The Lagrangian description of turbulence

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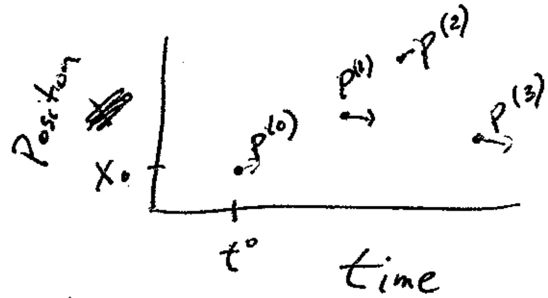
In the first of his 1941 papers, Kolmogorov proposes similarity hypotheses to be applied to velocity differences between space-time points in turbulence

$$\vec{u}(P) = \vec{u}(x, y, z, t)$$

$$\vec{w}(P) = \vec{u}(P) - \vec{u}(P_0)$$

The starting point is the distribution law F_n between n points $P^{(n)}$ and $P^{(0)}$

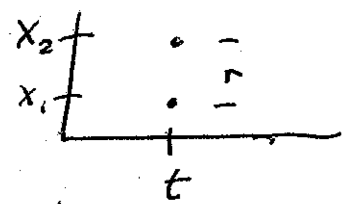
schematically in 1D

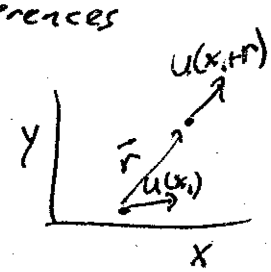


After some definitions, he drops the general case and studies the instantaneous 2point differences

- 2 point Eulerian Velocity differences

$$\overrightarrow{\Delta u_r} = \vec{u}(\vec{x}_1 + \vec{r}) - \vec{u}(\vec{x}_1)$$





In the first paper he only looks at the second moment of this which is a second rank tensor

$$B_{ij} = \langle (\Delta u_r)_i (\Delta u_r)_j \rangle$$ - Eulerian 2nd order Structure function

local homogeneity → no $\vec{x}_1$ dependence

local isotropy → choose to orient coordinate system with $\vec{r}$ along the x axis

$$\langle (\Delta u_r)_i (\Delta u_r)_j \rangle = \begin{bmatrix} B_{dd} & 0 & 0 \\ 0 & B_{nn} & 0 \\ 0 & 0 & B_{nn} \end{bmatrix}$$

B_{dd} - Longitudinal and B_{n} transverse component

Choose to look at the Longitudinal component

1^{st} similarity hypothesis - for sufficiently small $|\vec{r}|$ velocity differences are determined by

ν - viscosity $\left[\frac{m^2}{s}\right]$

ε - average energy dissipation rate per unit mass $\left[\frac{m^2}{s^3}\right]$

(and r)

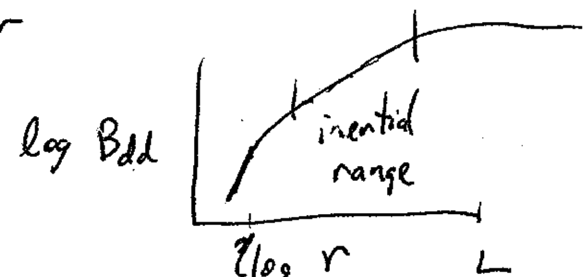
so

$$B_{dd} = \underbrace{(\varepsilon \nu)^{1/2}}_{\substack{\text{dimensional analysis} \Rightarrow \text{Kolmogorov velocity}}} \beta_{dd}\left(\frac{r}{\eta}\right) \qquad \eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \text{ Kolmogorov length scale}$$

or $(\varepsilon r)^{2/3} \beta'_{dd}(r/\eta)$

2^{nd} similarity hypothesis - if r is also $\gg \eta$ then viscosity does not matter

$$B_{dd} = C(\varepsilon r)^{2/3}$$



Kolmogorov's mistakes

1) K41 is a mean field theory - energy dissipation rate fluctuates in space and time
"intermittency problem"

2) 2^{nd} mistake - framing his hypotheses in terms of space time velocity differences is wrong

consider Eulerian temporal structure function

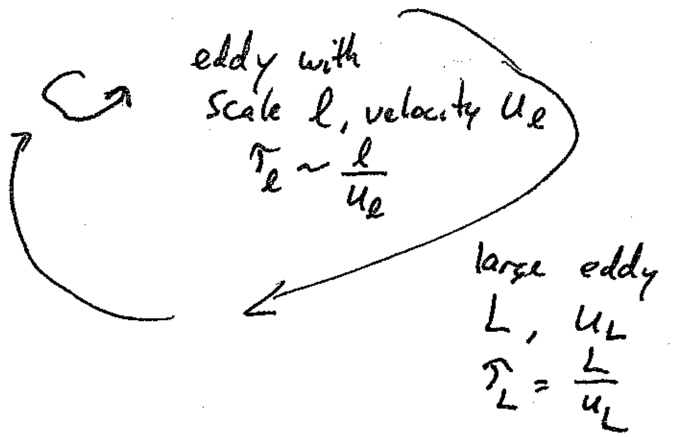
$$\left\langle \left(\vec{u}(\vec{x}, t_0) - \vec{u}(\vec{x}, t_0)\right)^2 \right\rangle \overset{?}{=} (\varepsilon \nu)^{1/2} f\left(\frac{t}{\tau_\eta}\right)$$

for small times

$$\tau_\eta = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$$

Kol time scale

No! sweeping by the large scales introduces a new quanity $\langle u^2 \rangle^{1/2}$



$$\tau_{sweeping} = \frac{\ell}{u_L} \ll \tau_\ell$$

So short time scales are dominated by sweeping from the largest eddys - can't use similarity hypotheses

~~What is~~ Kolmogorov apparently attempted to address this problem by going into a reference frame moving with the initial velocity of one point, but that fails except at $t=0$ since velocities decorrelate from the initial velocity and sweeping reappears

Tennekes 1975 JFM

What is the solution?

Go into a Lagrangian reference frame

$u^+(\vec{X}_0, t_0, t)$ = velocity at time t of a fluid particle that was at $\vec{X}_0$ at time t_0 → simplify to $u^+(t)$

$$\Delta u_t^+ = u^+(x_0, t_0, t+\tau) - u^+(x_0, t_0, t)$$

$D^L(\tau) = \langle (\Delta u_t^+)^2 \rangle$ Lagrangian 2nd order structure function

What is the K41 prediction for ~~inertial~~ scaling of $D^L(\tau)$

small scales
$$D^L(\tau) = \begin{matrix} (\varepsilon\nu)^{1/2} f(\tau/\tau_\eta) \\ (\varepsilon\tau) f'(\tau/\tau_\eta) \end{matrix}$$

inertial range

$$D^L(\tau) = C_0 \varepsilon \tau$$

for higher orders

K41 Experiment

$$D_p^L(\tau) = \langle |\delta u^+(\tau)|^p \rangle \sim (\varepsilon\tau)^{p/2} \quad (\varepsilon\tau)^{\zeta_p^L/2}$$

Now consider the fluid particle acceleration

$$a = \lim_{\tau\to 0} \left(\frac{u^+(t+\tau) - u^+(t)}{\tau} \right)$$

so $\langle a^2 \rangle = \lim_{\tau\to 0} \frac{D^L(\tau)}{\tau^2} =$

if $\langle a^2 \rangle$ is finite then $\lim_{\tau\to 0} D^L(\tau) \sim \tau^2$

$$D^L(\tau) = (\varepsilon\nu)^{1/2} f(\tau/\tau_\eta) \quad \text{so} \quad f = a_0 \left(\frac{\tau}{\tau_\eta}\right)^2$$

and $\langle a^2 \rangle = a_0 \frac{(\varepsilon\nu)^{1/2}}{\tau_\eta^2} = \frac{a_0\, \varepsilon^{3/2}}{\nu^{1/2}}$

Other Lagrangian quantities of interest

- two particle dispersion

  if $r(t)$ is the distance between two particles

  $\langle r^2 \rangle = C\varepsilon t^3$ Richardson's t^3 law

- multipoint statics - deformation of tetrahedra, etc

- Lagrangian Pressure statistics