are constrained to move along one particular direction) typically has two low-energy sectors, both of which are Luttinger liquids (except at special densities like half-filling). One of these is the spin sector which again has $K = 1/2$. The other one is the charge sector whose K value depends on a smooth way on the different interactions present in the system. Finally, the edge states in a fractional quantum Hall system behave as a chiral Luttinger liquid with K taking certain discrete rational values; the value of K can be changed by altering the electron density and the magnetic field in the bulk of the system. For all these systems, many properties have been measured such as the response to external electric and magnetic fields (conductivity or susceptibility) and to disorder, scattering of neutrons or photons from these systems, and specific heat; so the two Luttinger parameters can be extracted from the experimental data. The measurements clearly indicate the Luttinger liquid-like behavior of these systems with various critical exponents depending in a non-universal way on the interactions in the system.

On the theoretical side, a large number of exactly solvable models in one dimension have been shown to behave as Luttinger liquids at low energies [2, 12]. These include
(i) models with short range interactions which are solvable by the Bethe ansatz, such as the XXZ spin-1/2 chain (where K can take a range of values from 1/2 to ∞; this includes the XY model with $K = 1$ and the isotropic antiferromagnet with $K = 1/2$ as special cases), and the repulsive δ-function Bose gas (where K can go from 1 in the limit of infinite repulsion to ∞ in the limit of zero repulsion), and
(ii) models with inverse-square interactions such as the Calogero-Sutherland model (where K can go from 0 to ∞) and the Haldane-Shastry spin-1/2 model (where $K = 1/2$).
The models of type (ii) are *ideal* Luttinger liquids in the sense that they are scale invariant; the coefficients of all the marginal operators vanish, and therefore their correlation functions and excitation energies contain no logarithmic corrections. This property makes it particularly easy to study these systems numerically since the asymptotic behaviors are reached even for fairly small system sizes.

What has been left out?

Finally, let us mention the various important things in this field which has been left out. We have only worked with spinless fermions in the transport analysis. When we include spin and do not destroy the $SU(2)$ spin symmetry of the system, the results are very similar to the spinless fermion case. For repulsive interactions, the barrier becomes infinite and for attractive interaction, the barrier is healed. However, when the $SU(2)$ symmetry is destroyed, there exists possibilities of intermediate (non-trivial) fixed points where either spin or charge can be transmitted and the other reflected. The other thing that has been left out is the phenomenon of resonant tunneling with two impurities. This is an interesting result, because it says that for repulsive interactions, a single impurity cuts the wire, but

with two impurities, one can have particular energies, where there can be transmission. The reason, of course, is quantum mechanical tunneling. Here, the energy levels, are the energy levels of the quantum dot that is formed by the two impurities and one can have resonant tunneling at these energy levels. If we include interactions between the electrons on the island, (which is naturally included in the bosonized formalism), we can obtain the physics of the Coulomb blockade. The other important thing that we have left out, from a physical point of view, is what happens if there is a finite density of random impurities. In general, one would expect Anderson localization and no transport. But there are regimes of delocalization as well in the phase diagram. Finally, a very important application where the physics of the Luttinger liquids has actually been experimentally seen is in the edge states of the fractional Quantum Hall fluid. Since here, the edge states are chiral, a lot of the complications of backscattering due to impurities are avoided and it is possible to explicitly construct constrictions and allow tunneling through them. Here, both at the theoretical and experimental level, there are a lot of beautiful results that are worth understanding.

Another important topic not covered here is non-abelian bosonization [3]. This is a powerful technique for studying one-dimensional quantum systems with a continuous global symmetry such as $SU(2)$. For instance, isotropic Heisenberg antiferromagnets and Kondo systems are invariant under spin rotations, and they can be studied more efficiently using non-abelian bosonization.

To conclude, let us just say that low dimensional systems and mesoscopic systems have gained in importance in the last few years. Although currently, much of the theoretical work in mesoscopic systems has only involved conventional Fermi liquid theories, it is clear that there are regimes where strong interactions are very important. We expect that bosonization will be one of the important non-perturbative tools to analyze such problems for a few more years to come.

Acknowledgments

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Figure Captions

1. One-particle momentum distribution function. (a) shows the finite discontinuity at the Fermi momentum k_F for a system of interacting fermions in more than one dimension. (b) shows the absence of a discontinuity in an interacting system in one dimension.
2. Picture of the Fermi sea of a lattice model; the momentum lies in the range $[-\pi, \pi]$. The occupied states (filled circles) below the Fermi energy $E_F = 0$ and the two Fermi points at momenta $\pm k_F$ are shown.
3. The one-particle states of a right-moving fermion showing the occupied states (filled circles) below zero energy and the unoccupied states above zero energy.
4. Two possible particle-hole excitations of a right-moving fermionic system showing the occupied states.
5. The one-particle states of a left-moving fermion showing the occupied states below zero energy and unoccupied states above zero energy. Note that the momentum label k increases towards the left.
6. The single channel quantum wire with Fermi liquid leads on the left and the right.
7. Renormalization group flow diagram for a quantum wire with repulsive interactions in the presence of an impurity or barrier. In the absence of any non-trivial fixed points, the stable fixed point is the strong coupling fixed point. But perturbative analyses at the strong and weak coupling fixed points cannot rule out a pair of non-trivial fixed points at intermediate strengths of the barrier potential.

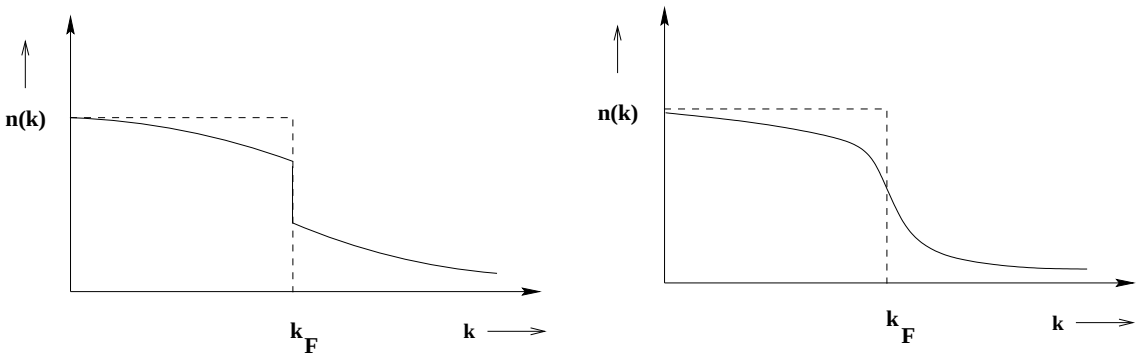

Fig. 1

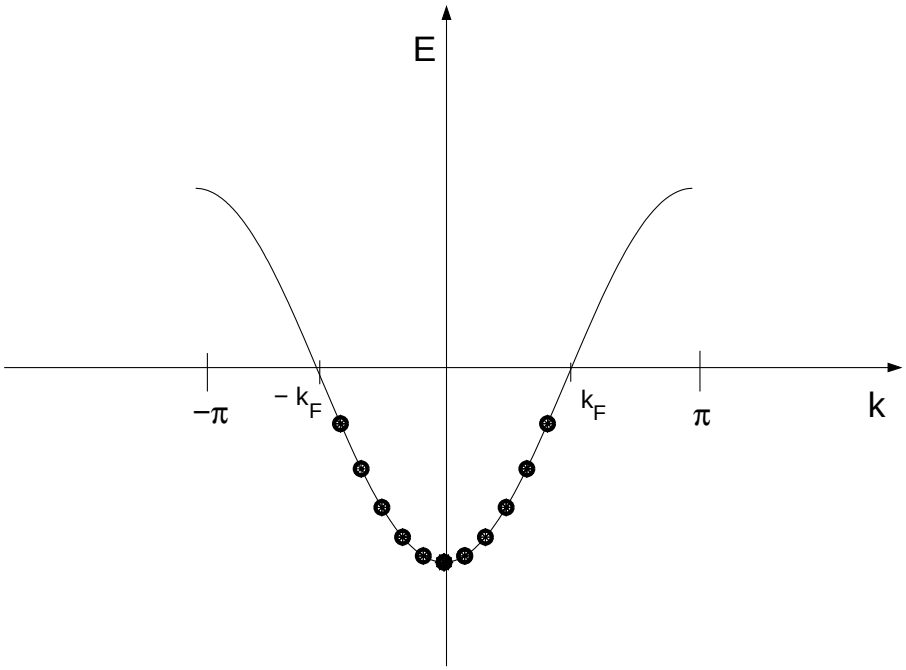

Fig. 2

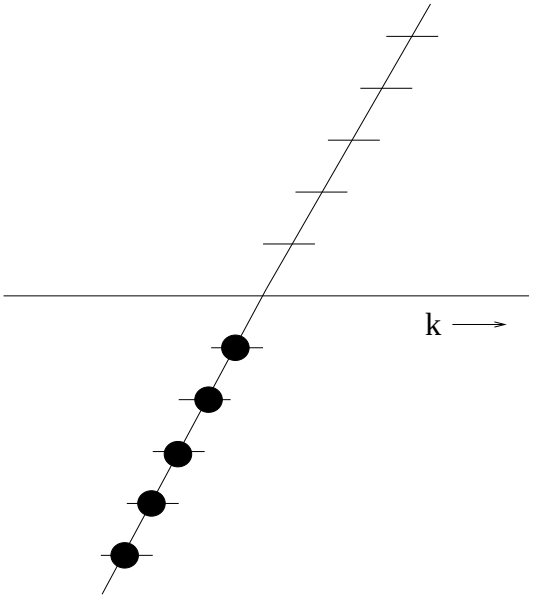

Fig. 3

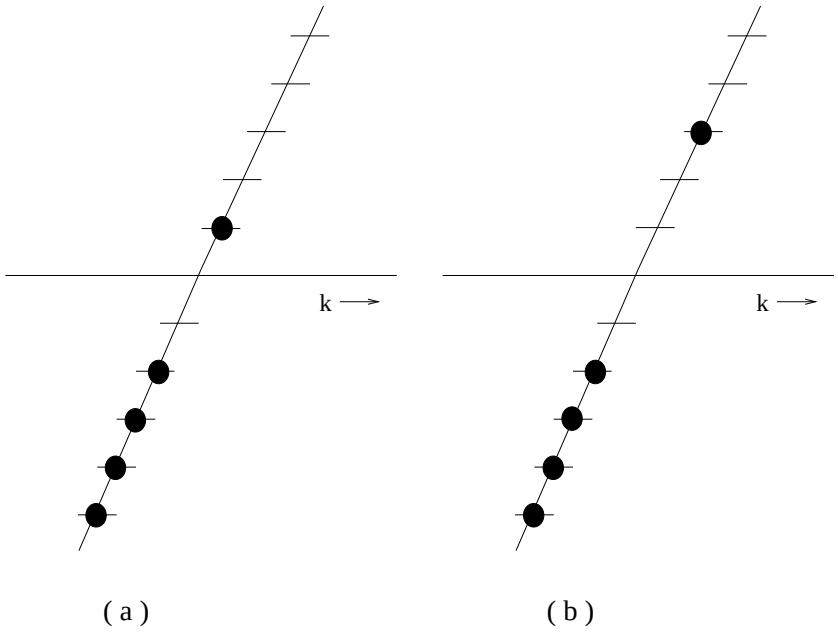

Fig. 4

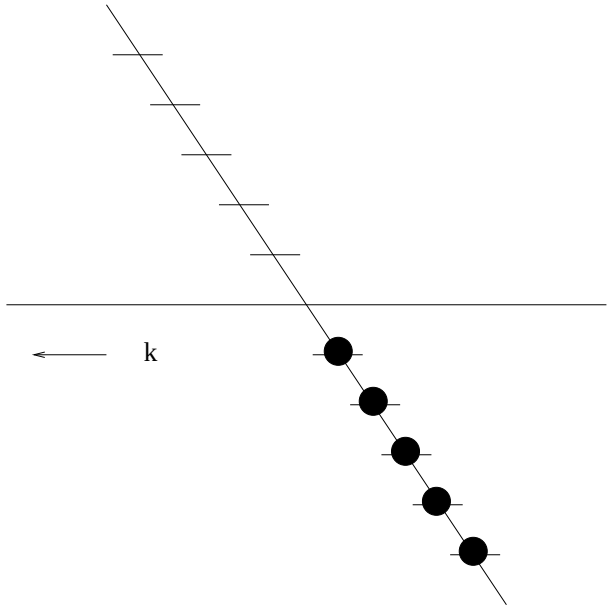

Fig. 5

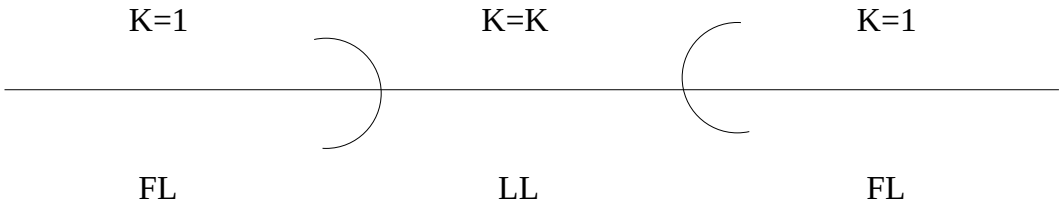
K=1
K=K
K=1
FL
LL
FL

Fig. 6

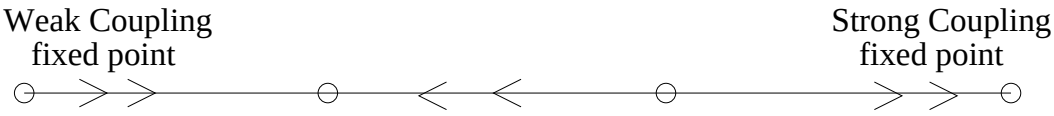
Weak Coupling
fixed point
Strong Coupling
fixed point

Fig. 7