Introduction to [Theoretical] Quantum Optics

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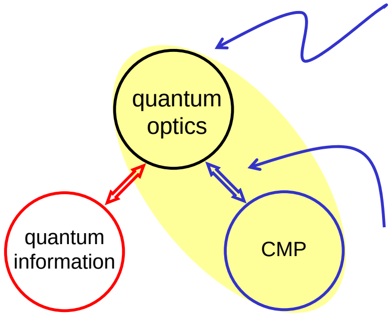


QO features:

- control / timedependence
- quantum state engineering / computing / nonclassical states
- coupling to the environment
- decoherence, measurement, preparation

Merging of QO and CMP …

- quantum degenerate atomic gases & strongly correlated systems
- quantum optics with a few atoms and photons & mesoscopic physics:
  trapped ions, atoms, CQED, …

These lectures:

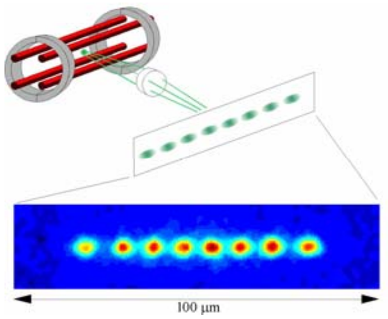
- qo systems
- qo ideas
- qo methods

Outline

- What is quantum optics?
  - a qualitative tour of quantum optical systems & problems
  - a conceptual overview
- Elementary (atomic) quantum optics Hamiltonian
  - trapped atoms, ions etc.
- Quantum state engineering
  - illustrating quantum state engineering with trapped ions: fast gates etc.
- Quantum Noise
  - …

What is quantum optics? … a zoo of quantum optical systems

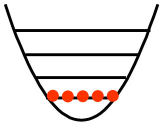
- trapped and laser cooled ions

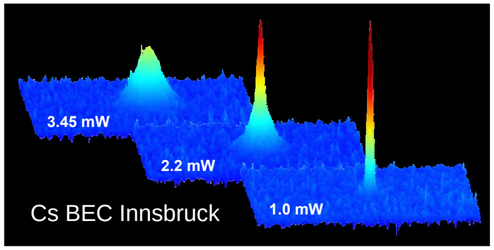


Few particle system with complete quantum control.

- quantum state engineering: quantum computing
- state preparation & measurement

- neutral atoms: BEC





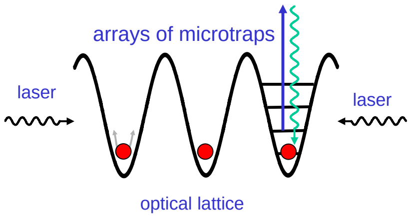
65,000 atoms (thermal) | 40,000 atoms (onset of condensation) | 16,000 atoms (BEC)

quantum optical problems:

- condensate growth … in analogy to switching on a laser
- coherence properties

… *strongly* interacting systems with *dilute* atomic gases

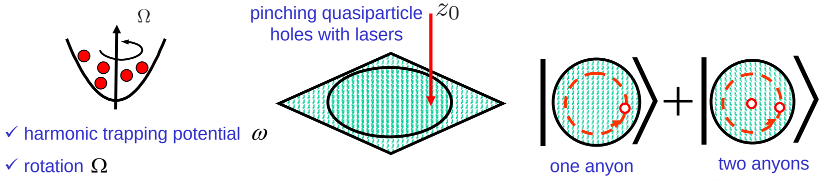
- Hubbard models with controllable parameters in optical lattices



qo features:

- engineering optical potential
- time dependent problem (non-equilibrium)

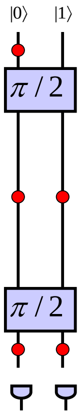
- Laughlin states & manipulating anyons (in small systems)



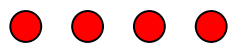
✓ harmonic trapping potential ω

✓ rotation Ω

… measurements beyond standard quantum limit



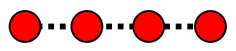
- N independent atoms



$$\Delta\omega_{\text{SQL}} = \frac{1}{T\sqrt{n_{\text{rep}}}}\frac{1}{\sqrt{N}}$$

standard quantum noise limit

- N entangled atoms

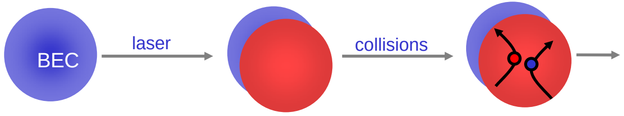


$$\Delta\omega_{\text{ent}} = \frac{1}{T\sqrt{n_{\text{rep}}}}\frac{1}{f(N)} \geq \frac{1}{T\sqrt{n_{\text{rep}}}}\frac{1}{N}$$

Heisenberg limit: maximally entangled state

$|0000\rangle + |1111\rangle$

- Entanglement via collisions

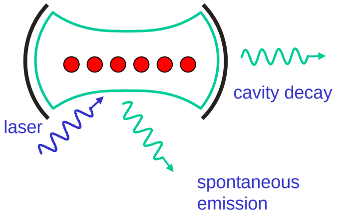


$|0\rangle^{\otimes N}$ product state

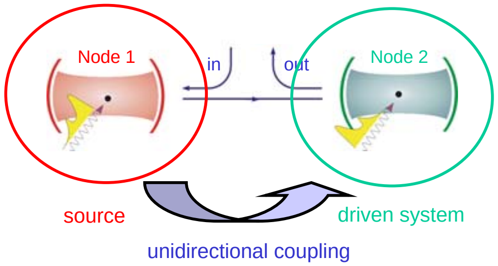
$[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)]^{\otimes N}$ product state

$\sum c_n |0\rangle^n |1\rangle^{N-n}$

- Cavity QED



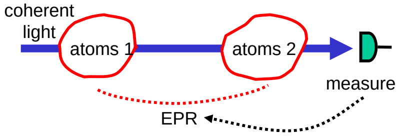
- cascaded quantum system: transmission in a quantum network



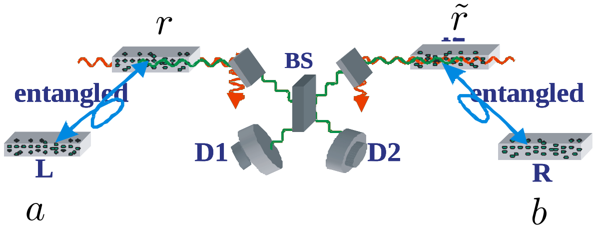
- quantum feedback

- atomic ensembles [see M. Lukin's lectures]

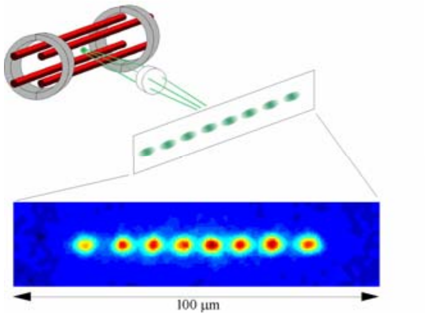
atomic / spin squeezing



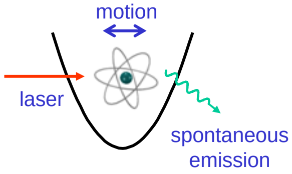
quantum repeater: establishing long distance EPR pairs

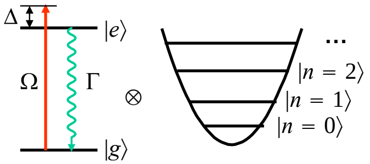


Example: trapped ions

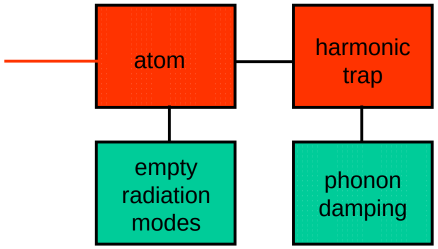


Single trapped ion



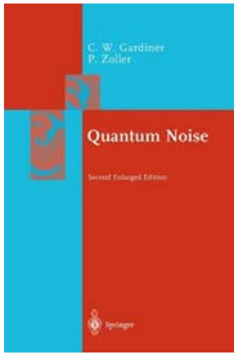


$$H_{\text{sys}} = -\tfrac{1}{2}\Delta\sigma_z + \left(\frac{\hat{p}^2}{2m} + \tfrac{1}{2}m\nu^2\hat{x}^2\right) + \tfrac{1}{2}\Omega(e^{ik\hat{x}}\sigma_+ + e^{ik\hat{x}}\sigma_-)$$

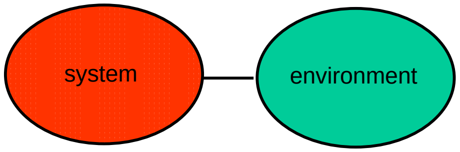


Quantum Optical Systems … a formal point of view

- composite systems
  - quantum state engineering
- open quantum systems
  - decoherence
  - measurement



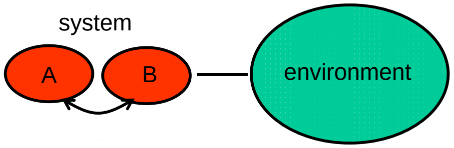
System + Environment



$$H_{\text{tot}} = H_{\text{sys}} + H_B + H_{\text{int}}$$

- microscopic understanding of Hamiltonians
- controllable parameters

Composite systems



coherent coupling

quantum engineering:

- engineer interesting quantum states of atoms, photons etc.
- examples / applications
  - quantum information = entangled state engineering
  - precision measurement
- requirement: strong coupling limit coherent coupling >> dissipation

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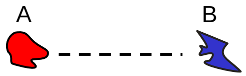
Entangled States & Engineering Entangled States

- entanglement



states: $|0\rangle \otimes |0\rangle$

$|1\rangle \otimes |1\rangle$ ... product states

but also ...

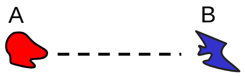
$\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$ ... entangled

- fundamental aspects of quantum mechanics
  - incompatibility of QM with LHVT
  - decoherence
  - measurement theory (?)
- applications
  - quantum communications & computing
  - precision measurement

Entangled States & Engineering Entangled States

We need ...

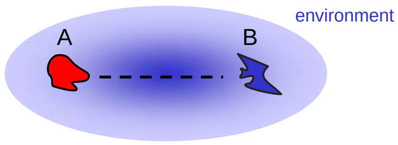
- "quantum engineering"



$$|a\rangle_A|b\rangle_B \to \sum c_{ab}|a\rangle_A|b\rangle_B$$

Hamiltonian evolution

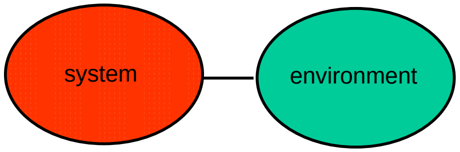
- isolation



$$|\phi\rangle_A|\varphi\rangle_B|E\rangle \to |\Psi\rangle_{ABE}$$

$$\rho_{AB} = \mathrm{tr}_E|\Psi\rangle_{ABE}\langle\Psi| \neq |\Psi\rangle_{AB}\langle\Psi|$$

Quantum optical systems as *Open* Quantum Systems



role of coupling to environment:

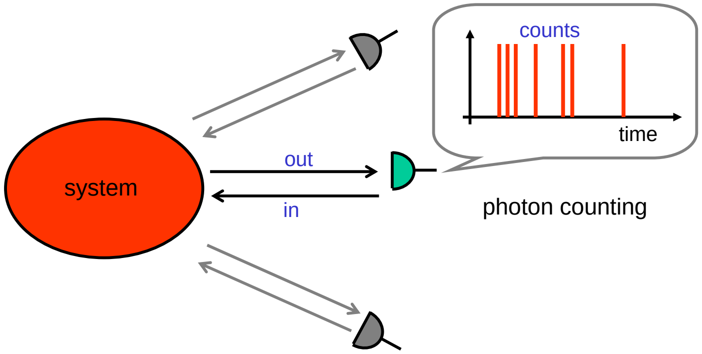
- noise / dissipation (decoherence)
- quantum optics … state preparation (e.g. laser cooling)

this *is* valid
in quantum optics

Quantum Markov processes:

- quantum stochastic Heisenberg and Schrödinger equations
- master equations etc.

Quantum optical systems as *Open* Quantum Systems



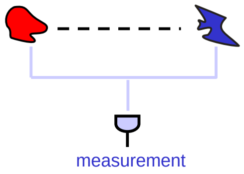
role of coupling to environment:

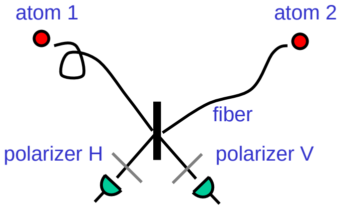
- continuous observation:

  clicks ↔ quantum jumps
  & preparation

Preparation of Entangled States via Measurement

- “quantum gambling”



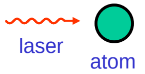


$|\mathrm{EPR}\rangle \sim |0\rangle|1\rangle + |1\rangle|0\rangle$

Quantum optical (atomic) Hamiltonians & Quantum State Engineering

Elementary atomic QO Hamiltonians (without dissipation)

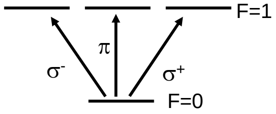
- atom interacting with laser light



$$H = H_{0A} - \vec{\mu}\cdot\vec{E}_{\mathrm{cl}}(\vec{x}=0,t)$$

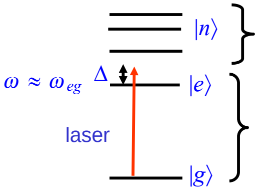
dipole interaction

- laser: electric field



$$\vec{E}_{\mathrm{cl}}(\vec{x}=0,t) = \mathcal{E}(t)\vec{\epsilon}e^{-i\omega t} + \mathrm{c.c.}$$

- two-level system + rotating wave approximation



non-resonant states

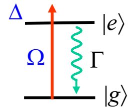
two-level system

step 1: resonant couplings

$$-\vec{\mu}\cdot\vec{E}_{\mathrm{cl}} \rightarrow -\vec{\mu}_{eg}\vec{\epsilon}\mathcal{E}(t)e^{-i\omega t}|e\rangle\langle g| - \vec{\mu}_{ge}\vec{\epsilon}\mathcal{E}^*(t)e^{i\omega t}|g\rangle\langle e|$$

step 2: rest in perturbation theory

- Two-level atom as effective Hamiltonian

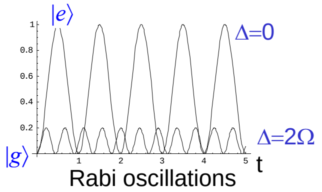


$$H = \hbar\omega_{eg}|e\rangle\langle e| - \vec{\mu}_{eg}\vec{\epsilon}\mathcal{E}(t)e^{-i\omega t}|e\rangle\langle g| - \vec{\mu}_{ge}\vec{\epsilon}\mathcal{E}^*(t)e^{i\omega t}|g\rangle\langle e|$$

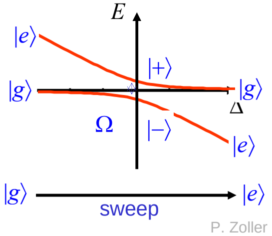
$$H_{\mathrm{TLS+RWA}} = -\tfrac{1}{2}\hbar\Delta\sigma_z + \tfrac{1}{2}\hbar\Omega e^{i\varphi}\sigma_- + \tfrac{1}{2}\hbar\Omega e^{-i\varphi}\sigma_+$$

Remarks:
- optical frequencies transformed away
- validity $|\vec{\mu}_{ng,e}\vec{\epsilon}\mathcal{E}(t)| \ll$ |detunings off-resonant states|

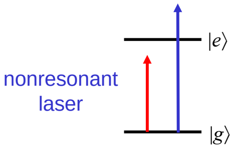
- Dynamics: Rabi oscillations vs. adiabatic sweep



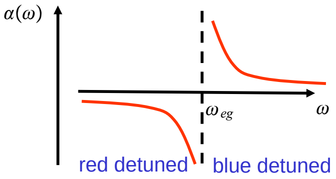
Rabi oscillations



- perturbation theory for the non-resonant states:
  example: AC Starkshift

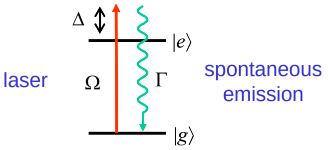


AC Stark shift



$$H = [\hbar\omega_g + \hbar\delta\omega_g(t)]|g\rangle\langle g| + \ldots \qquad \hbar\delta\omega_g(t) = \sum_n \frac{|\vec{\mu}_{ng}\vec{\epsilon}\mathcal{E}|^2}{\hbar(\omega_{gn}+\omega)} + \frac{|\vec{\mu}_{ng}\vec{\epsilon}\mathcal{E}|^2}{\hbar(\omega_{gn}-\omega)} \equiv \alpha(\omega)|\mathcal{E}|^2$$

- decoherence: spontaneous emission



$$\Delta E_g = \frac{1}{4}\frac{\Omega^2}{\Delta - \frac{1}{2}i\Gamma} = \delta E_g - i\frac{1}{2}\gamma_g$$

$$\frac{\text{good}}{\text{bad}} = \frac{\delta E_g}{\gamma_g} \sim \frac{|\Delta|}{\Gamma} \gg 1$$

typical off-resonant lattice : $\gamma \sim$ sec^{-1}

In a blue detuned lattice this can be strongly suppressed

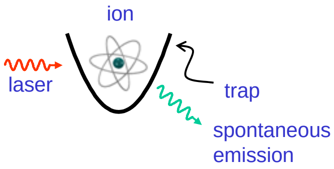
- including the center-of-mass motion

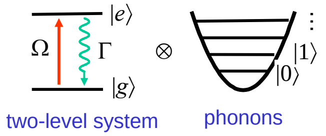
$$H = \frac{\vec{p}^2}{2M} + V_T(\vec{x}) + H_{0A} - \vec{\mu}\cdot\vec{E}(\vec{x},t)$$

atomic motion

coupling internal – external dynamics

- example 1: trapped ion





two-level system

phonons

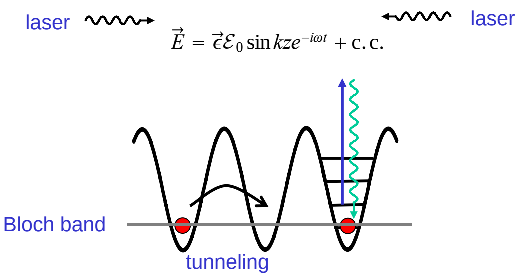
$$H = \frac{\hat{p}^2}{2M} + \frac{1}{2}M\nu^2\hat{x}^2 + \hbar\omega_{eg}|e\rangle\langle e| - \hbar\left(\frac{1}{2}\Omega e^{ik\hat{x}-i\omega t}|e\rangle\langle g| + \mathrm{h.c.}\right)$$

trap

atom – laser coupling

atom – laser coupling

- example 2: atom in optical trap / lattice



$$\vec{E} = \vec{\epsilon}\mathcal{E}_0 \sin kz e^{-i\omega t} + \text{c.c.}$$

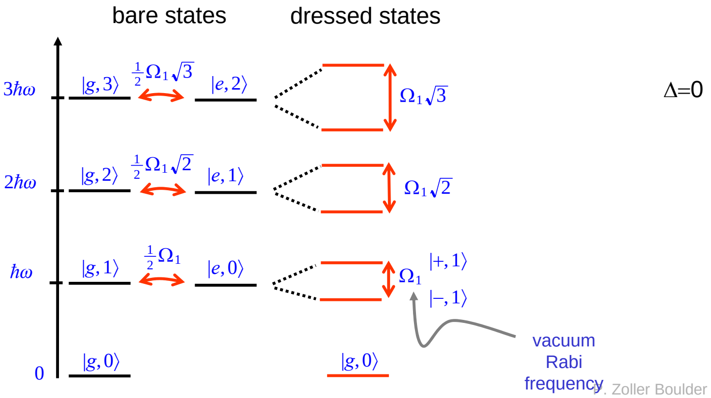
optical lattice as array of microtraps

$$V(x) = V_0 \sin^2 kx \qquad (k = \tfrac{2\pi}{\lambda})$$

- Cavity QED: Jaynes-Cummings model [see S. Haroche's lecture]

$$H = \hbar\omega_{eg}|e\rangle\langle e| + \hbar\omega b^\dagger b - (\vec{\mu}\vec{g}\sigma_+ b + \vec{\mu}^*\vec{g}^*\sigma_- b^\dagger)$$

- dressed states

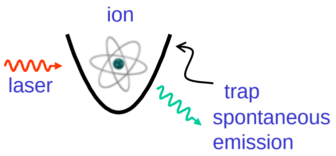


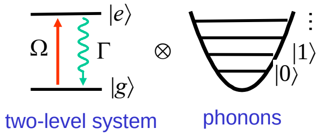
Intermezzo:
Analogies with Condensed Matter Hamiltonians

- [Cavity QED: optical / microwave CQED / ion trap *vs.* JJ + transmission line] see Girvin, Schoelkopf et al.
- Trapped Ion *vs.* Nanomechanical Systems + Quantum Dot

Trapped ion

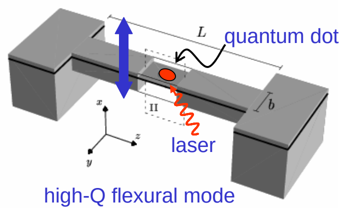
- trapped ion driven by laser





Nano-mechanical system

- quantum dot in a phonon cavity

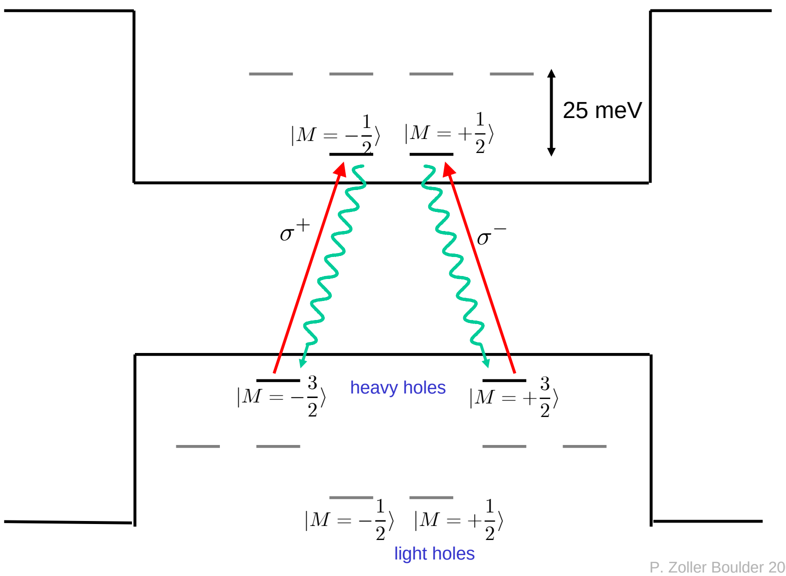


high-Q flexural mode

I. Wilson-Rae, PZ, A. Imamoglu, PRL 2004

Spectroscopy of Quantum Dots

see A. Imamoglu's lecture



25 meV

$|M=-\frac{1}{2}\rangle$ $|M=+\frac{1}{2}\rangle$

σ^+ σ^-

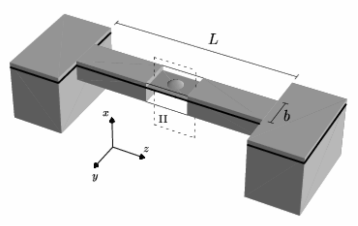
$|M=-\frac{3}{2}\rangle$ heavy holes $|M=+\frac{3}{2}\rangle$

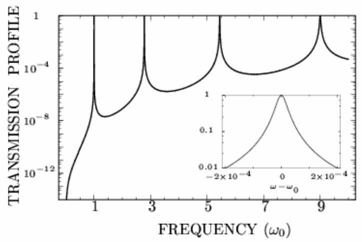
$|M=-\frac{1}{2}\rangle$ $|M=+\frac{1}{2}\rangle$

light holes

Quantum dot in a phonon cavity

- system





Q = 25,000 has been measured for modes with ω = 2 π x 200 MHz.

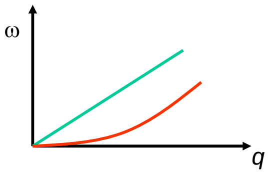
- Thin rod elasticity: $\lambda_p \sim$ L >> b,d

four branches with no infrared cutoff:

- ✓ flexural & in-plane bending $\omega \sim q^2$

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI_2}{\rho}\frac{\partial^4 u}{\partial y^4} = 0$$

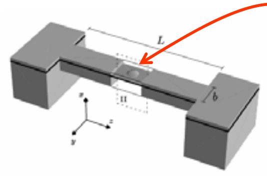
- ✓ torsional & compression modes $\omega \sim q$

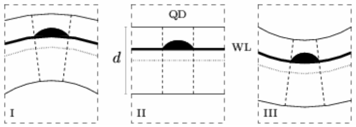


Hamiltonian

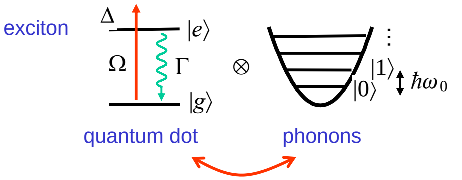
deformation coupling

$$H_{\mathrm{DP}} = \int d\bar{r}^3 [D_c \hat{\rho}_{\mathrm{el}}(\bar{r}) - D_v \hat{\rho}_{\mathrm{h}}(\bar{r})] \nabla \cdot \hat{u}(\bar{r})$$





- Hamiltonian: single mode coupled to a QD via deformation coupling



exciton

Δ $|e\rangle$ Ω Γ $|g\rangle$ $\otimes$ $|1\rangle$ $|0\rangle$ $\hbar\omega_0$

quantum dot phonons

$$H = \hbar\omega_0 b_0^\dagger b_0 + \hbar[-\Delta + \omega_0 \eta(b_0 + b_0^\dagger)]|e\rangle\langle e| + \hbar\frac{1}{2}\Omega(|e\rangle\langle g| + \mathrm{h.c.})$$

mode

laser driven quantum dot

deformation potential coupling: spin-boson model

- unitary transformation to polaron representation: $B = e^{\eta(b_0 - b_0^\dagger)}$

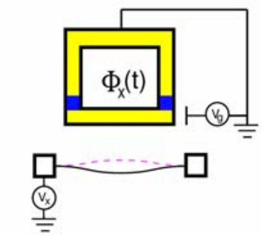
NMS + QD $H = \hbar\omega_0 b_0^\dagger b_0 - \hbar\Delta|e\rangle\langle e| + \hbar\frac{1}{2}\Omega\left(e^{\eta(b_0 - b_0^\dagger)}|e\rangle\langle g| + \mathrm{h.c.}\right)$

looks like ion trap Hamiltonian with effective Lamb-Dicke parameter (replacing the recoil) : η ~ 0.1

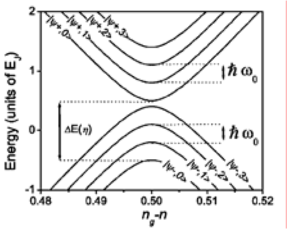
ion trap $H = \frac{P^2}{2M} + \frac{1}{2}M\nu^2X^2 - \Delta|e\rangle\langle e| - \frac{1}{2}\Omega(e^{ik_LX}|e\rangle\langle g| + \mathrm{h.c.})$

$\equiv e^{i\eta(a+a^\dagger)}$

- another example: Cooper pair box



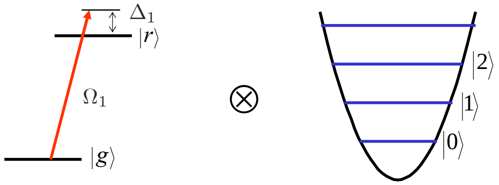
cooling: I.Martin, S.Shnirman; L. Tian, …



"cavity QED": K. Schwab et al.

Quantum State Engineering: Trapped Ions

Hamiltonians for quantum state engineering: g - r



- bare atom and trap

$$H_{0A} = -\Delta_1 |r\rangle\langle r|$$

$$H_{0T} = \frac{P^2}{2M} + \frac{1}{2} M\nu^2 X^2$$
$$\equiv \nu(a^\dagger a + 1/2)$$

- interaction atom - laser

$$H^{\pm}_{A_1 L} = \tfrac{1}{2}\Omega_1 \{|r\rangle\langle g| e^{\pm i\eta_1 (a+a^\dagger)} + \text{h.c.}\}$$

couple motion - internal

$$H = H_{0T} + H_{0A_1} + H_{A_1 L}$$

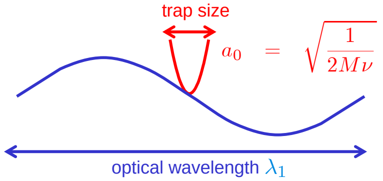
- Lamb-Dicke limit

$$\eta_1 = 2\pi a_0/\lambda_1 \ll 1$$

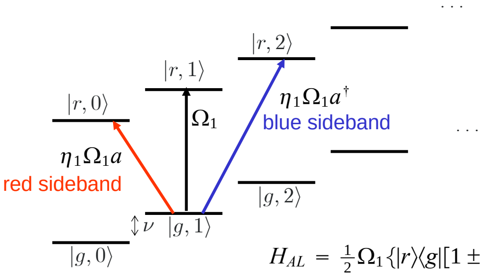
Lamb-Dicke parameter

$$e^{ikX} = 1 + ikX + ...$$

Lamb-Dicke expansion

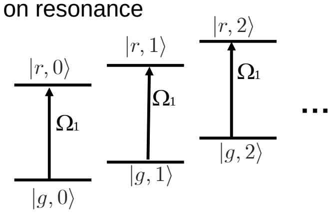


- Lamb-Dicke expansion of the Hamiltonian



$$H_{AL} = \tfrac{1}{2}\Omega_1 \{|r\rangle\langle g|[1 \pm i\eta_1(a+a^\dagger) + O(\eta_1^2)] + \text{h.c.}\}$$

- on resonance

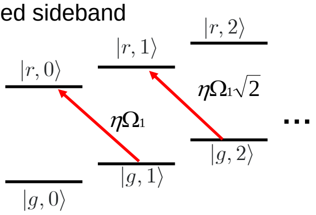


$$H_1 = \nu a^\dagger a - \tfrac{1}{2}\Delta\sigma_z + \tfrac{\Omega}{2}(\sigma_+ + \text{h.c.})$$

$$\sigma_+ = (\sigma_-)^\dagger = |r\rangle\langle g|$$

$$\sigma_z = |r\rangle\langle r| - |g\rangle\langle g|$$

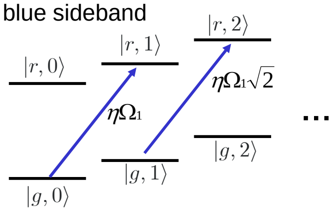
- red sideband



$$H_{JC} = \nu a^\dagger a - \tfrac{1}{2}\Delta\sigma_z + \tfrac{\Omega}{2}(i\eta_1\sigma_+ a + \text{h. c.})$$

Jaynes-Cummings Hamiltonian

- blue sideband



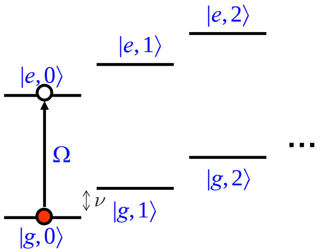
$$H_{AJC} = \nu a^\dagger a - \tfrac{1}{2}\Delta\sigma_z + \tfrac{\Omega}{2}(i\eta_1\sigma_+ a^\dagger + \text{h.c.})$$

anti-Jaynes-Cummings Hamiltonian

Exercises in quantum state engineering

- engineering harmonic oscillator states:
  - phonons: single trapped ion
  - [phonons: cantilever]
  - [photons: CQED]
- engineering entanglement
  - a fast two-qubit gate with ions

- atomic superposition state



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1. Quantum State Engineering with Single Trapped Ions

- design a sequence of unitary transformations (or Hamiltonians) so that

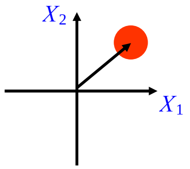
$$|i\rangle \longrightarrow |f\rangle = U|i\rangle \equiv \ldots U_2 U_1 |i\rangle \qquad |\psi\rangle_{\text{ph}} = \sum_{n=0}^{\infty} c_n |n\rangle$$

example: phonons

- examples of interesting harmonic oscillator states

$a =: \frac{1}{2}(X_1 + iX_2)$ quadrature components

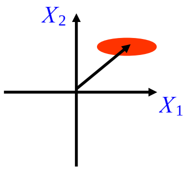
$[X_1, X_2] = 2i$ from which follows: $\Delta X_1 \Delta X_2 \geq 1$



coherent state $|\alpha\rangle$

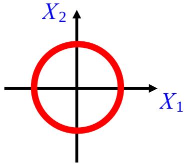
$\Delta X_1 = \Delta X_2 = 1$

$\frac{1}{2}\langle X_1 + iX_2\rangle = \alpha$

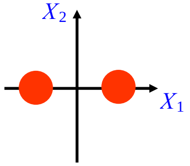


squeezed states $|\epsilon, r\rangle$

$\Delta X_1 < 1 < \Delta X_2$



Fock state $|n\rangle$

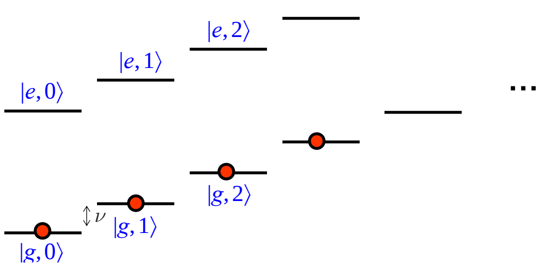


Schrödinger cat $|\alpha\rangle + |{-\alpha}\rangle$

- **Goal:** engineer an arbitrary superposition state of phonon states

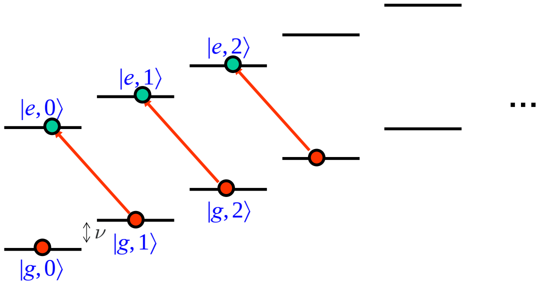
$$|g\rangle \otimes |0\rangle \rightarrow |\Psi\rangle = |g\rangle \otimes \sum_{n=0}^{N} c_n |n\rangle$$

for given coefficients c_n.



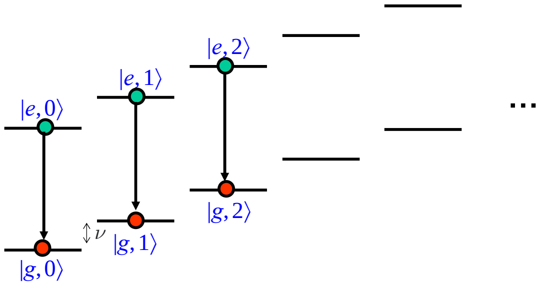
- **Idea:** let us first consider the inverse of the problem - given the above superposition state we can want to find unitary transformations to obtain $|g\rangle \otimes |0\rangle$.

Law & Eberly, Gardiner et al., Wineland et al.

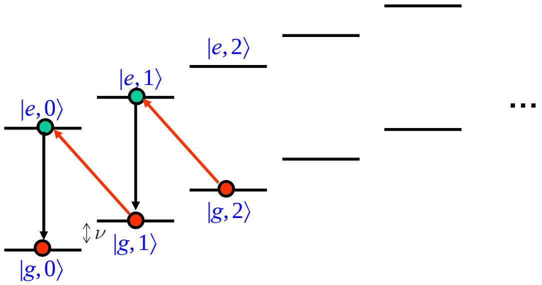


Procedure: Applying a laser on the red sideband we couple the states $|g\rangle|n\rangle \leftrightarrow |e\rangle|n-1\rangle$.

As a first step we apply a π-pulse so that we make the amplitude of $|g\rangle|N\rangle$ equal to zero by transferring the amplitude c_N to $|e\rangle|N-1\rangle$. But we now have a superposition of ground and excited state.



In the second step we apply a resonant laser so that we transform the known! superposition of $|g\rangle|N-1\rangle$, $|e\rangle|N-1\rangle$ to $|g\rangle|N-1\rangle$ with no amplitude left in $|e\rangle|N-1\rangle$. Now we repeat the argument until we have transformed the state to $|g\rangle|0\rangle$.



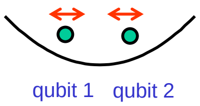
In the second step we apply a resonant laser so that we transform the known! superposition of $|g\rangle|N-1\rangle$, $|e\rangle|N-1\rangle$ to $|g\rangle|N-1\rangle$ with no amplitude left in $|e\rangle|N-1\rangle$. Now we repeat the argument until we have transformed the state to $|g\rangle|0\rangle$.

The inverse transformation produces the desired state starting from the ground state.

2. Entanglement engineering … a fast 2-qubit gate

J. Garcia-Ripoll
J.I.Cirac
PZ

- illustrate the quantum optical ideas behind a fast 2-qubit gate between two ions
- the “fastest” 2-qubit gate with ions
- phase gate



$$\begin{array}{ccc} |0\rangle|0\rangle & & |0\rangle|0\rangle \\ |0\rangle|1\rangle & \xrightarrow{\mathcal{U}} & |0\rangle|1\rangle \\ |1\rangle|0\rangle & & |1\rangle|0\rangle \\ |1\rangle|1\rangle & & -|1\rangle|1\rangle \end{array}$$

- “engineering task”

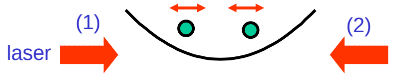
$|\psi\rangle\langle\psi|\otimes\rho_{\text{motion}} \to$ entangle qubits via motion $\to |\psi\rangle\langle\psi|\otimes\rho_{\text{motion}}$

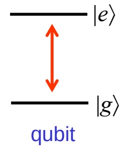
qubits motional state: e.g. thermal

motional state factors out

Model: 2 ions

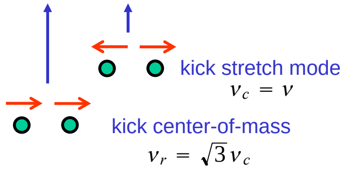
- 2 ions in a 1D trap kicked by laser beams





$$H = \nu_c a^\dagger a + \nu_r b^\dagger b$$
$$+ \frac{1}{2}\Omega(t)\sigma_1^+ e^{i\eta_c(a^\dagger+a)+\frac{1}{2}\eta_r(b^\dagger+b)} + \frac{1}{2}\Omega(t)\sigma_2^+ e^{i\eta_c(a^\dagger+a)-\frac{1}{2}\eta_r(b^\dagger+b)} + \text{h.c}$$

NO addressing



Remarks:

- Rabi frequency Ω is the same for both ions
- $\eta \to -\eta$ is reversing the direction of the laser beams

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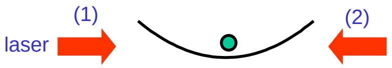
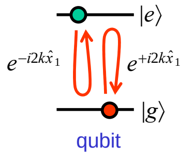
Dynamics

- kicking the ions (illustrated with a single ion)





$|g\rangle|\text{motion}\rangle \to |g\rangle e^{+i2k\hat{x}_1}|\text{motion}\rangle$

$|e\rangle|\text{motion}\rangle \to |e\rangle e^{-i2k\hat{x}_1}|\text{motion}\rangle$

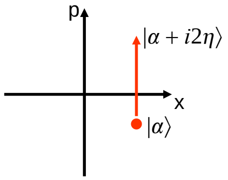
kicking the ion depending on the internal state

$|g\rangle|\alpha\rangle \to |g\rangle|\alpha + i2\eta\rangle$

$|e\rangle|\alpha\rangle \to |e\rangle|\alpha - i2\eta\rangle$

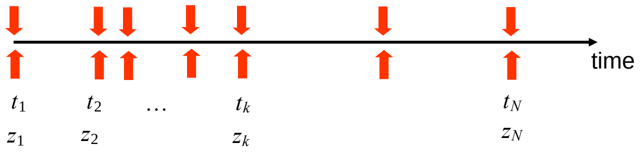
coherent state

phase space



Dynamics

- series of kicks by laser pulses and interspersed free evolution:



z_k are integer numbers (sign = direction of the laser pulses)

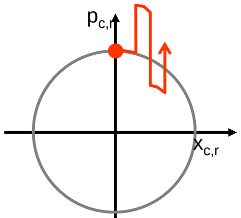
- action of time evolution operator on qubit & motion:

evolution operator $\mathcal{U} = \mathcal{U}_c\mathcal{U}_r$ with $\mathcal{U}_{c,r} = \prod_{k=1}^{N} U_{c,r}(\Delta t_k, z_k)$

com $U_c(\Delta t_k, z_k) = e^{-i2z_k\eta_c(a+a^\dagger)(\sigma_1^z+\sigma_2^z)}e^{-i\nu_c\Delta t_k a^\dagger a}$

stretchmode $U_r(\Delta t_k, z_k) = e^{-iz_k\eta_r(b+b^\dagger)(\sigma_1^z-\sigma_2^z)}e^{-i\nu_r\Delta t_k b^\dagger b}$

phase space



- Characterization of the action of $\mathcal{U}$ by

$$\mathcal{U}|i\rangle_1|j\rangle_2|\alpha\rangle_c|\beta\rangle_r = ?$$

qubits

coherent state: motion

Details …

- For a single kick the action on a coherent state is

$$e^{-ip_1(a+a^\dagger)}e^{-i\nu t_1\, a^\dagger a}|\alpha\rangle = |\alpha e^{-i\nu t_1} - ip_1\rangle \quad \textit{etc.}$$

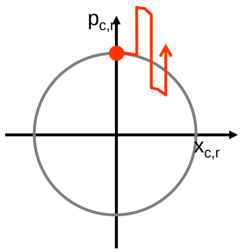
- and for many kicks ...

$$\mathcal{U}|\alpha\rangle = e^{i\xi}|\tilde{\alpha}\rangle$$

where

$$\tilde{\alpha} = (\alpha - i\sum_{k=1}^{N} p_k e^{i\nu t_k})e^{-i\nu t_N}$$

$$\xi = -\sum_{m=2}^{N}\sum_{k=1}^{m-1} p_m p_k \sin(\nu(t_k - t_m)) \; - \mathrm{Re}\left[\;\alpha\sum_{k=0}^{N} p_k e^{-i\nu t_k}\;\right]$$


phase space

✓ the initial coherent state evolves to a shifted coherent state

✓ state vector picks up a qubit dependent phase

- Commensurability Condition:

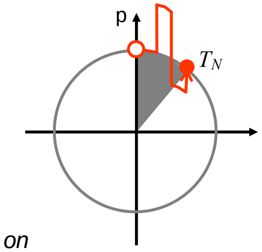
if

$$\boxed{\sum_{k=1}^{N} p_k e^{i\nu t_k} = 0}$$

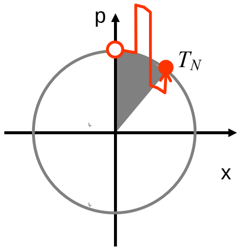
then

$$\mathcal{U}|\alpha\rangle = e^{i\xi}|\alpha e^{-i\nu t_N}\rangle$$

- the motional state is the one of a *free evolution*.
- a global phase ξ appears which *does not depend on the motional state*


phase space

Evolution Operator: Summary


com *and* stretch mode

- If (commensurability condition)

center of mass

$$C_c \equiv \sum_{k=1}^{N} z_k e^{-i\nu t_k} = 0,$$

stretch mode

$$C_r \equiv \sum_{k=1}^{N} z_k e^{-i\sqrt{3}\nu t_k} = 0.$$

the motional state of the ion will not depend on the qubits and the evolution operator will be given by

$$\mathcal{U}(\Theta)|\alpha\rangle_c|\beta\rangle_r = |\alpha e^{-i\nu_c T}\rangle_c|\beta e^{-i\nu_r T}\rangle_r \; e^{i\Theta\sigma_1^z\sigma_2^z}$$

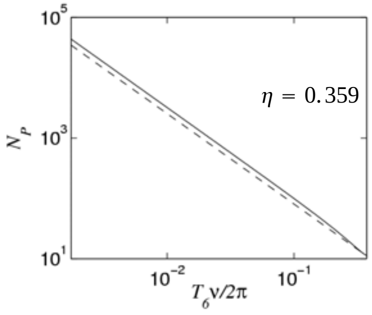
center-of mass and relative motion factors out

qubit phase (independent of motion)

- For $\Theta = \pi/4$ we will have a controlled–phase gate which is *completely independent of the initial motional state*, i.e. there is no temperature requirement

Performance

- Number of pulses for a given gate time



scaling as $(\nu T_N)^{-3/2}$

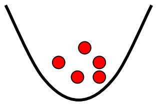
- insensitive to temperature

Engineering Hubbard Models

Jaksch et al 1998

1. Bose Hubbard in optical lattice: naïve derivation

- dilute bose gas



$$H = \int \psi^\dagger(\vec{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + V_T(\vec{x}) \right) \psi(\vec{x})\, d^3x$$

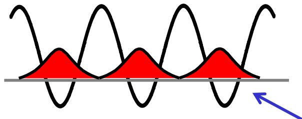
$$+\frac{1}{2} g \int \psi^\dagger(\vec{x}) \psi^\dagger(\vec{x}) \psi(\vec{x}) \psi(\vec{x})\, d^3x$$

collisions

- validity: dilute gas, $a_s \ll a_0 < \lambda/2$

$g = \frac{4\pi a_s \hbar^2}{m}$ scattering length

- optical lattice



$$\psi(\vec{x}) = \sum_\alpha w(\vec{x} - \vec{x}_\alpha)\, b_\alpha$$

Wannier functions

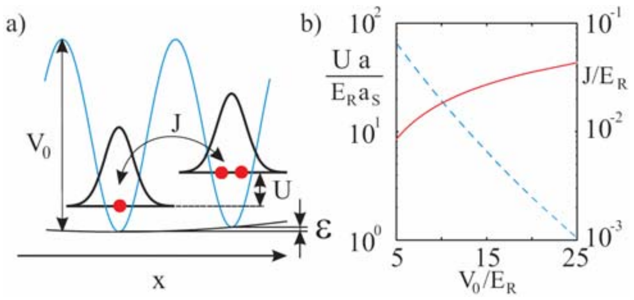
- Hubbard model

$$H = -J \sum_{<i,j>} b_i^\dagger b_j + \tfrac{1}{2} U \sum_i b_i^\dagger b_i^\dagger b_i b_i + \sum_i \epsilon_i b_i^\dagger b_i$$

kinetic engery: hopping

interaction: onsite repulsion

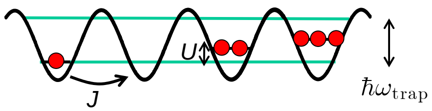
$$U = g \int |w(\vec{x})|^4 d^3x$$



- feature: (time dep) tunability from weakly to strongly interacting gas
- validity …

Hubbard model: microscopic picture

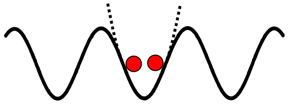
- Hubbard



✓ solve in n=1,2,3,… particle sector

✓ connect by tunneling (e.g in a tight binding approx)

- n=2 atoms on one lattice site: molecule



✓ molecular problem with added optical potential

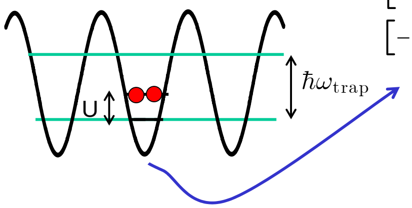
- n=3 atoms on one lattice site: … e.g. Efimov-type problem
- [n>n_{max}~3 killed by three body etc. loss]

- two atoms on one site

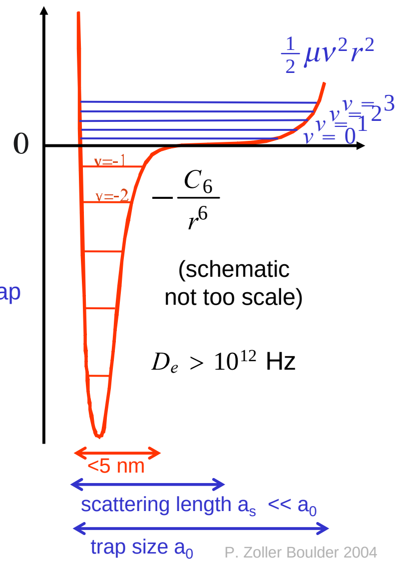
Julienne et al.

$$\left[-\frac{\hbar^2}{2(2m)}\nabla_R^2 + \frac{1}{2}(2m)\nu^2R^2\right]\psi_{cm}(R) = E_{cm}\psi_{cm}(R)$$

$$\left[-\frac{\hbar^2}{2\mu}\nabla_r^2 + \underbrace{\frac{1}{2}\mu\nu^2r^2 + V(r)}\right]\psi(r) = E\psi(r)$$



harmonic approximation

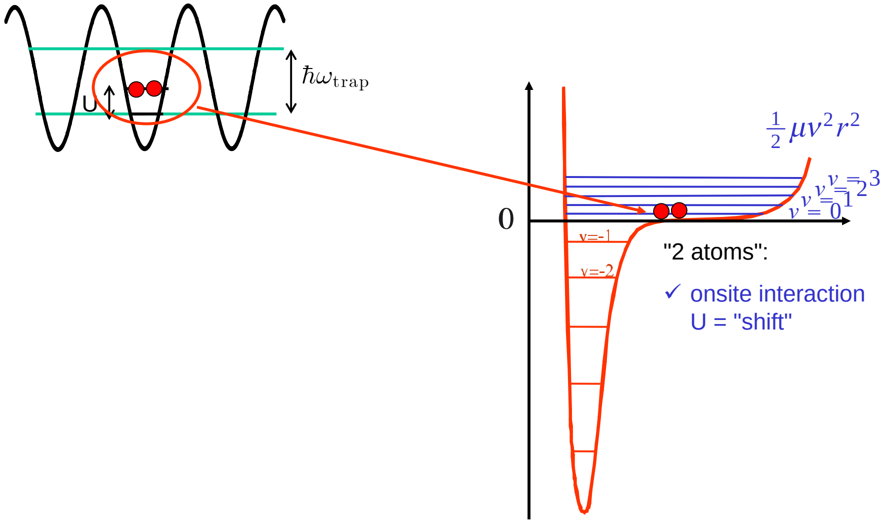


Born Oppenheimer potentials including trap

(schematic not too scale)

$D_e > 10^{12}$ Hz

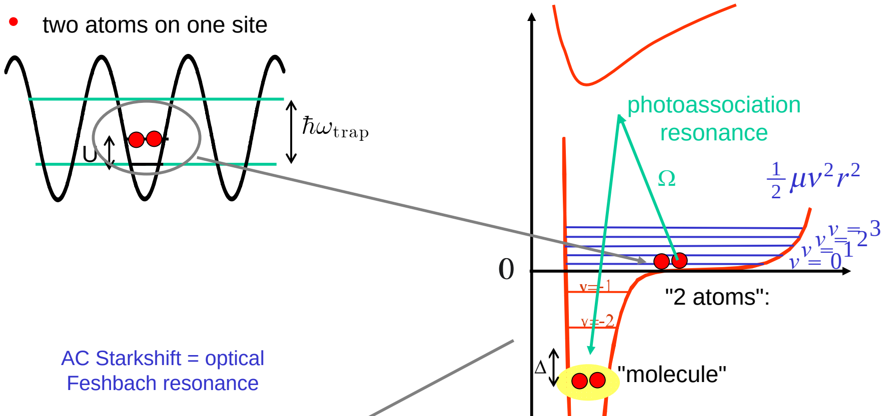
- two atoms on one site



"2 atoms":

✓ onsite interaction U = "shift"

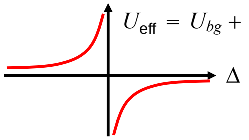
- two atoms on one site



"2 atoms":

"molecule"

AC Starkshift = optical Feshbach resonance

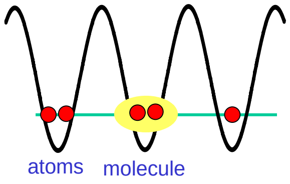


$$U_{\text{eff}} = U_{bg} + \frac{1}{4}\frac{\Omega^2}{\Delta}$$

$$H = (U_{bg} + \frac{1}{4}\frac{\Omega^2}{\Delta})b^{\dagger 2}b^2$$

Hubbard model including molecules

- Hamiltonian



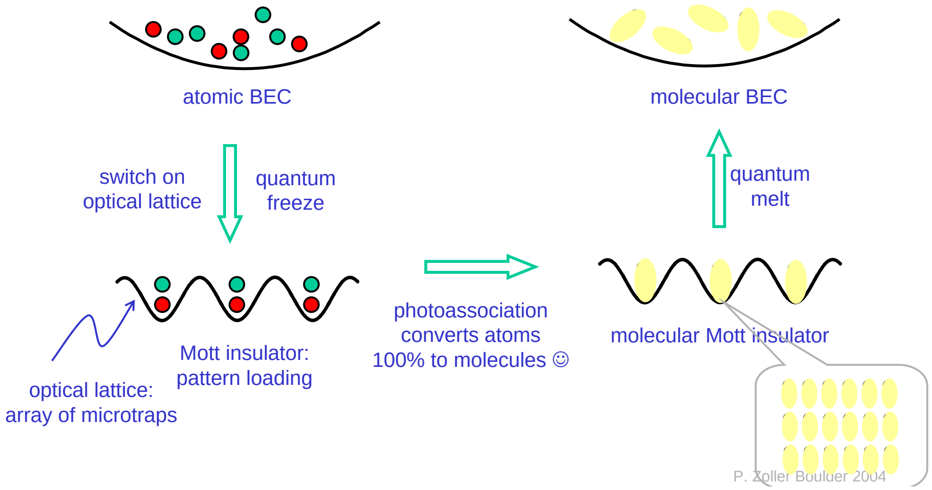
$$H = -J_b \sum_{<i,j>} b_i^\dagger b_j + \frac{1}{2} U_b \sum_i b_i^\dagger b_i^\dagger b_i b_i$$
$$- J_m \sum_{<i,j>} m_i^\dagger m_j + \frac{1}{2} U_m \sum_i m_i^\dagger m_i^\dagger m_i m_i - \sum_i \Delta m_i^\dagger m_i$$
$$+ \frac{1}{2} \Omega \sum_i m_i^\dagger b_i b_i + \text{h.c.}$$

Remarks:

- ✓ we have derived this only for sector: 2 atoms or 1 molecule
- ✓ inelastic collisions / loss for >2 atoms and >1 molecules (?)

Remark: quantum phases of "composite objects"

- molecular BEC via a quantum phase transition



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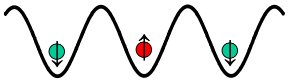
e.g. Duan et al.

Spin models

- optical lattice



bosons in a Mott phase

$$\langle n_{i\uparrow} \rangle + \langle n_{i\downarrow} \rangle \approx 1$$

$$H = -J \sum_{<i,j>,\sigma} b_{i\sigma}^\dagger b_{j\sigma} + \tfrac{1}{2} U \sum_{i,\sigma} n_{i\sigma}(n_{i\sigma} - 1) + U_{\uparrow\downarrow} \sum_i n_{i\uparrow} n_{i\downarrow}$$

$$\Downarrow$$

$$H = \sum_{\langle i,j \rangle} \left[\lambda_z \sigma_i^z \sigma_j^z \pm \lambda_\perp (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) \right]$$ XXZ-model

$$\lambda \sim \frac{J^2}{U}$$

pretty small ☹

$$\sigma_i^z = n_{\uparrow i} - n_{i\downarrow}$$
$$\sigma_i^x = b_{\uparrow i}^\dagger b_{i\downarrow} + b_{i\downarrow}^\dagger b_{\uparrow i}$$
$$\sigma_i^y = -i(b_{\uparrow i}^\dagger b_{i\downarrow} - b_{i\downarrow}^\dagger b_{\uparrow i})$$

- ideas for higher order H= σ σ σ interactions …

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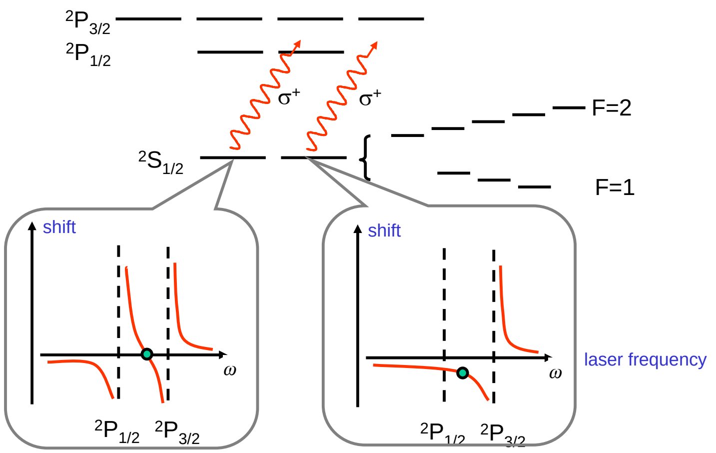
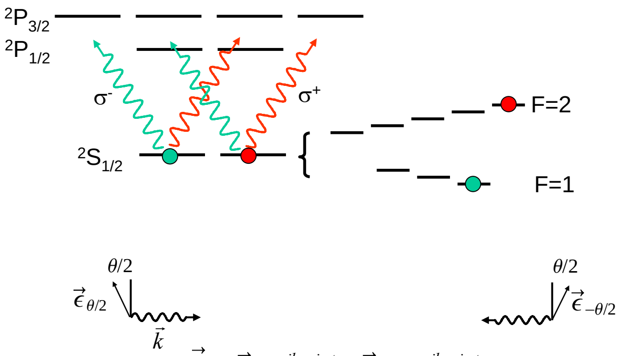
2. Optical Lattices … continued

- multiple ground states & spin-dependent lattices

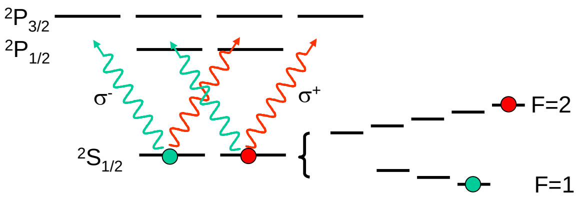


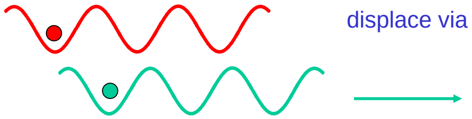
- multiple ground states & spin-dependent lattices



$$\vec{E} \sim \vec{\epsilon}_{\theta/2} e^{ikz-i\omega t} + \vec{\epsilon}_{-\theta/2} e^{-ikz-i\omega t}$$
$$\sim \vec{\epsilon}_{\sigma^+} \cos(kz-\theta/2) + \vec{\epsilon}_{\sigma^-} \sin(kz+\theta/2)$$

- multiple ground states & spin-dependent lattices





displace via polarization angle