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This lecture is based on Symmetry Fractionalization in Two Dimensional Topological Phases, arXiv:1606.07569.

I.1. Introduction

Let $g \in G$ be an element of a symmetry group G. A linear **representation** of G is a faithful representation which follows the group algebra rules exactly, i.e.,

$$M(g_1) M(g_2) = M(g_1 g_2) .$$

A **fractional representation** follows the group algebra rules up to a phase factor from any abelian group, here $\alpha(g_1, g_2) \in U(1)$:

$$\tilde{M}(g_1) \tilde{M}(g_2) = \alpha(g_1, g_2) \tilde{M}(g_1 g_2) .$$

A nice example of this is the $SU(2)$ group, which is a fractional representation of $SO(3)$ since going around the Bloch sphere gives you a -1:

$$\tilde{M}(\pi) \tilde{M}(\pi) = -\tilde{M}(0) .$$

I.2. Gauging

Let's illustrate this with a $\mathbb{Z}_2$-symmetric Ising model

$$H = -\sum_i \sigma_i^x + J \sum_i \sigma_i^z \sigma_{i+1}^z$$

on a 2D square lattice. A global symmetry is

$$U = \prod_{m,n} \sigma_{m,n}^x .$$

There is also a local symmetry σ_i^x and other symmetries, but focus on the global symmetry U.

Now let's introduce a $\mathbb{Z}_2$ **gauge field** τ with two states $|0\rangle, |1\rangle$, which will live on links between sites, just like a Peierls phase on a hopping term (e.g., $\tau_{m-1,n\to m,n}$ is on the link between sites $m-1, n$ and m, n). This time however, the phase is operator-valued, i.e., a **dynamical gauge field** (vs. a background gauge field). Just like E&M, we introduce a local symmetry transformation

$$U_{m,n} = \sigma_{m,n}^x \tau_{m-1,n\to m,n}^x \tau_{m,n\to m+1,n}^x \tau_{m,n-1\to m,n}^x \tau_{m,n\to m,n+1}^x \equiv A_{\text{vertex } v}$$

which, when made global produces the global symmetry:

$$\prod_{m,n} U_{m,n} = U .$$

Then, we make sure that our system is invariant under $U_{m,n}$. The Hamiltonian then is modified to

$$H = -\sum_i \sigma_i^x + J \sum_i \sigma_i^z \tau_{i\to i+1}^z \sigma_{i+1}^z .$$

Given a plaquette, we can calculate the total flux through it by multiplying the τ's on the plaquette. We can think of $\tau^z \sim e^{iA}$ as a Peierls phase and $\tau^x \sim e^{iE}$, but here these are only $\mathbb{Z}_2$-valued.

Now we want to enforce zero gauge flux in the ground state. To do this, we add

$$B_p \equiv \prod_{\text{plaquette } p} \tau^z$$

to the Hamiltonian. We also want to impose Gauss's law on our model. This is done by adding A_v to the Hamiltonian.

After this procedure, we obtain a new Hamiltonian on the expanded Hilbert space of the form

$$H' = -\sum_i \sigma_i^x + J\sum_i \sigma_i^z \tau_{i\to i+1}^z \sigma_{i+1}^z - \sum_p B_p - \sum_v A_v \,.$$

The matter field is σ_i, and we can integrate it out if we want to look at the low-energy excitations of the gauge field. This just means we project onto the $+1$ eigenspace of all σ^x's and let $J \to 0$, thereby getting rid of the $\sigma_{m,n}^x$ term in A_v and leaving us with the toric code. **$\mathbb{Z}_2$-gauging the Ising paramagnet and looking at low-energy excitations of the gauge field produces the toric code Hamiltonian**. Keeping $J \neq 0$ sufficiently small keeps the same topological phase, as long as the gap doesn't close.

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We notice from above that the matter field term $-\sum_i \sigma_i^x + J\sum_i \sigma_i^z \tau_{i\to i+1}^z \sigma_{i+1}^z$ is isospectral to the original Hamiltonian $-\sum_i \sigma_i^x + J\sum_i \sigma_i^z \sigma_{i+1}^z$, demonstrating that coupling gauge fields should not "change" the matter field. In addition, if the original Hamilonian is gapped, then the gauged one is gapped. Gauging Hamiltonians offers another way to characterize phases: **if two ungauged Hamiltonians can be adiabatically connected without closing the excitation gap, then their respective gauged versions can**. The contrapositive can be used to tell that two Hamiltonians are not in the same phase. For example, there is another SPT phase in 2D with $\mathbb{Z}_2$ symmetry, and it's gauged version,

$$\begin{array}{cc} H_0\,(\text{Ising } \mathbb{Z}_2) & H_0\,(\text{SPT } \mathbb{Z}_2) \\ \downarrow & \downarrow \\ H_0\,(\text{Toric code}) & H_0\,(\text{Double semion}) \end{array} ,$$

has excitations with different statistics than those of the toric code. Since there is no way to adiabatically connect models with different excitation statistics, the Ising and SPT phases are different. This technique applies to local Hamiltonians with global symmetries. This only applies to gauging discrete symmetries since continuous symmetries will give rise to gapless photon modes.

Since the τ's can be thought of as $\mathbb{Z}_2$-valued Peierels phase, we can relate the qparticles of the gauged model ("gauge charge" e and "gauge flux" m) to "symmetry charges" or "symmetry fluxes" of the ungauged model. A string excitation of the toric code (with two e's at the end) corresponds to adding a Peierls phase of -1 to all hopping terms $\sigma_i^z \sigma_{i+1}^z$ of the *ungauged* Hamiltonian whose sites form a line which crosses the string. This is equivalent to conjugating (only!) those terms with a local symmetry σ_i^x, i.e., flipping the sign of *only* those terms (a non-unitary transformation).

II.1. Gauging a topological superconductor with $\mathbb{Z}_2$ gauge field

Gauging an s-wave (trivial) superconductor using a $\mathbb{Z}_2$ gauge field, we obtain the toric code. This is not so trivial since we need to map fermions to spins, but it's possible. Thus, the toric code is the gauged analogue of a topologically trivial phase. The "gauge charge" $f = em$ corresponds to the Bogoluibov quasiparticle in the ungauged model. This charge exhibits symmetry fractionalization: $T^2 = -1$ when acting on f. However, T acts trivially on e, m (which we will see will be different for another model below).

Not let's take two layers of topological superconductor, $p \pm ip$, an SPT phase. This is clearly invariant under time reversal T,

$$T\left(f_{\uparrow,\downarrow}\right) = \pm f_{\downarrow,\uparrow} \,,$$

which in the many body case squares to fermion parity:

$$T^2 = (-)^{\hat{N}} \,.$$

We will gauge the fermion parity $(-)^{\hat{N}}$, which yields once again the toric code, but this time it is **enriched** by time reversal symmetry. We now will see how the charges e, m transform under time reversal symmetry. These correspond "symmetry fluxes" in the ungauged model — a Majorana mode $\gamma_{\uparrow/\downarrow}$ with up/down spin on each layer.

Two of them combined give you a fermion (a two dimensional Hilbert space) with parity $P_\pi = i\gamma_\uparrow\gamma_\downarrow$, corresponding to the two different possibilities of excitations ("gauge fluxes") in the toric code — the qparticles e and m. Under T (remembering that γ's anti-commute),

$$P_\pi \to (-i)\,(\gamma_\downarrow)\,(-\gamma_\uparrow) = i\gamma_\downarrow\gamma_\uparrow = -i\gamma_\uparrow\gamma_\downarrow = -P_\pi\,,$$

Majorana fermion parity is flipped. Correspondingly, T maps $e \leftrightarrow m$. To distinguish the regular trivial s-wave phase (for which T does not exchange e, m in the gauged version) with the non-trivial T-invariant p-wave phase (where T exchange e, m), we call the toric code with TRS a **symmetry enriched topological (SET) phase**.

More generally, given a symmetry group G, we can gauge a unitary subgroup of it and then see how the quotient group acts on the excitations of the gauged theory. This procedure can be used to distinguish different phases, irrespective of whether G contains unitary or antiunitary elements.

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Returning to symmetry fractionalization, let's start with a $\nu = \frac{1}{3}$ filled fQHE state with an additional $U(1)$ symmetry. Due to the additional symmetry, besides qparticles, we have new types of "excitations" added to the model — symmetry fluxes. Gauging the symmetry will help us understand how the fluxes braid/fuse with the anyons. Namely, two 180 degree rotations of a quasiparticle multiply projectively:

$$M(\pi)\,M(\pi) = e^{i\frac{2\pi}{3}}M(0)\ .$$

But bringing a qparticle of charge $\frac{1}{3}$ around a flux $\phi(2\pi)$ also gives a phase of $e^{i\frac{2\pi}{3}}$. Therefore, a quasiparticle is equivalent to a 2π symmetry flux:

$$a \sim \phi(2\pi)\ .$$

Fusing two fluxes then gives us a trivial flux, up to a quasiparticle:

$$\phi(\pi) \times \phi(\pi) = \phi(2\pi) = a\phi(0)\ .$$

This is a fractional representation since a is an element of an Abelian group.

Now let's take a TRS $\mathbb{Z}_2$ gauge theory, which has TRS flux $\phi(T)$. Now, $\phi(T)\,\phi(T)$ could give $\phi(0)$ times any one of the four toric code qparticles, and we need the additional associativity conditions for fractional representations to figure out which one. These are

$$a(g_1, g_2)\,a(g_1g_2, g_3) = a(g_1, g_2g_3)\,a(g_2, g_3)\ .$$

Returning to our previous discussion regarding creation of string excitations being equivalent to conjugation of the *ungauged* Hamiltonian terms on the string with the local symmetry, we now state that braiding a symmetry flux around the boundary of a region is equivalent to applying a symmetry to that region. In other words, let's create two symmetry fluxes, move them around, and annihilate them in a different spot. Moving them is equivalent to applying the local symmetry to each element of the ungauged Hamiltonian on the boundary of the region formed by the lines of the two symmetry fluxes. However, doing so is equivalent to applying local symmetry transformations to *all sites within* the region since the spins inside can't tell whether the symmetry applied was global or local. The spins on the boundary are the only ones that can tell this.

Instead of the Hamiltonian, now let's look at braiding symmetry fluxes on the ground state. Braiding two symmetry fluxes $g_{1,2}$ is equivalent to applying a unitary $U(g_{1,2})$ to the spins along the respective paths of the two fluxes. In this case however, this is not equivalent to applying the symmetry to all sites within the two regions. Instead,

$$U(g_1)\,U(g_2) = a(g_1, g_2)\,U(g_1g_2)\ .$$

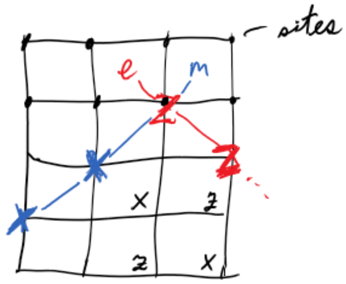
Fusion of symmetry fluxes is done "up to an anyon."

III.1. Examples

1. Consider the usual toric code model on a square lattice, but now flip the sign of the plaquette terms. Now, the ground state will have flux in all of the plaquettes and e excitations will move in a background of m charges — the lattice version of a background magnetic field. Specifically, if e (which is on a vertex) goes around a plaquette, it will get a -1:

$$T^e_xT^e_yT^e_{-x}T^e_{-y} = -1\,.$$

2. Now consider Wen's version of the toric code, in which the spins are on nodes and the unit cell is 2D (with X and Z being on the two nodes).



In this version of the toric code, translations map $e \leftrightarrow m$.

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IV.1. Multi Chern-Simons theory

Let's study multi-layer Chern-Simons theory and letting $I = 1, 2, ..., N$ (Read, Wen/Zee 1990s), we have

$$S = \int \frac{1}{4\pi} K_{IJ} a_\mu^I \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda^J + j_I^\mu a_\mu^I \,,$$

where $\lambda, \mu, \nu \in \{0, 1, 2\}$, $I \in \mathbb{Z}_p$, K_{IJ} a $p \times p$ matrix. This generalizes the IQH. The exchange phase is then $\theta_{\ell,\ell'} = 2\pi \ell^T K^{-1} \ell'$ and ground state degeneracy (on a torus) is $|\det K|$. Examples of this are in Chetan Nayak's notes.

On the edge of these theories, we can construct a Lagrangian

$$\mathcal{L}_e = \frac{1}{4\pi} K_{IJ} \partial_x \phi_I \partial_t \phi_J \qquad\qquad \phi_I \in [0, 2\pi) \,.$$

The anyon creation operator is $e^{i\ell^T \phi}$, creating an anyon described by the p-dimensional vector ℓ on the edge. Under a global symmetry transformation g,

$$\phi \to W_g \phi + \delta\phi_g$$

with integer matrix W_g and shift $\delta\phi_g$. In order for $\mathcal{L}_e$ to be invariant under g, we want

$$W_g^T K W_g = K \,.$$

Recall from Nayak's lecture that transformations $K \to W^T K W$ and $\ell \to M^T \ell$ (where the integer matrix W has determinant ± 1) are changes of basis for the anyons. So two Chern-Simons theories are equivalent up to the W's. *However*, once we pick a form for K, it may still have symmetries as above. Under an anti-unitary symmetry $W_t K$ with K conjugation, then the ∂_t will produce a minus sign which needs to be canceled by the unitary part:

$$W_t^T K W_t = -K \,.$$

- For example, consider the toric code with

$$e = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad m = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad f = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad K = 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,.$$

  This has a unitary $\mathbb{Z}_2$ symmetry with $M = I$ and $\delta\phi = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$. That way

$$e : e^{i\phi_1} \to -e^{i\phi_1} \qquad\qquad m : e^{i\phi_2} \to +e^{i\phi_2} \,,$$

  representing a addition of integer charges. Now if we instead have $\delta\phi = \begin{pmatrix} \pi/2 \\ 0 \end{pmatrix}$,

$$e : e^{i\phi_1} \to i e^{i\phi_1} \qquad\qquad m : e^{i\phi_2} \to +e^{i\phi_2} \,,$$

and we have fractionalization. More generally,

$$\mathbb{Z}_2 : \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + \frac{\pi}{2}\begin{pmatrix}1\\ 1\end{pmatrix} .$$

This is actually a $\mathbb{Z}_2$ and not a $\mathbb{Z}_4$ symmetry. This is because qparticles always for in pairs, so when two anyons each switch sign, the system remains invariant. This is why, for a $\mathbb{Z}_2$ symmetry, we can only have $\delta\phi$ be a multiple of $\frac{\pi}{2}$.
Under time reversal, we want W to anticommute with K.

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}\phi_1\\ -\phi_2\end{pmatrix} + \begin{pmatrix}\frac{\pi}{2}\\ 0\end{pmatrix} \qquad W = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} .$$

Then acting twice with the symmetry gives you a minus on m:

$$\begin{aligned} e : e^{i\phi_1} &\to We^{-i\phi_1} = e^{-i\left(\phi_1+\frac{\pi}{2}\right)} = -ie^{-i\phi_1} \to ie^{i\left(\phi_1+\frac{\pi}{2}\right)} = -e^{i\phi_1} \\ m : e^{i\phi_2} &\to We^{-i\phi_2} = e^{-i(-\phi_2)} = e^{i\phi_2} \to e^{i\phi_2} . \end{aligned}$$

So e transforms to $-e$ under T^2 and m remains invariant. Performing

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}\phi_1\\ -\phi_2\end{pmatrix} + \frac{\pi}{2}\begin{pmatrix}1\\ 1\end{pmatrix} \qquad W = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

does not make both of them transform to minus themselves. In fact, we can only make one flip sign under twice the TRS, so both e and m are not Kramer's doublets. So we cannot write a Chern-Simons theory for the case when both e, m transform as we want, and so it is not anomalous.

IV.2. Anomaly detection

Let's consider the toric code with a $U(1)$ symmetry ($eCmC$). Under the $U(1)$ rotation,

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + \frac{\alpha}{2}\begin{pmatrix}1\\ 1\end{pmatrix} .$$

The qparticles e, m have charge half and f has charge zero. We need to give charges to the a's, which can be done by coupling them to a field A:

$$S = \int \frac{1}{4\pi} K_{IJ} a^I_\mu \epsilon^{\mu\nu\lambda} \partial_\nu a^J_\lambda + j^\mu_I a^I_\mu - \frac{e}{2\pi}\tau_I \epsilon^{\lambda\mu\nu} A_\lambda \partial_\mu a_{I\nu} ,$$

where the charge vector is $\tau^T = \begin{pmatrix}1 & 1\end{pmatrix}$. The charges are then

$$q_\ell = \tau^T K^{-1} \ell$$

and the response (Hall conductance) is

$$\sigma_{xy} = \tau^T K^{-1} \tau = 1 .$$

This means that the system violates TRS at the edge, i.e., time-reversal is anomalous. Equivalently, you cannot realize this theory with TRS in 2D. However, it can be realized on the surface of a 3D system since the boundary of the 3D system does not have a boundary and we do not see the Hall conductance. In that case, our symmetry is $U(1) \ltimes T$.

As another example, consider the $eCmT$ toric code, which has $U(1) \times T$ symmetry. Now, the symmetry responsible for charge commutes with T. As the name suggests, the e particle transforms non-trivially under charge and m under time-reversal. Now, we will not get any Hall conductance on the edge of a disk if we repeat the above procedure. So we have to introduce symmetry fluxes. Let's put a flux of 2π in our system and braid particles around it. Braiding e gives -1 and m gives $+1$. But braiding an e around an m also gives -1. So

$$\phi(\pi) \times \phi(\pi) = m\phi(0) .$$

Since the symmetry is a direct product of charge and time-reversal, ϕ transforms under T^2 to plus or minus itself. We already know that m transforms to $-m$ under T^2. But since two ϕ's give an m, its impossible to make sure that m transforms as we want. Thus, the theory is anomalous.

Symmetry fluxes are also known as **topological defects**, and how they talk to the original anyons in a theory is discussed in arXiv:1410.4540.