Atmospheric and Oceanic Fluid Dynamics

Fundamentals and Large-scale Circulation

Geoffrey K. Vallis

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Chapter

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If a body is moving in any direction, there is a force, arising from the Earth's rotation, which always deflects it to the right in the northern hemisphere, and to the left in the southern.

William Ferrel, *The influence of the Earth's rotation upon the relative motion of bodies near its surface,* 1858.

CHAPTER TWO

Effects of Rotation and Stratification

THE ATMOSPHERE AND OCEAN are shallow layers of fluid on a sphere in that their thickness or depth is much less than their horizontal extent. Furthermore, their motion is strongly influenced by two effects: rotation and stratification, the latter meaning that there is a mean vertical gradient of (potential) density that is often large compared with the horizontal gradient. Here we consider how the equations of motion are affected by these effects. First, we consider some elementary effects of rotation on a fluid and derive the Coriolis and centrifugal forces, and then we write down the equations of motion appropriate for motion on a sphere. Then we discuss some approximations to the equations of motion that are appropriate for large-scale flow in the ocean and atmosphere, in particular the hydrostatic and geostrophic approximations. Following this we discuss gravity waves, a particular kind of wave motion that is enabled by the presence of stratification, and finally we talk about how rotation leads to the production of certain types of boundary layers — Ekman layers — in rotating fluids.

2.1 THE EQUATIONS OF MOTION IN A ROTATING FRAME OF REFERENCE

Newton's second law of motion, that the acceleration on a body is proportional to the imposed force divided by the body's mass, applies in so-called inertial frames of reference. The Earth rotates with a period of almost 24 hours (23h 56m) relative to the distant stars, and the surface of the Earth manifestly is not, in that sense, an inertial frame. Nevertheless, because the surface of the Earth is moving (in fact at speeds of up to a few hundreds of metres per second) it is very convenient to describe the flow relative to the Earth's surface, rather than in some inertial frame. This necessitates recasting the equations into a form that is appropriate for a rotating frame of reference, and that is the subject of this section.

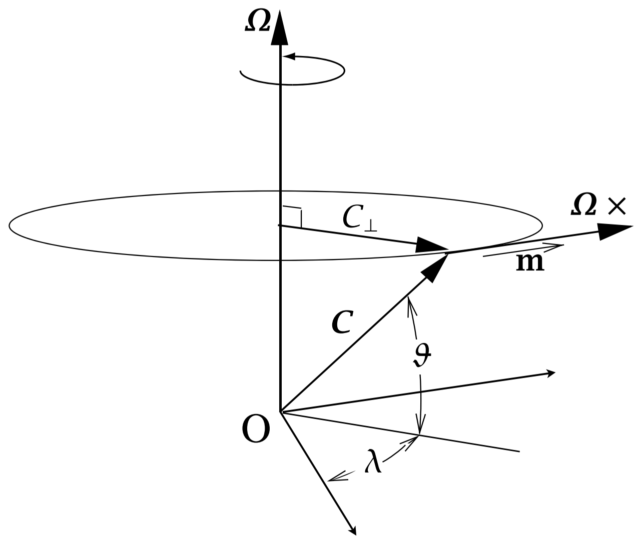

Fig. 2.1 A vector $\boldsymbol{C}$ rotating at an angular velocity $\boldsymbol{\Omega}$. It appears to be a constant vector in the rotating frame, whereas in the inertial frame it evolves according to $(\mathrm{d}\boldsymbol{C}/\mathrm{d}t)_I = \boldsymbol{\Omega}\times\boldsymbol{C}$.

2.1.1 Rate of change of a vector

Consider first a vector $\boldsymbol{C}$ of constant length rotating relative to an inertial frame at a constant angular velocity $\boldsymbol{\Omega}$. Then, in a frame rotating with that same angular velocity it appears stationary and constant. If in a small interval of time δt the vector $\boldsymbol{C}$ rotates through a small angle $\delta\lambda$ then the change in $\boldsymbol{C}$, as perceived in the inertial frame, is given by (see Fig. 2.1)

$$\delta\boldsymbol{C} = |\boldsymbol{C}|\cos\vartheta\,\delta\lambda\,\boldsymbol{m}, \tag{2.1}$$

where the vector $\boldsymbol{m}$ is the unit vector in the direction of change of $\boldsymbol{C}$, which is perpendicular to both $\boldsymbol{C}$ and $\boldsymbol{\Omega}$. But the rate of change of the angle λ is just, by definition, the angular velocity so that $\delta\lambda = |\boldsymbol{\Omega}|\delta t$ and

$$\delta\boldsymbol{C} = |\boldsymbol{C}||\boldsymbol{\Omega}|\sin\hat{\vartheta}\,\boldsymbol{m}\,\delta t = \boldsymbol{\Omega}\times\boldsymbol{C}\,\delta t. \tag{2.2}$$

using the definition of the vector cross product, where $\hat{\vartheta} = (\pi/2 - \vartheta)$ is the angle between $\boldsymbol{\Omega}$ and $\boldsymbol{C}$. Thus

$$\left(\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t}\right)_I = \boldsymbol{\Omega}\times\boldsymbol{C}, \tag{2.3}$$

where the left-hand side is the rate of change of $\boldsymbol{C}$ as perceived in the inertial frame.

Now consider a vector $\boldsymbol{B}$ that changes in the inertial frame. In a small time δt the change in $\boldsymbol{B}$ as seen in the rotating frame is related to the change seen in the inertial frame by

$$(\delta\boldsymbol{B})_I = (\delta\boldsymbol{B})_R + (\delta\boldsymbol{B})_{rot}, \tag{2.4}$$

where the terms are, respectively, the change seen in the inertial frame, the change due to the vector itself changing as measured in the rotating frame, and the change due to the rotation. Using (2.2) $(\delta\boldsymbol{B})_{rot} = \boldsymbol{\Omega}\times\boldsymbol{B}\,\delta t$, and so the rates of change of the vector $\boldsymbol{B}$ in the inertial and rotating frames are related by

$$\boxed{\left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{B}}. \tag{2.5}$$

This relation applies to a vector $\boldsymbol{B}$ that, as measured at any one time, is the same in both inertial and rotating frames.

2.1.2 Velocity and acceleration in a rotating frame

The velocity of a body is not measured to be the same in the inertial and rotating frames, so care must be taken when applying (2.5) to velocity. First apply (2.5) to $\boldsymbol{r}$, the position of a particle to obtain

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{r} \tag{2.6}$$

or

$$\boldsymbol{v}_I = \boldsymbol{v}_R + \boldsymbol{\Omega}\times\boldsymbol{r}. \tag{2.7}$$

We refer to $\boldsymbol{v}_R$ and $\boldsymbol{v}_I$ as the relative and inertial velocity, respectively, and (2.7) relates the two. Apply (2.5) again, this time to the velocity $\boldsymbol{v}_R$ to give

$$\left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{v}_R, \tag{2.8}$$

or, using (2.7)

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_I - \boldsymbol{\Omega}\times\boldsymbol{r})\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{v}_R, \tag{2.9}$$

or

$$\left(\frac{\mathrm{d}\boldsymbol{v}_I}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{v}_R + \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\times\boldsymbol{r} + \boldsymbol{\Omega}\times\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_I. \tag{2.10}$$

Then, noting that

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{r} = (\boldsymbol{v}_R + \boldsymbol{\Omega}\times\boldsymbol{r}), \tag{2.11}$$

and assuming that the rate of rotation is constant, (2.10) becomes

$$\left(\frac{\mathrm{d}\boldsymbol{v}_R}{\mathrm{d}t}\right)_R = \left(\frac{\mathrm{d}\boldsymbol{v}_I}{\mathrm{d}t}\right)_I - 2\boldsymbol{\Omega}\times\boldsymbol{v}_R - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}). \tag{2.12}$$

This equation may be interpreted as follows. The term on the left-hand side is the rate of change of the relative velocity as measured in the rotating frame. The first term on the right-hand side is the rate of change of the inertial velocity as measured in the inertial frame (or, loosely, the inertial acceleration). Thus, by Newton's second law, it is equal to the force on a fluid parcel divided by its mass. The second and third terms on the right-hand side (including the minus signs) are the *Coriolis force* and the *centrifugal force* per unit mass. Neither of these are true forces — they may be thought of as quasi-forces (i.e., 'as if' forces); that is, when a body is observed from a rotating frame it seems to behave as if unseen forces are present that affect its motion. If (2.12) is written, as is common, with the terms $+2\boldsymbol{\Omega}\times\boldsymbol{v}_r$ and $+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$ on the left-hand side then these terms should be referred to as the Coriolis and centrifugal *accelerations.*[1]

Centrifugal force

If $\boldsymbol{r}_\perp$ is the perpendicular distance from the axis of rotation (see Fig. 2.1 and substitute $\boldsymbol{r}$ for $\boldsymbol{C}$), then, because $\boldsymbol{\Omega}$ is perpendicular to $\boldsymbol{r}_\perp$, $\boldsymbol{\Omega}\times\boldsymbol{r} = \boldsymbol{\Omega}\times\boldsymbol{r}_\perp$. Then, using the vector identity $\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}_\perp) = (\boldsymbol{\Omega}\cdot\boldsymbol{r}_\perp)\boldsymbol{\Omega} - (\boldsymbol{\Omega}\cdot\boldsymbol{\Omega})\boldsymbol{r}_\perp$ and noting that the first term is zero, we see that the centrifugal force per unit mass is just given by

$$\boldsymbol{F}_{ce} = -\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) = \Omega^2\boldsymbol{r}_\perp. \tag{2.13}$$

This may usefully be written as the gradient of a scalar potential,

$$\boldsymbol{F}_{ce} = -\nabla\Phi_{ce}. \tag{2.14}$$

where $\Phi_{ce} = -(\Omega^2 r_\perp^2)/2 = -(\boldsymbol{\Omega}\times\boldsymbol{r}_\perp)^2/2$.

Coriolis force

The Coriolis force per unit mass is:

$$\boldsymbol{F}_{Co} = -2\boldsymbol{\Omega}\times\boldsymbol{v}_R. \tag{2.15}$$

It plays a central role in much of geophysical fluid dynamics and will be considered extensively later on. For now, we just note three basic properties.

(i) There is no Coriolis force on bodies that are stationary in the rotating frame.

(ii) The Coriolis force acts to deflect moving bodies at right angles to their direction of travel.

(iii) The Coriolis force does no work on a body because it is perpendicular to the velocity, and so $\boldsymbol{v}_R\cdot(\boldsymbol{\Omega}\times\boldsymbol{v}_R) = 0$.

2.1.3 Momentum equation in a rotating frame

Since (2.12) simply relates the accelerations of a particle in the inertial and rotating frames, then in the rotating frame of reference the momentum equation may be written

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega}\times\boldsymbol{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi, \tag{2.16}$$

incorporating the centrifugal term into the potential, Φ. We have dropped the subscript R; henceforth, unless we need to be explicit, all velocities without a subscript will be considered to be relative to the rotating frame.

2.1.4 Mass and tracer conservation in a rotating frame

Let ϕ be a scalar field that, in the inertial frame, obeys

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} + \phi\nabla\cdot\boldsymbol{v}_I = 0. \tag{2.17}$$

Now, observers in both the rotating and inertial frame measure the same value of ϕ. Further, $\mathrm{D}\phi/\mathrm{D}t$ is simply the rate of change of ϕ associated with a material parcel, and therefore is reference frame invariant. Thus,

$$\left(\frac{\mathrm{D}\phi}{\mathrm{D}t}\right)_R = \left(\frac{\mathrm{D}\phi}{\mathrm{D}t}\right)_I, \tag{2.18}$$

where $(\mathrm{D}\phi/\mathrm{D}t)_R = (\partial\phi/\partial t)_R + \boldsymbol{v}_R \cdot \nabla\phi$ and $(\mathrm{D}\phi/\mathrm{D}t)_I = (\partial\phi/\partial t)_I + \boldsymbol{v}_I \cdot \nabla\phi$ and the local temporal derivatives $(\partial\phi/\partial t)_R$ and $(\partial\phi/\partial t)_I$ are evaluated at fixed locations in the rotating and inertial frames, respectively.

Further, since $\boldsymbol{v} = \boldsymbol{v}_I - \boldsymbol{\Omega}\times\boldsymbol{r}$, we have that

$$\nabla\cdot\boldsymbol{v}_I = \nabla\cdot(\boldsymbol{v}_I - \boldsymbol{\Omega}\times\boldsymbol{r}) = \nabla\cdot\boldsymbol{v}_R \tag{2.19}$$

since $\nabla\cdot(\boldsymbol{\Omega}\times\boldsymbol{r}) = 0$. Thus, using (2.18) and (2.19), (2.17) is equivalent to

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} + \phi\nabla\cdot\boldsymbol{v} = 0, \tag{2.20}$$

where all observables are measured in the *rotating* frame. Thus, the equation for the evolution of a scalar whose measured value is the same in rotating and inertial frames is unaltered by the presence of rotation. In particular, the mass conservation equation is unaltered by the presence of rotation.

Although we have taken (2.18) as true a priori, the individual components of the material derivative differ in the rotating and inertial frames. In particular

$$\left(\frac{\partial\phi}{\partial t}\right)_I = \left(\frac{\partial\phi}{\partial t}\right)_R - (\boldsymbol{\Omega}\times\boldsymbol{r})\cdot\nabla\phi \tag{2.21}$$

because $\boldsymbol{\Omega}\times\boldsymbol{r}$ is the velocity, in the inertial frame, of a uniformly rotating body. Similarly,

$$\boldsymbol{v}_I\cdot\nabla\phi = (\boldsymbol{v}_R + \boldsymbol{\Omega}\times\boldsymbol{r})\cdot\nabla\phi. \tag{2.22}$$

Adding the last two equations reprises and confirms (2.18).

2.2 EQUATIONS OF MOTION IN SPHERICAL COORDINATES

The Earth is very nearly spherical and it might appear obvious that we should cast our equations in spherical coordinates. Although this does turn out to be true, the presence of a centrifugal force causes some complications which we must first discuss. The reader who is willing ab initio to treat the Earth as a perfect sphere and to neglect the horizontal component of the centrifugal force may skip the next section.

2.2.1 * The centrifugal force and spherical coordinates

The centrifugal force is a potential force, like gravity, and so we may therefore define an 'effective gravity' equal to the sum of the true, or Newtonian, gravity and the centrifugal force. The Newtonian gravitational force is directed approximately toward the centre of the Earth, with small deviations due mainly to the Earth's oblateness. The line of action of the effective gravity will in general differ slightly from this, and therefore have a component in the 'horizontal' plane, that is the plane perpendicular to the radial direction. The magnitude of the centrifugal force is $\Omega^2 r_\perp$, and so the effective gravity is given by

$$\boldsymbol{g} \equiv \boldsymbol{g}_{eff} = \boldsymbol{g}_{grav} + \Omega^2\boldsymbol{r}_\perp, \tag{2.23}$$

where $\boldsymbol{g}_{grav}$ is the Newtonian gravitational force due to the gravitational attraction of the Earth and $\boldsymbol{r}_\perp$ is normal to the rotation vector (in the direction $\boldsymbol{C}$ in Fig. 2.2), with $r_\perp = r\cos\vartheta$.

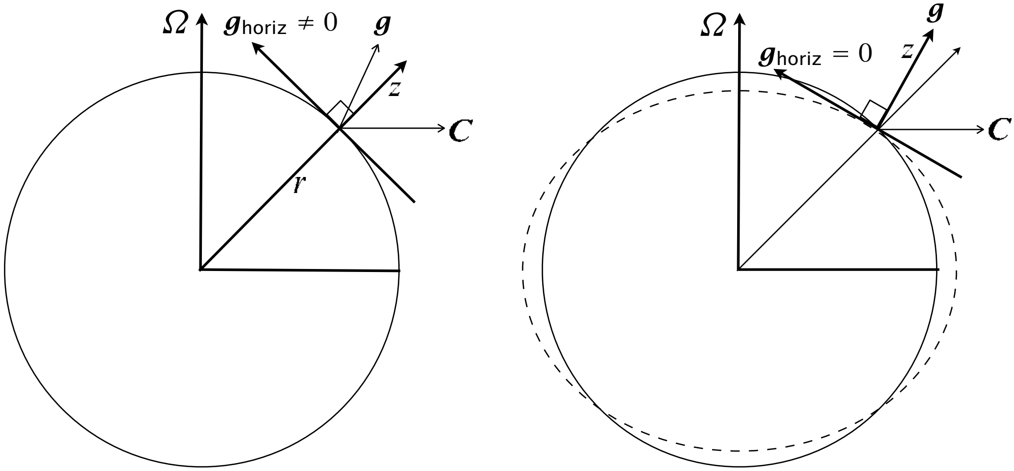

Fig. 2.2 Left: directions of forces and coordinates in true spherical geometry. $\boldsymbol{g}$ is the effective gravity (including the centrifugal force, $\boldsymbol{C}$) and its horizontal component is evidently non-zero. Right: a modified coordinate system, in which the vertical direction is defined by the direction of $\boldsymbol{g}$, and so the horizontal component of $\boldsymbol{g}$ is identically zero. The dashed line schematically indicates a surface of constant geopotential. The differences between the direction of $\boldsymbol{g}$ and the direction of the radial coordinate, and between the sphere and the geopotential surface, are much exaggerated and in reality are similar to the thickness of the lines themselves.

Both gravity and centrifugal force are potential forces and therefore we may define the *geopotential,* Φ, such that

$$\boldsymbol{g} = -\nabla\Phi. \tag{2.24}$$

Surfaces of constant Φ are not quite spherical because $r_\perp$, and hence the centrifugal force, vary with latitude (Fig. 2.2); this has certain ramifications, as we now discuss.

The components of the centrifugal force parallel and perpendicular to the radial direction are $\Omega^2 r\cos^2\vartheta$ and $\Omega^2 r\cos\vartheta\sin\vartheta$. Newtonian gravity is much larger than either of these, and at the Earth's surface the ratio of centrifugal to gravitational terms is approximately, and no more than,

$$\alpha \approx \frac{\Omega^2 a}{g} \approx \frac{(7.27\times10^{-5})^2\times6.4\times10^6}{10} \approx 3\times10^{-3}. \tag{2.25}$$

(Note that at the equator and pole the horizontal component of the centrifugal force is zero and the effective gravity is aligned with Newtonian gravity.) The angle between $\boldsymbol{g}$ and the line to the centre of the Earth is given by a similar expression and so is also small, typically around 3×10^{-3} radians. However, the horizontal component of the centrifugal force is still large compared to the Coriolis force, their ratio in mid-latitudes being given by

$$\frac{\text{horizontal centrifugal force}}{\text{Coriolis force}} \approx \frac{\Omega^2 a\cos\vartheta\sin\vartheta}{2\Omega u} \approx \frac{\Omega a}{4|u|} \approx 10, \tag{2.26}$$

using $u = 10\,\mathrm{m\,s^{-1}}$. The centrifugal term therefore dominates over the Coriolis term, and is largely balanced by a pressure gradient force. Thus, if we adhered to true spherical coordinates, both the horizontal and radial components of the momentum equation would be dominated by a static balance between a pressure gradient and gravity or centrifugal terms. Although in principle there is nothing wrong with writing the equations this way, it obscures the dynamical balances involving the Coriolis force and pressure that determine the large-scale horizontal flow.

A way around this problem is to use the direction of the geopotential force to *define* the vertical direction, and then for all geometric purposes to regard the surfaces of constant Φ as if they were true spheres.[2] The horizontal component of effective gravity is then identically zero, and we have traded a potentially large dynamical error for a very small geometric error. In fact, over time, the Earth has developed an equatorial bulge to compensate for and neutralize the centrifugal force, so that the effective gravity does act in a direction virtually normal to the Earth's surface; that is, the surface of the Earth is an oblate spheroid of nearly constant geopotential. The geopotential Φ is then a function of the vertical coordinate alone, and for many purposes we can just take $\Phi = gz$; that is, the direction normal to geopotential surfaces, the local vertical, is, in this approximation, taken to be the direction of increasing r in spherical coordinates. It is because the oblateness is very small (the polar diameter is about 12 714 km, whereas the equatorial diameter is about 12 756 km) that using spherical coordinates is a very accurate way to map the spheroid, and if the angle between effective gravity and a natural direction of the coordinate system were not small then more heroic measures would be called for.

If the solid Earth did not bulge at the equator, the *behaviour* of the atmosphere and ocean would differ significantly from that of the present system. For example, the surface of the ocean is nearly a geopotential surface, and if the solid Earth were exactly spherical then the ocean would perforce become much deeper at low latitudes and the ocean basins would dry out completely at high latitudes. We could still choose to use the spherical coordinate system discussed above to describe the dynamics, but the shape of the surface of the solid Earth would have to be represented by a topography, with the topographic height increasing monotonically polewards nearly everywhere.

2.2.2 Some identities in spherical coordinates

The location of a point is given by the coordinates (λ, ϑ, r) where λ is the angular distance eastwards (i.e., longitude), ϑ is angular distance polewards (i.e., latitude) and r is the radial distance from the centre of the Earth — see Fig. 2.3. (In some other fields of study co-latitude is used as a spherical coordinate.) If a is the radius of the Earth, then we also define $z = r - a$. At a given location we may also define the Cartesian increments $(\delta x, \delta y, \delta z) = (r \cos\vartheta\, \delta\lambda, r\,\delta\vartheta, \delta r)$.

For a scalar quantity ϕ the material derivative in spherical coordinates is

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{\partial \phi}{\partial t} + \frac{u}{r\cos\vartheta}\frac{\partial \phi}{\partial \lambda} + \frac{v}{r}\frac{\partial \phi}{\partial \vartheta} + w\frac{\partial \phi}{\partial r}, \tag{2.27}$$

where the velocity components corresponding to the coordinates (λ, ϑ, r) are

$$(u, v, w) \equiv \left(r\cos\vartheta \frac{\mathrm{D}\lambda}{\mathrm{D}t}, r\frac{\mathrm{D}\vartheta}{\mathrm{D}t}, \frac{\mathrm{D}r}{\mathrm{D}t} \right). \tag{2.28}$$

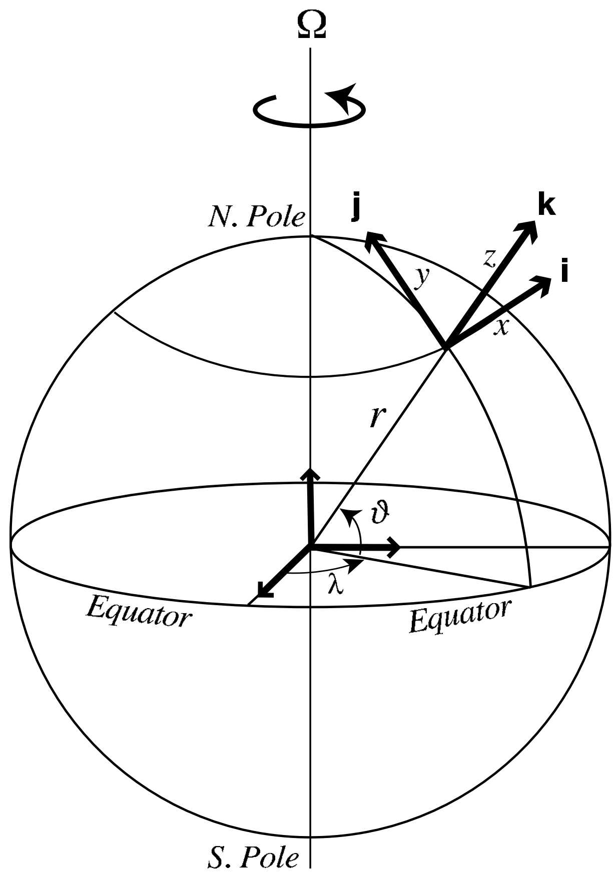

Fig. 2.3 The spherical coordinate system. The orthogonal unit vectors **i**, **j** and **k** point in the direction of increasing longitude λ, latitude ϑ, and altitude z. Locally, one may apply a Cartesian system with variables x, y and z measuring distances along **i**, **j** and **k**.

That is, u is the zonal velocity, v is the meridional velocity and w is the vertical velocity. If we define $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ to be the unit vectors in the direction of increasing (λ, ϑ, r) then

$$\boldsymbol{v} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w. \tag{2.29}$$

Note also that $\mathrm{D}r/\mathrm{D}t = \mathrm{D}z/\mathrm{D}t$.

The divergence of a vector $\boldsymbol{B} = \mathbf{i}\,B^\lambda + \mathbf{j}\,B^\vartheta + \mathbf{k}\,B^r$ is

$$\nabla \cdot \boldsymbol{B} = \frac{1}{\cos\vartheta}\left[\frac{1}{r}\frac{\partial B^\lambda}{\partial \lambda} + \frac{1}{r}\frac{\partial}{\partial \vartheta}(B^\vartheta \cos\vartheta) + \frac{\cos\vartheta}{r^2}\frac{\partial}{\partial r}(r^2 B^r)\right]. \tag{2.30}$$

The vector gradient of a scalar is:

$$\nabla\phi = \mathbf{i}\frac{1}{r\cos\vartheta}\frac{\partial\phi}{\partial\lambda} + \mathbf{j}\frac{1}{r}\frac{\partial\phi}{\partial\vartheta} + \mathbf{k}\frac{\partial\phi}{\partial r}. \tag{2.31}$$

The Laplacian of a scalar is:

$$\nabla^2\phi \equiv \nabla\cdot\nabla\phi = \frac{1}{r^2\cos\vartheta}\left[\frac{1}{\cos\vartheta}\frac{\partial^2\phi}{\partial\lambda^2} + \frac{\partial}{\partial\vartheta}\left(\cos\vartheta\frac{\partial\phi}{\partial\vartheta}\right) + \cos\vartheta\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)\right]. \tag{2.32}$$

The curl of a vector is:

$$\operatorname{curl} \boldsymbol{B} = \nabla \times \boldsymbol{B} = \frac{1}{r^2 \cos\vartheta} \begin{vmatrix} \mathbf{i}\, r\cos\vartheta & \mathbf{j}\, r & \mathbf{k} \\ \partial/\partial\lambda & \partial/\partial\vartheta & \partial/\partial r \\ B^\lambda r\cos\vartheta & B^\vartheta r & B^r \end{vmatrix}. \tag{2.33}$$

The vector Laplacian $\nabla^2 \boldsymbol{B}$ (used for example when calculating viscous terms in the momentum equation) may be obtained from the vector identity:

$$\nabla^2 \boldsymbol{B} = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla \times (\nabla \times \boldsymbol{B}). \tag{2.34}$$

Only in Cartesian coordinates does this take the simple form:

$$\nabla^2 \boldsymbol{B} = \frac{\partial^2 \boldsymbol{B}}{\partial x^2} + \frac{\partial^2 \boldsymbol{B}}{\partial y^2} + \frac{\partial^2 \boldsymbol{B}}{\partial z^2}. \tag{2.35}$$

The expansion in spherical coordinates is, to most eyes, rather uninformative.

Rate of change of unit vectors

In spherical coordinates the defining unit vectors are **i**, the unit vector pointing eastwards, parallel to a line of latitude; **j** is the unit vector pointing polewards, parallel to a meridian; and **k**, the unit vector pointing radially outward. The directions of these vectors change with location, and in fact this is the case in nearly all coordinate systems, with the notable exception of the Cartesian one, and thus their material derivative is not zero. One way to evaluate this is to consider geometrically how the coordinate axes change with position. Another way, and the way that we shall proceed, is to first obtain the effective rotation rate $\boldsymbol{\Omega}_{flow}$, relative to the Earth, of a unit vector as it moves with the flow, and then apply (2.3). Specifically, let the fluid velocity be $\boldsymbol{v} = (u, v, w)$. The meridional component, v, produces a displacement $r\delta\vartheta = v\,\delta t$, and this gives rise a local effective vector rotation rate around the local zonal axis of $-(v/r)\mathbf{i}$, the minus sign arising because a displacement in the direction of the north pole is produced by negative rotational displacement around the **i** axis. Similarly, the zonal component, u, produces a displacement $\delta\lambda r\cos\vartheta = u\delta t$ and so an effective rotation rate, about the Earth's rotation axis, of $u/(r\cos\vartheta)$. Now, a rotation around the Earth's rotation axis may be written as (see Fig. 2.4)

$$\boldsymbol{\Omega} = \Omega(\mathbf{j}\cos\vartheta + \mathbf{k}\sin\vartheta). \tag{2.36}$$

If the scalar rotation rate is not Ω but is $u/(r\cos\vartheta)$, then the vector rotation rate is

$$\frac{u}{r\cos\vartheta}(\mathbf{j}\cos\vartheta + \mathbf{k}\sin\vartheta) = \mathbf{j}\frac{u}{r} + \mathbf{k}\frac{u\tan\vartheta}{r}. \tag{2.37}$$

Thus, the total rotation rate of a vector that moves with the flow is

$$\boldsymbol{\Omega}_{flow} = -\mathbf{i}\frac{v}{r} + \mathbf{j}\frac{u}{r} + \mathbf{k}\frac{u\tan\vartheta}{r}. \tag{2.38}$$

Applying (2.3) to (2.38), we find

$$\frac{\mathrm{D}\mathbf{i}}{\mathrm{D}t} = \boldsymbol{\Omega}_{flow} \times \mathbf{i} = \frac{u}{r\cos\vartheta}(\mathbf{j}\sin\vartheta - \mathbf{k}\cos\vartheta), \tag{2.39a}$$

$$\frac{\mathrm{D}\mathbf{j}}{\mathrm{D}t} = \boldsymbol{\Omega}_{flow} \times \mathbf{j} = -\mathbf{i}\frac{u}{r}\tan\vartheta - \mathbf{k}\frac{v}{r}, \tag{2.39b}$$

$$\frac{\mathrm{D}\mathbf{k}}{\mathrm{D}t} = \boldsymbol{\Omega}_{flow} \times \mathbf{k} = \mathbf{i}\frac{u}{r} + \mathbf{j}\frac{v}{r}. \tag{2.39c}$$

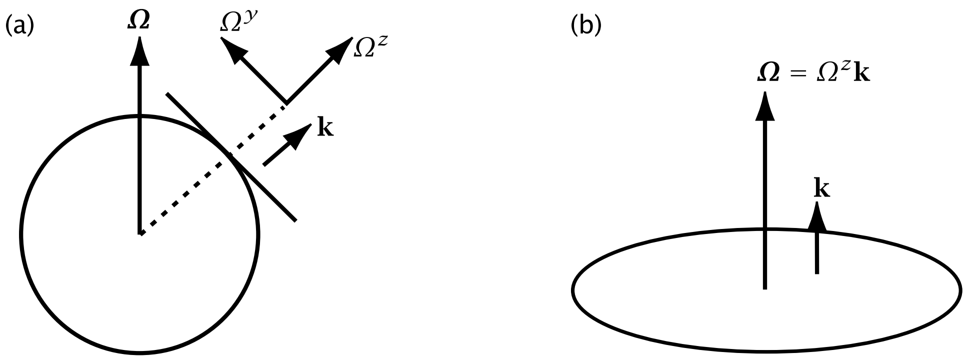

Fig. 2.4 (a) On the sphere the rotation vector $\boldsymbol{\Omega}$ can be decomposed into two components, one in the local vertical and one in the local horizontal, pointing toward the pole. That is, $\boldsymbol{\Omega} = \Omega_y \mathbf{j} + \Omega_z \mathbf{k}$ where $\Omega_y = \Omega \cos\vartheta$ and $\Omega_z = \Omega \sin\vartheta$. In geophysical fluid dynamics, the rotation vector in the local vertical is often the more important component in the horizontal momentum equations. On a rotating disk, (b), the rotation vector $\boldsymbol{\Omega}$ is parallel to the local vertical $\mathbf{k}$.

2.2.3 Equations of motion

Mass Conservation and Thermodynamic Equation

The mass conservation equation, (1.36a), expanded in spherical co-odinates, is

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \frac{u}{r\cos\vartheta}\frac{\partial \rho}{\partial \lambda} + \frac{v}{r}\frac{\partial \rho}{\partial \vartheta} + w\frac{\partial \rho}{\partial r} \\ + \frac{\rho}{r\cos\vartheta}\left[\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \vartheta}(v\cos\vartheta) + \frac{1}{r}\frac{\partial}{\partial r}(wr^2\cos\vartheta)\right] = 0.\end{aligned} \tag{2.40}$$

Equivalently, using the form (1.36b), this is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r\cos\vartheta}\frac{\partial (u\rho)}{\partial \lambda} + \frac{1}{r\cos\vartheta}\frac{\partial}{\partial \vartheta}(v\rho\cos\vartheta) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 w\rho) = 0. \tag{2.41}$$

The thermodynamic equation, (1.108), is a tracer advection equation. Thus, using (2.27), its (adiabatic) spherical coordinate form is

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{\partial \theta}{\partial t} + \frac{u}{r\cos\vartheta}\frac{\partial \theta}{\partial \lambda} + \frac{v}{r}\frac{\partial \theta}{\partial \vartheta} + w\frac{\partial \theta}{\partial r} = 0, \tag{2.42}$$

and similarly for tracers such as water vapour or salt.

Momentum Equation

Recall that the inviscid momentum equation is:

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega}\times\boldsymbol{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi, \tag{2.43}$$

where Φ is the geopotential. In spherical coordinates the directions of the coordinate axes change with position and so the component expansion of (2.43) is

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \frac{\mathrm{D}u}{\mathrm{D}t}\mathbf{i} + \frac{\mathrm{D}v}{\mathrm{D}t}\mathbf{j} + \frac{\mathrm{D}w}{\mathrm{D}t}\mathbf{k} + u\frac{\mathrm{D}\mathbf{i}}{\mathrm{D}t} + v\frac{\mathrm{D}\mathbf{j}}{\mathrm{D}t} + w\frac{\mathrm{D}\mathbf{k}}{\mathrm{D}t} \tag{2.44a}$$

$$= \frac{\mathrm{D}u}{\mathrm{D}t}\mathbf{i} + \frac{\mathrm{D}v}{\mathrm{D}t}\mathbf{j} + \frac{\mathrm{D}w}{\mathrm{D}t}\mathbf{k} + \boldsymbol{\Omega}_{flow} \times \boldsymbol{v}, \tag{2.44b}$$

using (2.39). Using either (2.44a) and the expressions for the rates of change of the unit vectors given in (2.39), or (2.44b) and the expression for $\boldsymbol{\Omega}_{flow}$ given in (2.38), (2.44) becomes

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \mathbf{i}\left(\frac{\mathrm{D}u}{\mathrm{D}t} - \frac{uv\tan\vartheta}{r} + \frac{uw}{r}\right) + \mathbf{j}\left(\frac{\mathrm{D}v}{\mathrm{D}t} + \frac{u^2\tan\vartheta}{r} + \frac{vw}{r}\right) + \mathbf{k}\left(\frac{\mathrm{D}w}{\mathrm{D}t} - \frac{u^2+v^2}{r}\right). \tag{2.45}$$

Using the definition of a vector cross product the Coriolis term is:

$$2\boldsymbol{\Omega}\times\boldsymbol{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2\Omega\cos\vartheta & 2\Omega\sin\vartheta \\ u & v & w \end{vmatrix}$$
$$= \mathbf{i}\,(2\Omega w\cos\vartheta - 2\Omega v\sin\vartheta) + \mathbf{j}\,2\Omega u\sin\vartheta - \mathbf{k}\,2\Omega u\cos\vartheta. \tag{2.46}$$

Using (2.45) and (2.46), and the gradient operator given by (2.31), the momentum equation (2.43) becomes:

$$\frac{\mathrm{D}u}{\mathrm{D}t} - \left(2\Omega + \frac{u}{r\cos\vartheta}\right)(v\sin\vartheta - w\cos\vartheta) = -\frac{1}{\rho r\cos\vartheta}\frac{\partial p}{\partial\lambda}, \tag{2.47a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + \frac{wv}{r} + \left(2\Omega + \frac{u}{r\cos\vartheta}\right)u\sin\vartheta = -\frac{1}{\rho r}\frac{\partial p}{\partial\vartheta}, \tag{2.47b}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} - \frac{u^2+v^2}{r} - 2\Omega u\cos\vartheta = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g. \tag{2.47c}$$

The terms involving Ω are called Coriolis terms, and the quadratic terms on the left-hand sides involving $1/r$ are often called metric terms.

2.2.4 The primitive equations

The so-called *primitive equations* of motion are simplifications of the above equations frequently used in atmospheric and oceanic modelling.[3] Three related approximations are involved.

(i) The hydrostatic approximation. In the vertical momentum equation the gravitational term is assumed to be balanced by the pressure gradient term, so that

$$\frac{\partial p}{\partial z} = -\rho g. \tag{2.48}$$

The advection of vertical velocity, the Coriolis terms, and the metric term $(u^2+v^2)/r$ are all neglected.

(ii) The shallow-fluid approximation. We write $r = a + z$ where the constant a is the radius of the Earth and z increases in the radial direction. The coordinate r is then replaced by a except where it used as the differentiating argument. Thus, for example,

$$\frac{1}{r^2}\frac{\partial(r^2 w)}{\partial r} \to \frac{\partial w}{\partial z}. \tag{2.49}$$

(iii) The traditional approximation. Coriolis terms in the horizontal momentum equations involving the vertical velocity, and the still smaller metric terms uw/r and vw/r, are neglected.

The second and third of these approximations should be taken, or not, together, the underlying reason being that they both relate to the presumed small aspect ratio of the motion, so the approximations succeed or fail together. If we make one approximation but not the other then we are being asymptotically inconsistent, and angular momentum and energy conservation are not assured (see section 2.2.7 and problem 2.13). The hydrostatic approximation also depends on the small aspect ratio of the flow, but in a slightly different way. For large-scale flow in the terrestrial atmosphere and ocean all three approximations are in fact all very accurate approximations. We defer a more complete treatment until section 2.7, in part because a treatment of the hydrostatic approximation is done most easily in the context of the Boussinesq equations, derived in section 2.4.

Making these approximations, the momentum equations become

$$\frac{\mathrm{D}u}{\mathrm{D}t} - 2\Omega \sin\vartheta\, v - \frac{uv}{a}\tan\vartheta = -\frac{1}{a\rho\cos\vartheta}\frac{\partial p}{\partial\lambda}, \tag{2.50a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + 2\Omega \sin\vartheta\, u + \frac{u^2\tan\vartheta}{a} = -\frac{1}{\rho a}\frac{\partial p}{\partial\vartheta}, \tag{2.50b}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{2.50c}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + \frac{u}{a\cos\vartheta}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\vartheta} + w\frac{\partial}{\partial z}\right). \tag{2.51}$$

We note the ubiquity of the factor $2\Omega\sin\vartheta$, and take the opportunity to define the *Coriolis parameter*, $f \equiv 2\Omega\sin\vartheta$.

The corresponding mass conservation equation for a shallow fluid layer is:

$$\begin{aligned}\frac{\partial\rho}{\partial t} &+ \frac{u}{a\cos\vartheta}\frac{\partial\rho}{\partial\lambda} + \frac{v}{a}\frac{\partial\rho}{\partial\vartheta} + w\frac{\partial\rho}{\partial z}\\ &+ \rho\left[\frac{1}{a\cos\vartheta}\frac{\partial u}{\partial\lambda} + \frac{1}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}(v\cos\vartheta) + \frac{\partial w}{\partial z}\right] = 0,\end{aligned} \tag{2.52}$$

or equivalently,

$$\frac{\partial\rho}{\partial t} + \frac{1}{a\cos\vartheta}\frac{\partial(u\rho)}{\partial\lambda} + \frac{1}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}(v\rho\cos\vartheta) + \frac{\partial(w\rho)}{\partial z} = 0. \tag{2.53}$$

2.2.5 Primitive equations in vector form

The primitive equations may be written in a compact vector form provided we make a slight reinterpretation of the material derivative of the coordinate axes. Let $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j} + 0\,\mathbf{k}$ be the horizontal velocity. The primitive equations (2.50a) and (2.50b) may be written as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}\times\boldsymbol{u} = -\frac{1}{\rho}\nabla_z p, \tag{2.54}$$

where $\boldsymbol{f} = f\mathbf{k} = 2\Omega\sin\vartheta\mathbf{k}$ and $\nabla_z p = [(a\cos\vartheta)^{-1}\partial p/\partial\lambda, a^{-1}\partial p/\partial\vartheta]$, the gradient operator at constant z. In (2.54) the material derivative of the horizontal velocity is given by

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \mathbf{i}\frac{\mathrm{D}u}{\mathrm{D}t} + \mathbf{j}\frac{\mathrm{D}v}{\mathrm{D}t} + u\frac{\mathrm{D}\mathbf{i}}{\mathrm{D}t} + v\frac{\mathrm{D}\mathbf{j}}{\mathrm{D}t}, \tag{2.55}$$

where instead of (2.39) we have

$$\frac{\mathrm{D}\mathbf{i}}{\mathrm{D}t} = \widetilde{\boldsymbol{\Omega}}_{flow} \times \mathbf{i} = \mathbf{j}\frac{u\tan\vartheta}{a}, \tag{2.56a}$$

$$\frac{\mathrm{D}\mathbf{j}}{\mathrm{D}t} = \widetilde{\boldsymbol{\Omega}}_{flow} \times \mathbf{j} = -\mathbf{i}\frac{u\tan\vartheta}{a}, \tag{2.56b}$$

where $\widetilde{\boldsymbol{\Omega}}_{flow} = \mathbf{k}u\tan\vartheta/a$ [which is the vertical component of (2.38), with r replaced by a.]. The advection of the horizontal wind $\boldsymbol{u}$ is still by the three-dimensional velocity $\boldsymbol{v}$. The vertical momentum equation is the hydrostatic equation, (2.50c), and the mass conservation equation is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\boldsymbol{v} = 0 \qquad \text{or} \qquad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0, \tag{2.57}$$

where $\mathrm{D}/\mathrm{D}t$ on a scalar is given by (2.51), and the second expression is written out in full in (2.53).

2.2.6 The vector invariant form of the momentum equation

The 'vector invariant' form of the momentum equation is so-called because it appears to take the same form in all coordinate systems — there is no advective derivative of the coordinate system to worry about. With the aid of the identity $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\boldsymbol{v}\times\boldsymbol{\omega} + \nabla(\boldsymbol{v}^2/2)$, where $\boldsymbol{\omega} \equiv \nabla\times\boldsymbol{v}$ is the relative vorticity, the three-dimensional momentum equation may be written:

$$\frac{\partial\boldsymbol{v}}{\partial t} + (2\boldsymbol{\Omega} + \boldsymbol{\omega})\times\boldsymbol{v} = -\frac{1}{\rho}\nabla p - \frac{1}{2}\nabla\boldsymbol{v}^2 + \boldsymbol{g}. \tag{2.58}$$

In spherical coordinates the relative vorticity is given by:

$$\begin{aligned}\boldsymbol{\omega} = \nabla\times\boldsymbol{v} &= \frac{1}{r^2\cos\vartheta}\begin{vmatrix} \mathbf{i}\,r\cos\vartheta & \mathbf{j}\,r & \mathbf{k} \\ \partial/\partial\lambda & \partial/\partial\vartheta & \partial/\partial r \\ ur\cos\vartheta & rv & w \end{vmatrix} \\ &= \mathbf{i}\frac{1}{r}\left(\frac{\partial w}{\partial\vartheta} - \frac{\partial(rv)}{\partial r}\right) - \mathbf{j}\frac{1}{r\cos\vartheta}\left(\frac{\partial w}{\partial\lambda} - \frac{\partial}{\partial r}(ur\cos\vartheta)\right) \\ &\quad + \mathbf{k}\frac{1}{r\cos\vartheta}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\vartheta}(u\cos\vartheta)\right).\end{aligned} \tag{2.59}$$

Making the traditional and shallow fluid approximations, the horizontal components of (2.58) may be written

$$\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{f} + \mathbf{k}\zeta)\times\boldsymbol{u} + w\frac{\partial\boldsymbol{u}}{\partial z} = -\frac{1}{\rho}\nabla_z p - \frac{1}{2}\nabla\boldsymbol{u}^2, \tag{2.60}$$

where $\boldsymbol{u} = (u, v, 0)$, $\boldsymbol{f} = \mathbf{k}\,2\Omega\sin\vartheta$, ∇_z is the horizontal gradient operator (the gradient at a constant value of z), and using (2.59), ζ is given by

$$\zeta = \frac{1}{a\cos\vartheta}\frac{\partial v}{\partial\lambda} - \frac{1}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}(u\cos\vartheta) = \frac{1}{a\cos\vartheta}\frac{\partial v}{\partial\lambda} - \frac{1}{a}\frac{\partial u}{\partial\vartheta} + \frac{u}{a}\tan\vartheta. \tag{2.61}$$

The separate components of the momentum equation are given by:

$$\frac{\partial u}{\partial t} - (f + \zeta)v + w\frac{\partial u}{\partial z} = -\frac{1}{a\rho\cos\vartheta}\left(\frac{1}{\rho}\frac{\partial p}{\partial \lambda} + \frac{1}{2}\frac{\partial \boldsymbol{u}^2}{\partial \lambda}\right), \tag{2.62}$$

and

$$\frac{\partial v}{\partial t} + (f + \zeta)u + w\frac{\partial v}{\partial z} = -\frac{1}{a}\left(\frac{1}{\rho}\frac{\partial p}{\partial \vartheta} + \frac{1}{2}\frac{\partial \boldsymbol{u}^2}{\partial \vartheta}\right). \tag{2.63}$$

Related expressions are given in problem 2.3, and we treat vorticity at greater length in chapter 4.

2.2.7 Angular momentum

The zonal momentum equation can be usefully expressed as a statement about axial angular momentum; that is, angular momentum about the rotation axis. The zonal angular momentum per unit mass is the component of angular momentum in the direction of the axis of rotation and it is given by, without making any shallow atmosphere approximation,

$$m = (u + \Omega r\cos\vartheta)r\cos\vartheta. \tag{2.64}$$

The evolution equation for this quantity follows from the zonal momentum equation and has the simple form

$$\frac{\mathrm{D}m}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda}, \tag{2.65}$$

where the material derivative is

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \frac{u}{r\cos\vartheta}\frac{\partial}{\partial \lambda} + \frac{v}{r}\frac{\partial}{\partial \vartheta} + w\frac{\partial}{\partial r}. \tag{2.66}$$

Using the mass continuity equation, (2.65) can be written as

$$\frac{\mathrm{D}\rho m}{\mathrm{D}t} + \rho m\nabla\cdot\boldsymbol{v} = -\frac{\partial p}{\partial \lambda} \tag{2.67}$$

or

$$\frac{\partial \rho m}{\partial t} + \frac{1}{r\cos\vartheta}\frac{\partial(\rho u m)}{\partial \lambda} + \frac{1}{r\cos\vartheta}\frac{\partial}{\partial \vartheta}(\rho v m\cos\vartheta) + \frac{\partial}{\partial z}(\rho m w) = -\frac{\partial p}{\partial \lambda}. \tag{2.68}$$

This is an angular momentum conservation equation.

If the fluid is confined to a shallow layer near the surface of a sphere, then we may replace r, the radial coordinate, by a, the radius of the sphere, in the definition of m, and we define $\widetilde{m} \equiv (u + \Omega a\cos\vartheta)a\cos\vartheta$. Then (2.65) is replaced by

$$\frac{\mathrm{D}\widetilde{m}}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda}, \tag{2.69}$$

where now

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \frac{u}{a\cos\vartheta}\frac{\partial}{\partial \lambda} + \frac{v}{a}\frac{\partial}{\partial \vartheta} + w\frac{\partial}{\partial z}. \tag{2.70}$$

Using mass continuity, (2.69) may be written as

$$\frac{\partial \rho\widetilde{m}}{\partial t} + \frac{u}{a\cos\vartheta}\frac{\partial \widetilde{m}}{\partial \lambda} + \frac{v}{a}\frac{\partial \widetilde{m}}{\partial \vartheta} + w\frac{\partial \widetilde{m}}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda}, \tag{2.71}$$

which is the appropriate angular momentum conservation equation for a shallow atmosphere.

** From angular momentum to the spherical component equations*

An alternative way of deriving the three components of the momentum equation in spherical polar coordinates is to *begin* with (2.65) and the principle of conservation of energy. That is, we take the equations for conservation of angular momentum and energy as true a priori and demand that the forms of the momentum equation be constructed to satisfy these. Expanding the material derivative in (2.65), noting that $\mathrm{D}r/\mathrm{D}t = w$ and $\mathrm{D}\cos\vartheta/\mathrm{D}t = -(v/r)\sin\vartheta$, immediately gives (2.47a). Multiplication by u then yields

$$u\frac{\mathrm{D}u}{\mathrm{D}t} - 2\Omega uv\sin\vartheta + 2\Omega uw\cos\vartheta - \frac{u^2 v\tan\vartheta}{r} + \frac{u^2 w}{r} = -\frac{u}{\rho r\cos\vartheta}\frac{\partial p}{\partial\lambda}. \tag{2.72}$$

Now suppose that the meridional and vertical momentum equations are of the form

$$\frac{\mathrm{D}v}{\mathrm{D}t} + \text{Coriolis and metric terms} = -\frac{1}{\rho r}\frac{\partial p}{\partial\vartheta} \tag{2.73a}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} + \text{Coriolis and metric terms} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2.73b}$$

but that we do not know what form the Coriolis and metric terms take. To determine that form, construct the kinetic energy equation by multiplying (2.73) by v and w, respectively. Now, the metric terms must vanish when we sum the resulting equations along with (2.72), so that (2.73a) must contain the Coriolis term $2\Omega u\sin\vartheta$ as well as the metric term $u^2\tan\vartheta/r$, and (2.73b) must contain the term $-2\Omega u\cos\phi$ as well as the metric term u^2/r. But if (2.73b) contains the term u^2/r it must also contain the term v^2/r by isotropy, and therefore (2.73a) must also contain the term vw/r. In this way, (2.47) is precisely reproduced, although the sceptic might argue that the uniqueness of the form has not been demonstrated.

A particular advantage of this approach arises in determining the appropriate momentum equations that conserve angular momentum and energy in the shallow-fluid approximation. We begin with (2.69) and expand to obtain (2.50a). Multiplying by u gives

$$u\frac{\mathrm{D}u}{\mathrm{D}t} - 2\Omega uv\sin\vartheta - \frac{u^2 v\tan\vartheta}{a} = -\frac{u}{\rho a\cos\vartheta}\frac{\partial p}{\partial\lambda}. \tag{2.74}$$

To ensure energy conservation, the meridional momentum equation must contain the Coriolis term $2\Omega u\sin\vartheta$ and the metric term $u^2\tan\vartheta/a$, but the vertical momentum equation must have neither of the metric terms appearing in (2.47c). Thus we deduce the following equations:

$$\frac{\mathrm{D}u}{\mathrm{D}t} - \left(2\Omega\sin\vartheta + \frac{u\tan\vartheta}{a}\right)v = -\frac{1}{\rho a\cos\vartheta}\frac{\partial p}{\partial\lambda}, \tag{2.75a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + \left(2\Omega\sin\vartheta + \frac{u\tan\vartheta}{a}\right)u = -\frac{1}{\rho a}\frac{\partial p}{\partial\vartheta}, \tag{2.75b}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial \rho}{\partial r} - g. \tag{2.75c}$$

This equation set, when used in conjunction with the thermodynamic and mass continuity equations, conserves appropriate forms of angular momentum and energy. In the hydrostatic approximation the material derivative of w in (2.75c) is *additionally* neglected. Thus, the hydrostatic approximation is mathematically and physically consistent with the shallow-fluid approximation, but it is an additional approximation with slightly different requirements that one may choose, rather than being required, to make. From an asymptotic perspective, the difference lies in the small parameter necessary for either approximation to hold, namely:

$$\text{shallow fluid and traditional approximations:} \qquad \gamma \equiv \frac{H}{a} \ll 1, \tag{2.76a}$$

$$\text{small aspect ratio for hydrostatic approximation:} \qquad \alpha \equiv \frac{H}{L} \ll 1, \tag{2.76b}$$

where L is the horizontal scale of the motion and a is the radius of the Earth. For hemispheric or global scale phenomena $L \sim a$ and the two approximations coincide. (Requirement (2.76b) for the hydrostatic approximation is derived in section 2.7.)

2.3 CARTESIAN APPROXIMATIONS: THE TANGENT PLANE

2.3.1 The f-plane

Although the rotation of the Earth is central for many dynamical phenomena, the sphericity of the Earth is not always so. This is especially true for phenomena on a scale somewhat smaller than global where the use of spherical coordinates becomes awkward, and it is more convenient to use a locally Cartesian representation of the equations. Referring to Fig. 2.4 we will define a plane tangent to the surface of the Earth at a latitude ϑ_0, and then use a Cartesian coordinate system (x, y, z) to describe motion on that plane. For small excursions on the plane, $(x, y, z) \approx (a\lambda \cos\vartheta_0, a(\vartheta - \vartheta_0), z)$. Consistently, the velocity is $\boldsymbol{v} = (u, v, w)$, so that u,v and w are the components of the velocity *in the tangent plane,* in approximately in the east–west, north–south and vertical directions, respectively.

The momentum equations for flow in this plane are then

$$\frac{\partial u}{\partial t} + (\boldsymbol{v} \cdot \nabla)u + 2(\Omega^y w - \Omega^z v) = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.77a}$$

$$\frac{\partial v}{\partial t} + (\boldsymbol{v} \cdot \nabla)v + 2(\Omega^z u - \Omega^x w) = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2.77b}$$

$$\frac{\partial w}{\partial t} + (\boldsymbol{v} \cdot \nabla)w + 2(\Omega^x v - \Omega^y u) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{2.77c}$$

where the rotation vector $\boldsymbol{\Omega} = \Omega^x\mathbf{i} + \Omega^y\mathbf{j} + \Omega^z\mathbf{k}$ and $\Omega^x = 0$, $\Omega^y = \Omega\cos\vartheta_0$ and $\Omega^z = \Omega\sin\vartheta_0$. If we make the traditional approximation, and so ignore the components of $\boldsymbol{\Omega}$ not in the direction of the local vertical, then

$$\frac{\mathrm{D}u}{\mathrm{D}t} - f_0 v = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.78a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + f_0 u = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2.78b}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \tag{2.78c}$$

where $f_0 = 2\Omega^z = 2\Omega\sin\vartheta_0$. Defining the horizontal velocity vector $\boldsymbol{u} = (u, v, 0)$, the first two equations may also be written as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}_0 \times \boldsymbol{u} = -\frac{1}{\rho}\nabla_z p, \tag{2.79}$$

where $\mathrm{D}\boldsymbol{u}/\mathrm{D}t = \partial \boldsymbol{u}/\partial t + \boldsymbol{v}\cdot\nabla\boldsymbol{u}$, $\boldsymbol{f}_0 = 2\Omega \sin\vartheta_0 \mathbf{k} = f_0\mathbf{k}$, and $\mathbf{k}$ is the direction perpendicular to the plane (it does not change its orientation with latitude). These equations are, evidently, exactly the same as the momentum equations in a system in which the rotation vector is aligned with the local vertical, as illustrated in the right-hand panel in Fig. 2.4 (on page 60). They will describe flow on the surface of a rotating sphere to a good approximation provided the flow is of limited latitudinal extent so that the effects of sphericity are unimportant; we have made what is known as the *f-plane* approximation since the Coriolis parameter is a constant. We may in addition make the hydrostatic approximation, in which case (2.78c) becomes the familiar $\partial p/\partial z = -\rho g$.

2.3.2 The beta-plane approximation

The magnitude of the vertical component of rotation varies with latitude, and this has important dynamical consequences. We can approximate this effect by allowing the effective rotation vector to vary. Thus, noting that, for small variations in latitude,

$$f = 2\Omega \sin\vartheta \approx 2\Omega\sin\vartheta_0 + 2\Omega(\vartheta - \vartheta_0)\cos\vartheta_0, \tag{2.80}$$

then on the tangent plane we may mimic this by allowing the Coriolis parameter to vary as

$$\boxed{f = f_0 + \beta y}, \tag{2.81}$$

where $f_0 = 2\Omega\sin\vartheta_0$ and $\beta = \partial f/\partial y = (2\Omega\cos\vartheta_0)/a$. This important approximation is known as the *beta-plane*, or *β-plane,* approximation; it captures the the most important *dynamical* effects of sphericity, without the complicating *geometric* effects, which are not essential to describe many phenomena. The momentum equations (2.78) are unaltered except that f_0 is replaced by $f_0 + \beta y$ to represent a varying Coriolis parameter. Thus, sphericity combined with rotation is dynamically equivalent to a *differentially rotating* system. For future reference, we write down the β-plane horizontal momentum equations:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}\times\boldsymbol{u} = -\frac{1}{\rho}\nabla_z p, \tag{2.82}$$

where $\boldsymbol{f} = (f_0 + \beta y)\hat{\mathbf{k}}$. In component form this equation becomes

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{2.83a,b}$$

The mass conservation, thermodynamic and hydrostatic equations in the β-plane approximation are the same as the usual Cartesian, f-plane, forms of those equations.

2.4 EQUATIONS FOR A STRATIFIED OCEAN: THE BOUSSINESQ APPROXIMATION

The density variations in the ocean are quite small compared to the mean density, and we may exploit this to derive somewhat simpler but still quite accurate equations of motion. Let us first examine how much density does vary in the ocean.

2.4.1 Variation of density in the ocean

The variations of density in the ocean are due to three effects: the compression of water by pressure (which we denote as $\Delta_p \rho$), the thermal expansion of water if its temperature changes ($\Delta_T \rho$), and the haline contraction if its salinity changes ($\Delta_S \rho$). How big are these? An appropriate equation of state to approximately evaluate these effects is the linear one

$$\rho = \rho_0 \left[1 - \beta_T (T - T_0) + \beta_S (S - S_0) + \frac{p}{\rho_0 c_s^2} \right], \tag{2.84}$$

where $\beta_T \approx 2 \times 10^{-4}\,\mathrm{K}^{-1}$, $\beta_S \approx 10^{-3}\,\mathrm{psu}^{-1}$ and $c_s \approx 1500\,\mathrm{m\,s}^{-1}$ (see the table on page 35). The three effects may then be evaluated as follows.

Pressure compressibility. We have $\Delta_p \rho \approx \Delta p / c_s^2 \approx \rho_0 g H / c_s^2$. where H is the depth and the pressure change is quite accurately evaluated using the hydrostatic approximation. Thus,

$$\frac{|\Delta_p \rho|}{\rho_0} \ll 1 \qquad \text{if} \qquad \frac{gH}{c_s^2} \ll 1, \tag{2.85}$$

or if $H \ll c_s^2 / g$. The quantity $c_s^2/g \approx 200\,\mathrm{km}$ is the density scale height of the ocean. Thus, the pressure at the bottom of the ocean (say $H = 10\,\mathrm{km}$ in the deep trenches), enormous as it is, is insufficient to compress the water enough to make a significant change in its density. Changes in density due to dynamical variations of pressure are small if the Mach number is small, and this is also the case.

Thermal expansion. We have $\Delta_T \rho \approx -\beta_T \rho_0 \Delta T$ and therefore

$$\frac{|\Delta_T \rho|}{\rho_0} \ll 1 \qquad \text{if} \qquad \beta_T \Delta T \ll 1. \tag{2.86}$$

For $\Delta T = 20\,\mathrm{K}$, $\beta_T \Delta T \approx 4 \times 10^{-3}$, and evidently we would require temperature differences of order β_T^{-1}, or 5000 K to obtain order one variations in density.

Saline contraction. We have $\Delta_S \rho \approx \beta_S \rho_0 \Delta S$ and therefore

$$\frac{|\Delta_S \rho|}{\rho_0} \ll 1 \qquad \text{if} \qquad \beta_S \Delta S \ll 1. \tag{2.87}$$

As changes in salinity in the ocean rarely exceed 5 psu, for which $\beta_S \Delta S = 5 \times 10^{-3}$, the fractional change in the density of seawater is correspondingly very small.

Evidently, fractional density changes in the ocean are very small.

2.4.2 The Boussinesq equations

The *Boussinesq equations* are a set of equations that exploit the smallness of density variations in many liquids.[4] To set notation we write

$$\rho = \rho_0 + \delta\rho(x, y, z, t) \tag{2.88a}$$

$$= \rho_0 + \hat{\rho}(z) + \rho'(x, y, z, t) \tag{2.88b}$$

$$= \tilde{\rho}(z) + \rho'(x, y, z, t), \tag{2.88c}$$

where ρ_0 is a constant and we assume that

$$|\widehat{\rho}|, |\rho'|, |\delta\rho| \ll \rho_0. \tag{2.89}$$

We need not assume that $|\rho'| \ll |\widehat{\rho}|$, but this is often the case in the ocean. To obtain the Boussinesq equations we will just use (2.88a), but (2.88c) will be useful for the anelastic equations considered later.

Associated with the reference density is a reference pressure that is defined to be in hydrostatic balance with it. That is,

$$p = p_0(z) + \delta p(x, y, z, t) \tag{2.90a}$$
$$= \widetilde{p}(z) + p'(x, y, z, t), \tag{2.90b}$$

where $|\delta p| \ll p_0$, $|p'| \ll \widetilde{p}$ and

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} \equiv -g\rho_0, \qquad \frac{\mathrm{d}\widetilde{p}}{\mathrm{d}z} \equiv -g\widetilde{\rho}. \tag{2.91a,b}$$

Note that $\nabla_z p = \nabla_z p' = \nabla_z \delta p$ and that $p_0 \approx \widetilde{p}$ if $|\widehat{\rho}| \ll \rho_0$.

Momentum equations

To obtain the Boussinesq equations we use $\rho = \rho_0 + \delta\rho$, and assume $\delta\rho/\rho_0$ is small. Without approximation, the momentum equation can be written as

$$(\rho_0 + \delta\rho)\left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v}\right) = -\nabla\delta p - \frac{\partial p_0}{\partial z}\mathbf{k} - g(\rho_0 + \delta\rho)\mathbf{k}, \tag{2.92}$$

and using (2.91a) this becomes, again without approximation,

$$(\rho_0 + \delta\rho)\left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v}\right) = -\nabla\delta p - g\delta\rho\mathbf{k}. \tag{2.93}$$

If density variations are small this becomes

$$\boxed{\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\nabla\phi + b\mathbf{k}}, \tag{2.94}$$

where $\phi = \delta p/\rho_0$ and $b = -g\,\delta\rho/\rho_0$ is the *buoyancy*. Note that we should not and do not neglect the term $g\,\delta\rho$, for there is no reason to believe it to be small ($\delta\rho$ may be small, but g is big). Equation (2.94) is the momentum equation in the Boussinesq approximation, and it is common to say that the Boussinesq approximation ignores all variations of density of a fluid in the momentum equation, except when associated with the gravitational term.

For most large-scale motions in the ocean the *deviation* pressure and density fields are also approximately in hydrostatic balance, and in that case the vertical component of (2.94) becomes

$$\frac{\partial\phi}{\partial z} = b. \tag{2.95}$$

A condition for (2.95) to hold is that vertical accelerations are small *compared to $g\,\delta\rho/\rho_0$, and not compared to the acceleration due to gravity itself.* For more discussion of this point, see section 2.7.

Mass Conservation

The unapproximated mass conservation equation is

$$\frac{D\delta\rho}{Dt} + (\rho_0 + \delta\rho)\nabla \cdot \boldsymbol{v} = 0. \tag{2.96}$$

Provided that time scales advectively — that is to say that D/Dt scales in the same way as $\boldsymbol{v} \cdot \nabla$ — then we may approximate this equation by

$$\boxed{\nabla \cdot \boldsymbol{v} = 0}, \tag{2.97}$$

which is the same as that for a constant density fluid. This *absolutely does not* allow one to go back and use (2.96) to say that $D\delta\rho/Dt = 0$; the evolution of density is given by the thermodynamic equation in conjunction with an equation of state, and this should not be confused with the mass conservation equation. Note also that in eliminating the time-derivative of density we eliminate the possibility of sound waves.

Thermodynamic equation and equation of state

The Boussinesq equations are closed by the addition of an equation of state, a thermodynamic equation and, as appropriate, a salinity equation. Neglecting salinity for the moment, a useful starting point is to write the thermodynamic equation, (1.116), as

$$\frac{D\rho}{Dt} - \frac{1}{c_s^2}\frac{Dp}{Dt} = \frac{\dot{Q}}{(\partial\eta/\partial\rho)_p T} \approx -\dot{Q}\left(\frac{\rho_0\beta_T}{c_p}\right) \tag{2.98}$$

using $(\partial\eta/\partial\rho)_p = (\partial\eta/\partial T)_p(\partial T/\partial\rho)_p \approx c_p/(T\rho_0\beta_T)$. Given the expansions (2.88a) and (2.90a), (2.98) can be written to a good approximation as

$$\frac{D\delta\rho}{Dt} - \frac{1}{c_s^2}\frac{Dp_0}{Dt} = -\dot{Q}\left(\frac{\rho_0\beta_T}{c_p}\right), \tag{2.99}$$

or, using (2.91a),

$$\frac{D}{Dt}\left(\delta\rho + \frac{\rho_0 g}{c_s^2}z\right) = -\dot{Q}\left(\frac{\rho_0\beta_T}{c_p}\right), \tag{2.100}$$

as in (1.119). The severest approximation to this is to neglect the second term in brackets on the left-hand side, and noting that $b = -g\delta\rho/\rho_0$ we obtain

$$\boxed{\frac{Db}{Dt} = \dot{b}}, \tag{2.101}$$

where $\dot{b} = g\beta_T\dot{Q}/c_p$. The momentum equation (2.94), mass continuity equation (2.97) and thermodynamic equation (2.101) then form a closed set, called the *simple Boussinesq equations*.

A somewhat more accurate approach is to include the compressibility of the fluid that is due to the hydrostatic pressure. From (2.100), the potential density is given by $\delta\rho_{\text{pot}} = \delta\rho + \rho_0 gz/c_s^2$, and so the *potential buoyancy,* that is the buoyancy based on potential density, is given by

$$b_\sigma \equiv -g\frac{\delta\rho_{\text{pot}}}{\rho_0} = -\frac{g}{\rho_0}\left(\delta\rho + \frac{\rho_0 gz}{c_s^2}\right) = b - g\frac{z}{H_\rho}, \tag{2.102}$$

where $H_\rho = c_s^2/g$. The thermodynamic equation, (2.100), may then be written

$$\frac{\mathrm{D}b_\sigma}{\mathrm{D}t} = \dot{b}_\sigma, \tag{2.103}$$

where $\dot{b}_\sigma = \dot{b}$. Buoyancy itself is obtained from b_σ by the 'equation of state', $b = b_\sigma + gz/H_\rho$.

In many applications we may need to use a still more accurate equation of state. In that case (and see section 1.6.2) we replace (2.101) by the thermodynamic equations

$$\boxed{\frac{\mathrm{D}\theta}{\mathrm{D}t} = \dot{\theta}, \qquad \frac{\mathrm{D}S}{\mathrm{D}t} = \dot{S}}\,, \tag{2.104a,b}$$

where θ is the potential temperature and S is salinity, along with an equation of state. The equation of state has the general form $b = b(\theta, S, p)$, but to be consistent with the level of approximation in the other Boussinesq equations we replace p by the hydrostatic pressure calculated with the reference density, that is by $-\rho_0 gz$, and the equation of state then takes the general form

$$\boxed{b = b(\theta, S, z)}\,. \tag{2.105}$$

An example of (2.105) is (1.156), taken with the definition of buoyancy $b = -g\delta\rho/\rho_0$. The closed set of equations (2.94), (2.97), (2.104) and (2.105) are the *general Boussinesq equations*. Using an accurate equation of state and the Boussinesq approximation is the procedure used in many comprehensive ocean general circulation models. The Boussinesq equations, which with the hydrostatic and traditional approximations are often considered to be the oceanic primitive equations, are summarized in the shaded box on the next page.

* *Mean stratification and the buoyancy frequency*

The processes that cause density to vary in the vertical often differ from those that cause it to vary in the horizontal. For this reason it is sometimes useful to write $\rho = \rho_0 + \widehat{\rho}(z) + \rho'(x, y, z, t)$ and define $\widetilde{b}(z) \equiv -g\widehat{\rho}/\rho_0$ and $b' \equiv -g\rho'/\rho_0$. Using the hydrostatic equation to evaluate pressure, the thermodynamic equation (2.98) becomes, to a good approximation,

$$\frac{\mathrm{D}b'}{\mathrm{D}t} + N^2 w = 0, \tag{2.106}$$

where $\mathrm{D}/\mathrm{D}t$ remains a three-dimensional operator and

$$N^2(z) = \left(\frac{\mathrm{d}\widetilde{b}}{\mathrm{d}z} - \frac{g^2}{c_s^2}\right) = \frac{\mathrm{d}\widetilde{b}_\sigma}{\mathrm{d}z}, \tag{2.107}$$

where $\widetilde{b}_\sigma = \widetilde{b} - gz/H_\rho$. The quantity N^2 is a measure of the mean stratification of the fluid, and is equal to the vertical gradient of the mean potential buoyancy. N is known as the buoyancy frequency, something we return to in section 2.9. Equations (2.106) and (2.107) also hold in the simple Boussinesq equations, but with $c_s^2 = \infty$.

Summary of Boussinesq Equations

The simple Boussinesq equations are, for an inviscid fluid:

$$\text{momentum equations:} \quad \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{v} = -\nabla\phi + b\mathbf{k}, \tag{B.1}$$

$$\text{mass conservation:} \quad \nabla \cdot \boldsymbol{v} = 0, \tag{B.2}$$

$$\text{buoyancy equation:} \quad \frac{\mathrm{D}b}{\mathrm{D}t} = \dot{b}. \tag{B.3}$$

A more general form replaces the buoyancy equation by:

$$\text{thermodynamic equation:} \quad \frac{\mathrm{D}\theta}{\mathrm{D}t} = \dot{\theta}, \tag{B.4}$$

$$\text{salinity equation:} \quad \frac{\mathrm{D}S}{\mathrm{D}t} = \dot{S}, \tag{B.5}$$

$$\text{equation of state:} \quad b = b(\theta, S, \phi). \tag{B.6}$$

Energy conservation is only assured if $b = b(\theta, S, z)$.

2.4.3 Energetics of the Boussinesq system

In a uniform gravitational field but with no other forcing or dissipation, we write the simple Boussinesq equations as

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = b\mathbf{k} - \nabla\phi, \qquad \nabla \cdot \boldsymbol{v} = 0, \qquad \frac{\mathrm{D}b}{\mathrm{D}t} = 0. \tag{2.108a,b,c}$$

From (2.108a) and (2.108b) the kinetic energy density evolution (cf. section 1.10) is given by

$$\frac{1}{2}\frac{\mathrm{D}v^2}{\mathrm{D}t} = bw - \nabla \cdot (\phi\boldsymbol{v}), \tag{2.109}$$

where the constant reference density ρ_0 is omitted. Let us now define the potential $\Phi \equiv -z$, so that $\nabla\Phi = -\mathbf{k}$ and

$$\frac{\mathrm{D}\Phi}{\mathrm{D}t} = \nabla \cdot (\boldsymbol{v}\Phi) = -w, \tag{2.110}$$

and using this and (2.108c) gives

$$\frac{\mathrm{D}}{\mathrm{D}t}(b\Phi) = -wb. \tag{2.111}$$

Adding (2.111) to (2.109) and expanding the material derivative gives

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{v}^2 + b\Phi\right) + \nabla \cdot \left[\boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2 + b\Phi + \phi\right)\right] = 0. \tag{2.112}$$

This constitutes an energy equation for the Boussinesq system, and may be compared to (1.186). (Also see problem 2.14.) The energy density (divided by ρ_0) is just $\boldsymbol{v}^2/2 + b\Phi$. What does the term $b\Phi$ represent? Its integral, multiplied by ρ_0, is the potential energy of the flow minus that of the basic state, or $\int g(\rho - \rho_0) z \, dz$. If there were a heating term on the right-hand side of (2.108c) this would directly provide a source of potential energy, rather than internal energy as in the compressible system. Because the fluid is incompressible, there is no conversion from kinetic and potential energy into internal energy.

** Energetics with a general equation of state*

Now consider the energetics of the general Boussinesq equations. Suppose first that we allow the equation of state to be a function of pressure; the equations are motion are then (2.108) except that (2.108c) is replaced by

$$\frac{D\theta}{Dt} = 0, \qquad \frac{DS}{Dt} = 0, \qquad b = b(\theta, S, \phi). \tag{2.113a,b,c}$$

A little algebraic experimentation will reveal that no energy conservation law of the form (2.112) generally exists for this system! The problem arises because, by requiring the fluid to be incompressible, we eliminate the proper conversion of internal energy to kinetic energy. However, if we use the approximation $b = b(\theta, S, z)$, the system does conserve an energy, as we now show.[5]

Define the potential, Π, as the integral of b at constant potential temperature and salinity; that is

$$\Pi(\theta, S, z) \equiv -\int_a^z b \, dz', \tag{2.114}$$

where a is any constant, so that $\partial \Pi / \partial z = -b$. Taking the material derivative of the left-hand side gives

$$\frac{D\Pi}{Dt} = \left(\frac{\partial \Pi}{\partial \theta}\right)_{S,z} \frac{D\theta}{Dt} + \left(\frac{\partial \Pi}{\partial S}\right)_{\theta,z} \frac{DS}{Dt} + \left(\frac{\partial \Pi}{\partial z}\right)_{\theta,S} \frac{Dz}{Dt} = -bw, \tag{2.115}$$

using (2.113a,b). Combining (2.115) and (2.109) gives

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{v}^2 + \Pi\right) + \nabla \cdot \left[\boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2 + \Pi + \phi\right)\right] = 0. \tag{2.116}$$

Thus, energetic consistency is maintained with an arbitrary equation of state, provided the pressure is replaced by a function of z. If b is not an explicit function of z in the equation of state, the conservation law is identical to (2.112).

2.5 EQUATIONS FOR A STRATIFIED ATMOSPHERE: THE ANELASTIC APPROXIMATION

2.5.1 Preliminaries

In the atmosphere the density varies significantly, especially in the vertical. However deviations of both ρ and p from a statically balanced state are often quite small, and the relative vertical variation of potential temperature is also small. We can usefully exploit these observations to give a somewhat simplified set of equations, useful both for theoretical and numerical analyses because sound waves are eliminated by way of an 'anelastic' approximation.[6] To begin we set

$$\rho = \widetilde{\rho}(z) + \delta\rho(x, y, z, t), \qquad p = \widetilde{p}(z) + \delta p(x, y, z, t), \tag{2.117a,b}$$

where we assume that $|\delta\rho| \ll |\widetilde{\rho}|$ and we define $\widetilde{p}$ such that

$$\frac{\partial \widetilde{p}}{\partial z} \equiv -g\widetilde{\rho}(z). \tag{2.118}$$

The notation is similar to that for the Boussinesq case except that, importantly, the density basic state is now a (given) function of vertical coordinate. As with the Boussinesq case, the idea is to ignore dynamic variations of density (i.e., of $\delta\rho$) except where associated with gravity. First recall a couple of ideal gas relationships involving potential temperature, θ, and entropy s (divided by c_p, so $s \equiv \log\theta$), namely

$$s \equiv \log\theta = \log T - \frac{R}{c_p}\log p = \frac{1}{\gamma}\log p - \log\rho, \tag{2.119}$$

where $\gamma = c_p/c_v$, implying

$$\delta s = \frac{1}{\theta}\delta\theta = \frac{1}{\gamma}\frac{\delta p}{p} - \frac{\delta\rho}{\rho} \approx \frac{1}{\gamma}\frac{\delta p}{\widetilde{p}} - \frac{\delta\rho}{\widetilde{\rho}}. \tag{2.120}$$

Further, if $\widetilde{s} \equiv \gamma^{-1}\log\widetilde{p} - \log\widetilde{\rho}$ then

$$\frac{\mathrm{d}\widetilde{s}}{\mathrm{d}z} = \frac{1}{\gamma\widetilde{p}}\frac{\mathrm{d}\widetilde{p}}{\mathrm{d}z} - \frac{1}{\widetilde{\rho}}\frac{\mathrm{d}\widetilde{\rho}}{\mathrm{d}z} = -\frac{g\widetilde{\rho}}{\gamma\widetilde{p}} - \frac{1}{\widetilde{\rho}}\frac{\mathrm{d}\widetilde{\rho}}{\mathrm{d}z}. \tag{2.121}$$

In the atmosphere, the left-hand side is, typically, much smaller than either of the two terms on the right-hand side.

2.5.2 The momentum equation

The exact inviscid horizontal momentum equation is

$$(\widetilde{\rho} + \rho')\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}\times\boldsymbol{u}\right) = -\nabla_z\delta p. \tag{2.122}$$

Neglecting ρ' where it appears with $\widetilde{\rho}$ leads to

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}\times\boldsymbol{u} = -\nabla_z\phi, \tag{2.123}$$

where $\phi = \delta p/\widetilde{\rho}$, and this is similar to the corresponding equation in the Boussinesq approximation.

The vertical component of the inviscid momentum equation is, without approximation,

$$(\widetilde{\rho} + \delta\rho)\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial\widetilde{p}}{\partial z} - \frac{\partial\delta p}{\partial z} - g\widetilde{\rho} - g\delta\rho = -\frac{\partial\delta p}{\partial z} - g\delta\rho. \tag{2.124}$$

using (2.118). Neglecting $\delta\rho$ on the left-hand side we obtain

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\widetilde{\rho}}\frac{\partial\delta p}{\partial z} - g\frac{\delta\rho}{\widetilde{\rho}} = -\frac{\partial}{\partial z}\left(\frac{\delta p}{\widetilde{\rho}}\right) - \frac{\delta p}{\widetilde{\rho}^2}\frac{\partial\widetilde{\rho}}{\partial z} - g\frac{\delta\rho}{\widetilde{\rho}}. \tag{2.125}$$

This is not a useful form for a gaseous atmosphere, since the variation of the mean density cannot be ignored. However, we may eliminate $\delta\rho$ in favour of δs using (2.120) to give

$$\frac{\mathrm{D}w}{\mathrm{D}t} = g\delta s - \frac{\partial}{\partial z}\left(\frac{\delta p}{\widetilde{\rho}}\right) - \frac{g}{\gamma}\frac{\delta p}{\widetilde{p}} - \frac{\delta p}{\widetilde{\rho}^2}\frac{\partial\widetilde{\rho}}{\partial z}, \tag{2.126}$$

and using (2.121) gives

$$\frac{Dw}{Dt} = g\delta s - \frac{\partial}{\partial z}\left(\frac{\delta p}{\widetilde{\rho}}\right) + \frac{d\widetilde{s}}{dz}\frac{\delta p}{\widetilde{\rho}}. \qquad (2.127)$$

What have these manipulations gained us? Two things:

(i) The gravitational term now involves δs rather than $\delta\rho$ which enables a more direct connection with the thermodynamic equation.

(ii) The potential temperature scale height ($\sim 100\,\text{km}$) in the atmosphere is much larger than the density scale height ($\sim 10\,\text{km}$), and so the last term in (2.127) is small.

The second item thus suggests that we choose our reference state to be one of constant potential temperature (see also problem 2.19). The term $d\widetilde{s}/dz$ then vanishes and the vertical momentum equation becomes

$$\boxed{\frac{Dw}{Dt} = g\delta s - \frac{\partial \phi}{\partial z}}, \qquad (2.128)$$

where $\phi = \delta p/\widetilde{\rho}$ and $\delta s = \delta\theta/\theta_0$, where θ_0 is a constant. If we define a buoyancy by $b_a \equiv g\delta s = g\delta\theta/\theta_0$, then (2.123) and (2.128) have the same form as the Boussinesq momentum equations, but with a slightly different definition of buoyancy.

2.5.3 Mass conservation

Using (2.117a) the mass conservation equation may be written, without approximation, as

$$\frac{\partial \delta\rho}{\partial t} + \nabla \cdot [(\widetilde{\rho} + \delta\rho)\boldsymbol{v}] = 0. \qquad (2.129)$$

We neglect $\delta\rho$ where it appears with $\widetilde{\rho}$ in the divergence term. Further, the local time derivative will be small if time itself is scaled advectively (i.e., $T \sim L/U$ and sound waves do not dominate), giving

$$\nabla \cdot \boldsymbol{u} + \frac{1}{\widetilde{\rho}}\frac{\partial}{\partial z}(\widetilde{\rho} w) = 0. \qquad (2.130)$$

It is here that the eponymous 'anelastic approximation' arises: the elastic compressibility of the fluid is neglected, and this serves to eliminate sound waves. For reference, in spherical coordinates the equation is

$$\frac{1}{a\cos\vartheta}\frac{\partial u}{\partial \lambda} + \frac{1}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}(v\cos\vartheta) + \frac{1}{\widetilde{\rho}}\frac{\partial(w\widetilde{\rho})}{\partial z} = 0. \qquad (2.131)$$

In an ideal gas, the choice of constant potential temperature determines how the reference density $\widetilde{\rho}$ varies with height. In some circumstances it is convenient to let $\widetilde{\rho}$ be a constant, ρ_0 (effectively choosing a different equation of state), in which case the anelastic equations become identical to the Boussinesq equations, albeit with the buoyancy interpreted in terms of potential temperature in the former and density in the latter. Because of their similarity, the Boussinesq and anelastic approximations are sometimes referred to as the strong and weak Boussinesq approximations, respectively.

2.5.4 Thermodynamic equation

The thermodynamic equation for an ideal gas may be written

$$\frac{\mathrm{D}\ln\theta}{\mathrm{D}t} = \frac{\dot{Q}}{Tc_p}. \tag{2.132}$$

In the anelastic equations, $\theta = \widetilde{\theta} + \delta\theta$, where $\widetilde{\theta}$ is constant, and the thermodynamic equation is

$$\frac{\mathrm{D}\delta s}{\mathrm{D}t} = \frac{\widetilde{\theta}}{Tc_p}\dot{Q}. \tag{2.133}$$

Summarizing, the complete set of anelastic equations, with rotation but with no dissipation or diabatic terms, is

$$\boxed{\begin{gathered} \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega}\times\boldsymbol{v} = \mathbf{k}b_a - \nabla\phi \\ \frac{\mathrm{D}b_a}{\mathrm{D}t} = 0 \\ \nabla\cdot(\widetilde{\rho}\boldsymbol{v}) = 0 \end{gathered}}, \tag{2.134a,b,c}$$

where $b_a = g\delta s = g\delta\theta/\widetilde{\theta}$. The main difference between the anelastic and Boussinesq sets of equations is in the mass continuity equation, and when $\widetilde{\rho} = \rho_0 =$ constant the two equation sets are formally identical. However, whereas the Boussinesq approximation is a very good one for ocean dynamics, the anelastic approximation is much less so for large-scale atmosphere flow: the constancy of the reference potential temperature state is not a particularly good approximation, and the deviations in density from its reference profile are not especially small, leading to inaccuracies in the momentum equation. Nevertheless, the anelastic equations have been used very productively in limited area 'large-eddy simulations' where one does not wish to make the hydrostatic approximation but where sound waves are unimportant.[7] The equations also provide a good jumping-off point for theoretical studies and for the still simpler models of chapter 5.

2.5.5 * Energetics of the anelastic equations

Conservation of energy follows in much the same way as for the Boussinesq equations, except that $\widetilde{\rho}$ enters. Take the dot product of (2.134a) with $\widetilde{\rho}\boldsymbol{v}$ to obtain

$$\widetilde{\rho}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\boldsymbol{v}^2\right) = -\nabla\cdot(\phi\widetilde{\rho}\boldsymbol{v}) + b_a\widetilde{\rho}w. \tag{2.135}$$

Now, define a potential $\Phi(z)$ such that $\nabla\Phi = -\mathbf{k}$, and so

$$\widetilde{\rho}\frac{\mathrm{D}\Phi}{\mathrm{D}t} = -w\widetilde{\rho}. \tag{2.136}$$

Combining this with the thermodynamic equation (2.134b) gives

$$\widetilde{\rho}\frac{\mathrm{D}(b_a\Phi)}{\mathrm{D}t} = -wb_a\widetilde{\rho}. \tag{2.137}$$

Adding this to (2.135) gives

$$\tilde{\rho}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}\boldsymbol{v}^2 + b_a\Phi\right) = -\nabla\cdot(\phi\tilde{\rho}\boldsymbol{v}), \tag{2.138}$$

or, expanding the material derivative,

$$\frac{\partial}{\partial t}\left[\tilde{\rho}\left(\frac{1}{2}\boldsymbol{v}^2 + b_a\Phi\right)\right] + \nabla\cdot\left[\tilde{\rho}\boldsymbol{v}\left(\frac{1}{2}\boldsymbol{v}^2 + b_a\Phi + \phi\right)\right] = 0. \tag{2.139}$$

This equation has the form

$$\frac{\partial E}{\partial t} + \nabla\cdot\left[\boldsymbol{v}(E + \tilde{\rho}\phi)\right] = 0, \tag{2.140}$$

where $E = \tilde{\rho}(\boldsymbol{v}^2/2 + b_a\Phi)$ is the energy density of the flow. This is a consistent energetic equation for the system, and when integrated over a closed domain the total energy is evidently conserved. The total energy density comprises the kinetic energy and a term $\tilde{\rho}b_a\Phi$, which is analogous to the potential energy of a simple Boussinesq system. However, it is not exactly equal to potential energy because b_a is the buoyancy based on potential temperature, not density; rather, the term combines contributions from both the internal energy and the potential energy into an enthalpy-like quantity.

2.6 CHANGING VERTICAL COORDINATE

Although using z as a vertical coordinate is a natural choice given our Cartesian worldview, it is not the only option, nor is it always the most useful one. Any variable that has a one-to-one correspondence with z in the vertical, so any variable that varies monotonically with z, could be used; pressure and, perhaps surprisingly, entropy, are common choices. In the atmosphere pressure almost always falls monotonically with height, and using it instead of z provides a useful simplification of the mass conservation and geostrophic relations, as well as a more direct connection with observations, which are often taken at fixed values of pressure. (In the ocean pressure coordinates are essentially almost the same as height coordinates, because density is almost constant.) Entropy seems an exotic vertical coordinate, but it is very useful in adiabatic flow and we consider it in chapter 3.

2.6.1 General relations

First consider a general vertical coordinate, ξ. Any variable Ψ that is a function of the coordinates (x, y, z, t) may be expressed instead in terms of (x, y, ξ, t) by considering z to be function of the independent variables (x, y, ξ, t); that is, we let $\Psi(x, y, \xi, t) = \Psi(x, y, z(x, y, \xi, t), t)$. Derivatives with respect to z and ξ are related by

$$\frac{\partial \Psi}{\partial \xi} = \frac{\partial \Psi}{\partial z}\frac{\partial z}{\partial \xi} \qquad \text{and} \qquad \frac{\partial \Psi}{\partial z} = \frac{\partial \Psi}{\partial \xi}\frac{\partial \xi}{\partial z}. \tag{2.141a,b}$$

Horizontal derivatives in the two coordinate systems are related by the chain rule,

$$\left(\frac{\partial \Psi}{\partial x}\right)_\xi = \left(\frac{\partial \Psi}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_\xi \frac{\partial \Psi}{\partial z}, \tag{2.142}$$

and similarly for time.

The material derivative in ξ coordinates may be derived by transforming the original expression in z coordinates using the chain rule, but because (x, y, t, ξ) are independent coordinates, and noting that the 'vertical velocity' in ξ coordinates is just $\dot{\xi}$ (i.e., $D\xi/Dt$, just as the vertical velocity in z coordinates is $w = Dz/Dt$), we can write down

$$\frac{D\Psi}{Dt} = \left.\frac{\partial \Psi}{\partial t}\right|_{x,y,\xi} + \boldsymbol{u} \cdot \nabla_\xi \Psi + \dot{\xi}\frac{\partial \Psi}{\partial \xi}, \tag{2.143}$$

where ∇_ξ is the gradient operator at constant ξ. The operator D/Dt is physically the same in z or ξ coordinates because it is the total derivative of some property of a fluid parcel, and this is independent of the coordinate system. However, the individual terms within it will differ between coordinate systems.

2.6.2 Pressure coordinates

Let us now transform the ideal gas primitive equations from height coordinates to pressure coordinates, (x, y, p, t). In z coordinates the equations are

$$\frac{D\boldsymbol{u}}{Dt} + \boldsymbol{f} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p, \qquad \frac{\partial p}{\partial z} = -\rho g, \tag{2.144a}$$

$$\frac{D\theta}{Dt} = 0, \qquad \frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{v} = 0, \tag{2.144b}$$

where $p = \rho R T$ and $\theta = T\left(p_R/p\right)^{R/c_p}$, and p_R is the reference pressure. These are respectively the horizontal momentum, hydrostatic, thermodynamic and mass continuity equations. The analogue of the vertical velocity is $\omega \equiv Dp/Dt$, and the advective derivative itself is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla_p + \omega \frac{\partial}{\partial p}. \tag{2.145}$$

To obtain an expression for the pressure force, now let $\xi = p$ in (2.142) and apply the relationship to p itself to give

$$0 = \left(\frac{\partial p}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_p \frac{\partial p}{\partial z}, \tag{2.146}$$

which, using the hydrostatic relationship, gives

$$\left(\frac{\partial p}{\partial x}\right)_z = \rho \left(\frac{\partial \Phi}{\partial x}\right)_p, \tag{2.147}$$

where $\Phi = gz$ is the *geopotential*. Thus, the horizontal pressure force in the momentum equations is

$$\frac{1}{\rho}\nabla_z p = \nabla_p \Phi, \tag{2.148}$$

where the subscripts on the gradient operator indicate that the horizontal derivatives are taken at constant z or constant p. Also, from (2.144a), the hydrostatic equation is just

$$\frac{\partial \Phi}{\partial p} = -\alpha. \tag{2.149}$$

The mass conservation equation simplifies attractively in pressure coordinates, if the hydrostatic approximation is used. Recall that the mass conservation equation can be derived from the material form

$$\frac{\mathrm{D}}{\mathrm{D}t}(\rho\,\delta V) = 0, \tag{2.150}$$

where $\delta V = \delta x\,\delta y\,\delta z$ is a volume element. But by the hydrostatic relationship $\rho \delta z = (1/g)\delta p$ and thus

$$\frac{\mathrm{D}}{\mathrm{D}t}(\delta x\,\delta y\,\delta p) = 0. \tag{2.151}$$

This is completely analogous to the expression for the material conservation of volume in an incompressible fluid, (1.15). Thus, without further ado, we write the mass conservation in pressure coordinates as

$$\nabla_p \cdot \boldsymbol{u} + \frac{\partial \omega}{\partial p} = 0, \tag{2.152}$$

where the horizontal derivative is taken at constant pressure. The primitive equations in pressure coordinates, equivalent to (2.144) in height coordinates, are thus:

$$\boxed{\begin{array}{cc} \dfrac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\nabla_p \Phi, & \dfrac{\partial \Phi}{\partial p} = -\alpha \\ \dfrac{\mathrm{D}\theta}{\mathrm{D}t} = 0, & \nabla_p \cdot \boldsymbol{u} + \dfrac{\partial \omega}{\partial p} = 0 \end{array}}, \tag{2.153}$$

where $\mathrm{D}/\mathrm{D}t$ is given by (2.145). The equation set is completed with the addition of the ideal gas equation and the definition of potential temperature. These equations are isomorphic to the hydrostatic general Boussinesq equations (see the shaded box on page 72) with $z \leftrightarrow -p$, $w \leftrightarrow -\omega$, $\phi \leftrightarrow \Phi$, $b \leftrightarrow \alpha$, and an equation of state $b = b(\theta, z) \leftrightarrow \alpha = \alpha(\theta, p)$. In an ideal gas, for example, $\alpha = (\theta R/p_R)(p_R/p)^{1/\gamma}$.

The main practical difficulty with the pressure-coordinate equations is the lower boundary condition. Using

$$w \equiv \frac{\mathrm{D}z}{\mathrm{D}t} = \frac{\partial z}{\partial t} + \boldsymbol{u} \cdot \nabla_p z + \omega \frac{\partial z}{\partial p}, \tag{2.154}$$

and (2.149), the boundary condition of $w = 0$ at $z = z_s$ becomes

$$\frac{\partial \Phi}{\partial t} + \boldsymbol{u} \cdot \nabla_p \Phi - \alpha \omega = 0 \tag{2.155}$$

at $p(x, y, z_s, t)$. In theoretical studies, it is common to assume that the lower boundary is in fact a constant pressure surface and simply assume that $\omega = 0$, or sometimes the condition $\omega = -\alpha^{-1}\partial\Phi/\partial t$ is used. For realistic studies (with general circulation models, say) the fact that the level $z = 0$ is not a coordinate surface must be properly accounted for. For this reason, and especially if the lower boundary is uneven because of the presence of topography, so-called *sigma coordinates* are sometimes used, in which the vertical coordinate is chosen so that the lower boundary is a coordinate surface. Sigma coordinates may use height itself as a vertical measure (typical in oceanic applications) or use pressure (typical in atmospheric applications). In the latter case the vertical coordinate is $\sigma = p/p_s$ where $p_s(x, y, t)$ is the surface pressure. The difficulty of applying (2.155) is replaced by a prognostic equation for the surface pressure, derived from the mass conservation equation (problem 2.24).

2.6.3 Log-pressure coordinates

A variant of pressure coordinates is *log-pressure* coordinates, in which the vertical coordinate is $Z = -H\ln(p/p_R)$ where p_R is a reference pressure (say 1000 mb) and H is a constant (for example the scale height RT_s/g) so that Z has units of length. (Uppercase letters are conventionally used for some variables in log-pressure coordinates, and these are not to be confused with scaling parameters.) The 'vertical velocity' for the system is now

$$W \equiv \frac{\mathrm{D}Z}{\mathrm{D}t}, \tag{2.156}$$

and the advective derivative is

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla_p + W\frac{\partial}{\partial Z}. \tag{2.157}$$

It is straightforward to show (problem 2.25) that the primitive equations of motion in these coordinates are:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f}\times\boldsymbol{u} = -\nabla_Z\Phi, \qquad \frac{\partial\Phi}{\partial Z} = \frac{RT}{H}, \tag{2.158a}$$

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = 0, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial W}{\partial Z} - \frac{W}{H} = 0. \tag{2.158b}$$

The last equation may be written $\nabla_Z\cdot\boldsymbol{u} + \rho_R^{-1}\partial(\rho_R W)/\partial z = 0$, where $\rho_R = \exp(-z/H)$, so giving a form similar to the mass conservation equation in the anelastic equations. Note that integrating the hydrostatic equation between two pressure levels gives, with $\Phi = gz$,

$$z(p_2) - z(p_1) = \frac{R}{g}\int_{p_1}^{p_2} T\,\mathrm{d}\ln p. \tag{2.159}$$

Thus, the thickness of the layer is proportional to the average temperature of the layer.

2.7 SCALING FOR HYDROSTATIC BALANCE

In this section we consider one of the most fundamental balances in geophysical fluid dynamics, hydrostatic balance, and in the next section we consider another fundamental balance, geostrophic balance. The corresponding states, hydrostasy and geostrophy, are not exactly realized, but their approximate satisfaction has profound consequences on the behaviour of the atmosphere and ocean. We first encountered hydrostatic balance in section 1.3.4; we now look in more detail at the conditions required for it to hold.

2.7.1 Preliminaries

Consider the relative sizes of terms in (2.77c):

$$\frac{W}{T} + \frac{UW}{L} + \frac{W^2}{H} + \Omega U \sim \left|\frac{1}{\rho}\frac{\partial p}{\partial z}\right| + g. \tag{2.160}$$

For most large-scale motion in the atmosphere and ocean the terms on the right-hand side are orders of magnitude larger than those on the left, and therefore must be approximately equal. Explicitly, suppose $W \sim 1\,\mathrm{cm\,s^{-1}}$, $L \sim 10^5\,\mathrm{m}$, $H \sim 10^3\,\mathrm{m}$, $U \sim 10\,\mathrm{m\,s^{-1}}$, $T = L/U$. Then by substituting into (2.160) it seems that the pressure term is the only one which could balance the gravitational term, and we are led to approximate (2.77c) by,

$$\frac{\partial p}{\partial z} = -\rho g. \tag{2.161}$$

This equation, which is a vertical momentum equation, is known as *hydrostatic balance*.

However, (2.161) is not always a useful equation! Let us suppose that the density is a constant, ρ_0 . We can then write the pressure as

$$p(x, y, z, t) = p_0(z) + p'(x, y, z, t), \tag{2.162}$$

where

$$\frac{\partial p_0}{\partial z} \equiv -\rho_0 g. \tag{2.163}$$

That is, p_0 and ρ_0 are in hydrostatic balance. The inviscid vertical momentum equation becomes, without approximation,

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z}. \tag{2.164}$$

Thus, for constant density fluids, the gravitational term has no dynamical effect: there is no buoyancy force, and the pressure term in the horizontal momentum equations can be replaced by p'. Hydrostatic balance, and in particular (2.163), is certainly not an appropriate vertical momentum equation in this case. If the fluid is stratified, we should therefore subtract off the hydrostatic pressure associated with the mean density before we can determine whether hydrostasy is a useful *dynamical* approximation, accurate enough to determine the horizontal pressure gradients. This is automatic in the Boussinesq equations, where the vertical momentum equation is

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial \phi}{\partial z} + b, \tag{2.165}$$

and the hydrostatic balance of the basic state is already subtracted out. In the more general equation,

$$\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{2.166}$$

we need to compare the advective term on the left-hand side with the pressure variations arising from horizontal flow in order to determine whether hydrostasy is an appropriate vertical momentum equation. Nevertheless, if we only need to determine the pressure for use in an equation of state then we simply need to compare the sizes of the dynamical terms in (2.77c) with g itself, in order to determine whether a hydrostatic approximation will suffice.

2.7.2 Scaling and the aspect ratio

In a Boussinesq fluid we write the horizontal and vertical momentum equations as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{f} \times \boldsymbol{u} = -\nabla \phi, \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial \phi}{\partial z} - b. \tag{2.167a,b}$$

With $\boldsymbol{f} = 0$, (2.167a) implies the scaling

$$\phi \sim U^2. \tag{2.168}$$

If we use mass conservation, $\nabla_z \cdot \boldsymbol{u} + \partial w/\partial z = 0$, to scale vertical velocity then

$$w \sim W = \frac{H}{L} U = \alpha U, \tag{2.169}$$

where $\alpha \equiv H/L$ is the aspect ratio. The advective terms in the vertical momentum equation all scale as

$$\frac{\mathrm{D}w}{\mathrm{D}t} \sim \frac{UW}{L} = \frac{U^2 H}{L^2}. \tag{2.170}$$

Using (2.168) and (2.170) the ratio of the advective term to the pressure gradient term in the vertical momentum equations then scales as

$$\frac{|\mathrm{D}w/\mathrm{D}t|}{|\partial\phi/\partial z|} \sim \frac{U^2 H/L^2}{U^2/H} \sim \left(\frac{H}{L}\right)^2. \tag{2.171}$$

Thus, the condition for hydrostasy, that $|\mathrm{D}w/\mathrm{D}t|/|\partial\phi/\partial z| \ll 1$, is:

$$\boxed{\alpha^2 \equiv \left(\frac{H}{L}\right)^2 \ll 1}. \tag{2.172}$$

The advective term in the vertical momentum may then be neglected. Thus, hydrostatic balance is a *small aspect ratio approximation.*

We can obtain the same result more formally by non-dimensionalizing the momentum equations. Using uppercase symbols to denote scaling values we write

$$\begin{gathered}
(x, y) = L(\hat{x}, \hat{y}), \qquad z = H\hat{z}, \qquad \boldsymbol{u} = U\hat{\boldsymbol{u}}, \qquad w = W\widehat{w} = \frac{HU}{L}\widehat{w}, \\
t = T\hat{t} = \frac{L}{U}\hat{t}, \qquad \phi = \Phi\hat{\phi} = U^2\hat{\phi}, \qquad b = B\hat{b} = \frac{U^2}{H}\hat{b},
\end{gathered} \tag{2.173}$$

where the hatted variables are non-dimensional and the scaling for w is suggested by the mass conservation equation, $\nabla_z \cdot \boldsymbol{u} + \partial w/\partial z = 0$. Substituting (2.173) into (2.167) (with $\boldsymbol{f} = 0$) gives us the non-dimensional equations

$$\frac{\mathrm{D}\hat{\boldsymbol{u}}}{\mathrm{D}\hat{t}} = -\nabla\hat{\phi}, \qquad \alpha^2 \frac{\mathrm{D}\widehat{w}}{\mathrm{D}\hat{t}} = -\frac{\partial\hat{\phi}}{\partial\hat{z}} - \hat{b}, \tag{2.174a,b}$$

where $\mathrm{D}/\mathrm{D}\hat{t} = \partial/\partial\hat{t} + \hat{u}\partial/\partial\hat{x} + \hat{v}\partial/\partial\hat{y} + \widehat{w}\partial/\partial\hat{z}$ and we use the convention that when ∇ operates on non-dimensional quantities the operator itself is non-dimensional. From (2.174b) it is clear that hydrostatic balance pertains when $\alpha^2 \ll 1$.

2.7.3 * Effects of stratification on hydrostatic balance

To include the effects of stratification we need to involve the thermodynamic equation, so let us first write down the complete set of non-rotating dimensional equations:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla_z\phi, \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial\phi}{\partial z} + b', \tag{2.175a,b}$$

$$\frac{\mathrm{D}b'}{\mathrm{D}t} + wN^2 = 0, \qquad \nabla\cdot\boldsymbol{v} = 0. \tag{2.176a,b}$$

We have written, without approximation, $b = b'(x,y,z,t) + \widetilde{b}(z)$, with $N^2 = \mathrm{d}\widetilde{b}/\mathrm{d}z$; this separation is useful because the horizontal and vertical buoyancy variations may scale in different ways, and often N^2 may be regarded as given. (We have also redefined ϕ by subtracting off a static component in hydrostatic balance with $\widetilde{b}$.) We non-dimensionalize (2.176) by first writing

$$\begin{gathered}(x,y) = L(\hat{x},\hat{y}), \qquad z = H\hat{z}, \qquad \boldsymbol{u} = U\hat{\boldsymbol{u}}, \quad w = W\widehat{w} = \epsilon\frac{HU}{L}\widehat{w},\\ t = T\hat{t} = \frac{L}{U}\hat{t}, \qquad \phi = U^2\hat{\phi}, \qquad b' = \Delta b\hat{b} = \frac{U^2}{H}\hat{b}', \qquad N^2 = \overline{N}^2\hat{N}^2,\end{gathered} \tag{2.177}$$

where ϵ is, for the moment, undetermined, $\overline{N}$ is a representative, constant, value of the buoyancy frequency and Δb scales only the horizontal buoyancy variations. Substituting (2.177) into (2.175) and (2.176) gives

$$\frac{\mathrm{D}\hat{\boldsymbol{u}}}{\mathrm{D}\hat{t}} = -\nabla_z\hat{\phi}, \qquad \epsilon\alpha^2\frac{\mathrm{D}\widehat{w}}{\mathrm{D}t} = -\frac{\partial\hat{\phi}}{\partial\hat{z}} + \hat{b}' \tag{2.178a,b}$$

$$\frac{U^2}{\overline{N}^2H^2}\frac{\mathrm{D}\hat{b}'}{\mathrm{D}\hat{t}} + \epsilon\widehat{w}\hat{N}^2 = 0, \qquad \nabla\cdot\hat{\boldsymbol{u}} + \epsilon\frac{\partial\widehat{w}}{\partial\hat{z}} = 0. \tag{2.179a,b}$$

where now $\mathrm{D}/\mathrm{D}\hat{t} = \partial/\partial\hat{t} + \hat{\boldsymbol{u}}\cdot\nabla_z + \epsilon\partial/\partial\hat{z}$. To obtain a non-trivial balance in (2.179a) we choose $\epsilon = U^2/(\overline{N}^2H^2) \equiv Fr^2$, where Fr is the *Froude number,* a measure of the stratification of the flow. The vertical velocity then scales as

$$W = \frac{Fr\,UH}{L} \tag{2.180}$$

and if the flow is highly stratified the vertical velocity will be even smaller than a pure aspect ratio scaling might suggest. (There must, therefore, be some cancellation in horizontal divergence in the mass continuity equation; that is, $|\nabla_z\cdot\boldsymbol{u}| \ll U/L$.) With this choice of ϵ the non-dimensional Boussinesq equations may be written:

$$\frac{\mathrm{D}\hat{\boldsymbol{u}}}{\mathrm{D}\hat{t}} = -\nabla_z\hat{\phi}, \qquad Fr^2\alpha^2\frac{\mathrm{D}\widehat{w}}{\mathrm{D}\hat{t}} = -\frac{\partial\hat{\phi}}{\partial\hat{z}} + \hat{b}' \tag{2.181a,b}$$

$$\frac{\mathrm{D}\hat{b}'}{\mathrm{D}\hat{t}} + \widehat{w}\hat{N}^2 = 0, \qquad \nabla\cdot\hat{\boldsymbol{u}} + Fr^2\frac{\partial\widehat{w}}{\partial\hat{z}} = 0. \tag{2.182a,b}$$

The non-dimensional parameters in the system are the aspect ratio and the Froude number (in addition to $\hat{N}$, but by construction this is just an order one function of z). From (2.181b) condition for hydrostatic balance to hold is evidently that

$$\boxed{Fr^2\alpha^2 \ll 1}, \tag{2.183}$$

so generalizing the aspect ratio condition (2.172) to a stratified fluid. Because Fr is a measure of stratification, (2.183) formalizes our intuitive expectation that the more stratified a fluid the more vertical motion is suppressed and therefore the more likely hydrostatic balance is to hold. Also note that (2.183) is equivalent to $U^2/(L^2\overline{N}^2) \ll 1$.

Suppose we solve the hydrostatic equations; that is, we omit the advective derivative in the vertical momentum equation, and by numerical integration we obtain $\boldsymbol{u}$, w and b. This flow is the solution of the non-hydrostatic equations in the small aspect ratio limit. The solution never violates the scaling assumptions, even if w seems large, because we can always rescale the variables in order that condition (2.183) is satisfied.

Why bother with any of this scaling? Why not just say that hydrostatic balance holds when $|Dw/Dt| \ll |\partial\phi/\partial z|$? One reason is that we do not have a good idea of the value of w from direct measurements, and it may change significantly in different oceanic and atmospheric parameter regimes. On the other hand the Froude number and the aspect ratio are familiar non-dimensional parameters with a wide applicability in other contexts, and which we can control in a laboratory setting or estimate in the ocean or atmosphere. Still, in scaling theory it is common that ascertaining which parameters are to be regarded as given and which should be derived is a choice, rather than being set a priori.

2.7.4 Hydrostasy in the ocean and atmosphere

Is the hydrostatic approximation in fact a good one in the ocean and atmosphere?

In the ocean

For the large-scale ocean circulation, let $N \sim 10^{-2}\,\mathrm{s}^{-1}$, $U \sim 0.1\,\mathrm{m\,s}^{-1}$ and $H \sim 1\,\mathrm{km}$. Then $Fr = U/(NH) \sim 10^{-2} \ll 1$. Thus, $Fr^2\alpha^2 \ll 1$ even for unit aspect-ratio motion. In fact, for larger scale flow the aspect ratio is also small; for basin-scale flow $L \sim 10^6\,\mathrm{m}$ and $Fr^2\alpha^2 \sim 0.01^2 \times 0.001^2 = 10^{-10}$ and hydrostatic balance is an extremely good approximation.

For intense convection, for example in the Labrador Sea, the hydrostatic approximation may be less appropriate, because the intense descending plumes may have an aspect ratio (H/L) of one or greater and the stratification is very weak. The hydrostatic condition then often becomes the requirement that the Froude number is small. Representative orders of magnitude are $U \sim W \sim 0.1\,\mathrm{m\,s}^{-1}$, $H \sim 1\,\mathrm{km}$ and $N \sim 10^{-3}\,\mathrm{s}^{-1}$ to $10^{-4}\,\mathrm{s}^{-1}$. For these values Fr ranges between 0.1 and 1, and at the upper end of this range hydrostatic balance is violated.

In the atmosphere

Over much of the troposphere $N \sim 10^{-2}\,\mathrm{s}^{-1}$ so that with $U = 10\,\mathrm{m\,s}^{-1}$ and $H = 1\,\mathrm{km}$ we find $Fr \sim 1$. Hydrostasy is then maintained because the aspect ratio H/L is much less than unity. For larger scale synoptic activity a larger vertical scale is appropriate, and with $H = 10\,\mathrm{km}$ both the Froude number and the aspect ratio are much smaller than one; indeed with $L = 1000\,\mathrm{km}$ we find $Fr^2\alpha^2 \sim 0.1^2 \times 0.1^2 = 10^{-4}$ and the flow is hydrostatic to a very good approximation indeed. However, for smaller scale atmospheric motions associated with fronts and, especially, convection, there can be little expectation that hydrostatic balance will be a good approximation.

For large-scale flows in both atmosphere and ocean, the conceptual simplifications afforded by the hydrostatic approximation can hardly be overemphasized.

Variable	Scaling symbol	Meaning	Atmos. value	Ocean value
(x, y)	L	Horizontal length scale	10^6 m	10^5 m
t	T	Time scale	1 day (10^5 s)	10 days (10^6 s)
(u, v)	U	Horizontal velocity	$10\,\mathrm{m\,s^{-1}}$	$0.1\,\mathrm{m\,s^{-1}}$
	Ro	Rossby number, U/fL	0.1	0.01

Table 2.1 Scales of large-scale flow in atmosphere and ocean. The choices given are representative of large-scale eddying motion in both systems.

2.8 GEOSTROPHIC AND THERMAL WIND BALANCE

We now consider the dominant dynamical balance in the horizontal components of the momentum equation. In the horizontal plane (meaning along geopotential surfaces) we find that the Coriolis term is much larger than the advective terms and the dominant balance is between it and the horizontal pressure force. This balance is called *geostrophic balance,* and it occurs when the Rossby number is small, as we now investigate.

2.8.1 The Rossby number

The *Rossby number* characterizes the importance of rotation in a fluid.[8] It is, essentially, the ratio of the magnitude of the relative acceleration to the Coriolis acceleration, and it is of fundamental importance in geophysical fluid dynamics. It arises from a simple scaling of the horizontal momentum equation, namely

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} + \boldsymbol{f}\times\boldsymbol{u} = -\frac{1}{\rho}\nabla_z p, \tag{2.184a}$$

$$U^2/L \qquad fU \tag{2.184b}$$

where U is the approximate magnitude of the horizontal velocity and L is a typical length scale over which that velocity varies. (We assume that $W/H \lesssim U/L$, so that vertical advection does not dominate the advection.) The ratio of the sizes of the advective and Coriolis terms is defined to be the Rossby number,

$$\boxed{Ro \equiv \frac{U}{fL}}\,. \tag{2.185}$$

If the Rossby number is small then rotation effects are important, and as the values in Table 2.1 indicate this is the case for large-scale flow in both ocean and atmosphere.

Another intuitive way to think about the Rossby number is in terms of time scales. The Rossby number based on a time scale is

$$Ro_T \equiv \frac{1}{fT}, \tag{2.186}$$

where T is a time scale associated with the dynamics at hand. If the time scale is an advective one, meaning that $T \sim L/U$, then this definition is equivalent to (2.185). Now, $f = 2\Omega \sin\vartheta$, where Ω is the angular velocity of the rotating frame and equal to $2\pi/T_p$ where T_p is the period of rotation (24 hours). Thus,

$$Ro_T = \frac{T_p}{4\pi T \sin\vartheta} = \frac{T_i}{T}, \tag{2.187}$$

where $T_i = 1/f$ is the 'inertial time scale', about three hours in mid-latitudes. Thus, for phenomena with time scales much longer than this, such as the motion of the Gulf Stream or a mid-latitude atmospheric weather system, the effects of the Earth's rotation can be expected to be important, whereas a short-lived phenomena, such as a cumulus cloud or tornado, may be oblivious to such rotation. The expressions (2.185) and (2.186) are, of course, just approximate measures of the importance of rotation.

2.8.2 Geostrophic balance

If the Rossby number is sufficiently small in (2.184a) then the rotation term will dominate the nonlinear advection term, and if the time period of the motion scales advectively then the rotation term also dominates the local time derivative. The only term that can then balance the rotation term is the pressure term, and therefore we must have

$$\boldsymbol{f} \times \boldsymbol{u} \approx -\frac{1}{\rho}\nabla_z p, \tag{2.188}$$

or, in Cartesian component form

$$fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad fv \approx \frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{2.189}$$

This balance is known as *geostrophic balance,* and its consequences are profound, giving geophysical fluid dynamics a special place in the broader field of fluid dynamics. We *define* the geostrophic velocity by

$$\boxed{fu_g \equiv -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad fv_g \equiv \frac{1}{\rho}\frac{\partial p}{\partial x}}, \tag{2.190}$$

and for low Rossby number flow $u \approx u_g$ and $v \approx v_g$. In spherical coordinates the geostrophic velocity is

$$fu_g = -\frac{1}{\rho a}\frac{\partial p}{\partial \vartheta}, \qquad fv_g = \frac{1}{a\rho\cos\vartheta}\frac{\partial p}{\partial \lambda}, \tag{2.191}$$

where $f = 2\Omega \sin\vartheta$. Geostrophic balance has a number of immediate ramifications:

⋆ Geostrophic flow is parallel to lines of constant pressure (isobars). If $f > 0$ the flow is anticlockwise round a region of low pressure and clockwise around a region of high pressure (see Fig. 2.5).

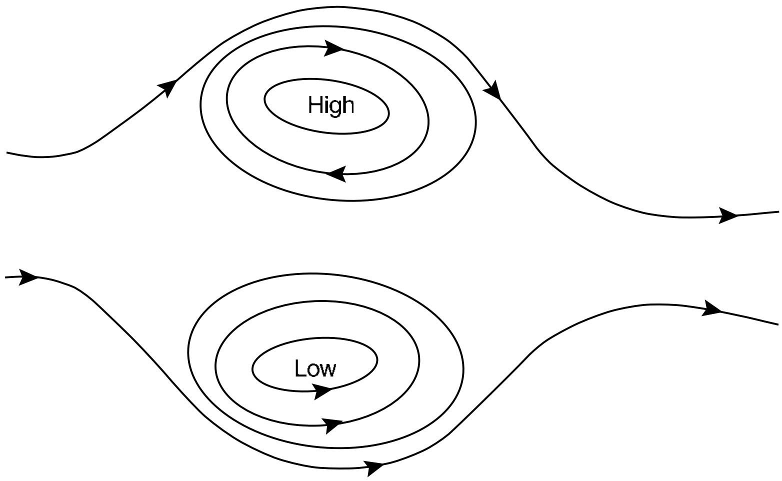

Fig. 2.5 Schematic of geostrophic flow with a positive value of the Coriolis parameter f. Flow is parallel to the lines of constant pressure (isobars). Cyclonic flow is anticlockwise around a low pressure region and anticyclonic flow is clockwise around a high. If f were negative, as in the Southern Hemisphere, (anti)cyclonic flow would be (anti)clockwise.

⋆ If the Coriolis force is constant and if the density does not vary in the horizontal the geostrophic flow is horizontally non-divergent and

$$\boxed{\nabla_z \cdot \boldsymbol{u}_g = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0}. \tag{2.192}$$

We may define the *geostrophic streamfunction,* ψ, by

$$\psi \equiv \frac{p}{f_0 \rho_0}, \tag{2.193}$$

whence

$$u_g = -\frac{\partial \psi}{\partial y}, \qquad v_g = \frac{\partial \psi}{\partial x}. \tag{2.194}$$

The vertical component of vorticity, ζ, is then given by

$$\zeta = \mathbf{k} \cdot \nabla \times \boldsymbol{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla_z^2 \psi. \tag{2.195}$$

⋆ If the Coriolis parameter is not constant, then cross-differentiating (2.190) gives, for constant density geostrophic flow,

$$v_g \frac{\partial f}{\partial y} + f \nabla_z \cdot \boldsymbol{u}_g = 0, \tag{2.196}$$

which implies, using mass continuity,

$$\beta v_g = f \frac{\partial w}{\partial z}. \tag{2.197}$$

where $\beta \equiv \partial f/\partial y = 2\Omega\cos\vartheta/a$. This geostrophic vorticity balance is sometimes known as 'Sverdrup balance', although the latter expression is better restricted to the case when the vertical velocity results from external agents, and specifically a wind stress, as considered in chapter 14.

2.8.3 Taylor–Proudman effect

If $\beta = 0$, then (2.197) implies that the vertical velocity is not a function of height. In fact, in that case none of the components of velocity vary with height if density is also constant. To show this, in the limit of zero Rossby number we first write the three-dimensional momentum equation as

$$\boldsymbol{f}_0 \times \boldsymbol{v} = -\nabla\phi - \nabla\chi, \tag{2.198}$$

where $\boldsymbol{f}_0 = 2\boldsymbol{\Omega} = 2\Omega\mathbf{k}$, $\phi = p/\rho_0$, and $\nabla\chi$ represents other potential forces. If $\chi = gz$ then the vertical component of this equation represents hydrostatic balance, and the horizontal components represent geostrophic balance. On taking the curl of this equation, the terms on the right-hand side vanish and the left-hand side becomes

$$(\boldsymbol{f}_0 \cdot \nabla)\boldsymbol{v} - \boldsymbol{f}_0 \nabla \cdot \boldsymbol{v} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{f}_0 + \boldsymbol{v}\nabla \cdot \boldsymbol{f}_0 = 0. \tag{2.199}$$

But $\nabla \cdot \boldsymbol{v} = 0$ by mass conservation, and because $\boldsymbol{f}_0$ is constant both $\nabla \cdot \boldsymbol{f}_0$ and $(\boldsymbol{v} \cdot \nabla)\boldsymbol{f}_0$ vanish. Thus

$$(\boldsymbol{f}_0 \cdot \nabla)\boldsymbol{v} = 0, \tag{2.200}$$

which, since $\boldsymbol{f}_0 = f_0\mathbf{k}$, implies $f_0 \partial \boldsymbol{v}/\partial z = 0$, and in particular we have

$$\frac{\partial u}{\partial z} = 0, \quad \frac{\partial v}{\partial z} = 0, \quad \frac{\partial w}{\partial z} = 0. \tag{2.201}$$

A different presentation of this argument proceeds as follows. If the flow is exactly in geostrophic and hydrostatic balance then

$$v = \frac{1}{f_0}\frac{\partial \phi}{\partial x}, \qquad u = -\frac{1}{f_0}\frac{\partial \phi}{\partial y}, \qquad \frac{\partial \phi}{\partial z} = -g. \tag{2.202a,b,c}$$

Differentiating (2.202a,b) with respect to z, and using (2.202c) yields

$$\frac{\partial v}{\partial z} = \frac{-1}{f_0}\frac{\partial g}{\partial x} = 0, \quad \frac{\partial u}{\partial z} = \frac{1}{f_0}\frac{\partial g}{\partial y} = 0. \tag{2.203}$$

Noting that the geostrophic velocities are horizontally non-divergent ($\nabla_z \cdot \boldsymbol{u} = 0$), and using mass continuity then gives $\partial w/\partial z = 0$, as before.

If there is a solid horizontal boundary anywhere in the fluid, for example at the surface, then $w = 0$ at that surface and thus $w = 0$ everywhere. Hence the motion occurs in planes that lie perpendicular to the axis of rotation, and the flow is effectively two dimensional. This result is known as the *Taylor–Proudman effect,* namely that for constant density flow in geostrophic and hydrostatic balance the vertical derivatives of the horizontal and the vertical velocities are zero.[9] At zero Rossby number, if the vertical velocity is zero somewhere in the flow, it is zero everywhere in that vertical column; furthermore, the horizontal flow has no vertical shear, and the fluid moves like a slab. The effects of rotation have provided a *stiffening* of the fluid in the vertical.

In neither the atmosphere nor the ocean do we observe precisely such vertically coherent flow, mainly because of the effects of stratification. However, it is typical of geophysical fluid dynamics that the assumptions underlying a derivation are not fully satisfied, yet there are manifestations of it in real flow. Thus, one might have naïvely expected, because $\partial w/\partial z = -\nabla_z \cdot \boldsymbol{u}$, that the scales of the various variables would be related by $W/H \sim U/L$. However, if the flow is rapidly rotating we expect that the horizontal flow will be in near geostrophic balance and therefore nearly divergence free; thus $\nabla_z \cdot \boldsymbol{u} \ll U/L$, and $W \ll HU/L$.

2.8.4 Thermal wind balance

Thermal wind balance arises by combining the geostrophic and hydrostatic approximations, and this is most easily done in the context of the anelastic (or Boussinesq) equations, or in pressure coordinates. For the anelastic equations, geostrophic balance may be written

$$-f v_g = -\frac{\partial \phi}{\partial x} = -\frac{1}{a\cos\vartheta}\frac{\partial \phi}{\partial \lambda}, \qquad f u_g = -\frac{\partial \phi}{\partial y} = -\frac{1}{a}\frac{\partial \phi}{\partial \vartheta}. \tag{2.204a,b}$$

Combining these relations with hydrostatic balance, $\partial\phi/\partial z = b$, gives

$$\boxed{\begin{aligned} -f\frac{\partial v_g}{\partial z} &= -\frac{\partial b}{\partial x} = -\frac{1}{a\cos\lambda}\frac{\partial b}{\partial \lambda} \\ f\frac{\partial u_g}{\partial z} &= -\frac{\partial b}{\partial y} = -\frac{1}{a}\frac{\partial b}{\partial \vartheta} \end{aligned}}\;. \tag{2.205a,b}$$

These equations represent *thermal wind balance,* and the vertical derivative of the geostrophic wind is the 'thermal wind'. Eq. (2.205b) may be written in terms of the zonal angular momentum as

$$\frac{\partial m_g}{\partial z} = -\frac{a}{2\Omega\tan\vartheta}\frac{\partial b}{\partial y}, \tag{2.206}$$

where $m_g = (u_g + \Omega a\cos\vartheta)a\cos\vartheta$. Potentially more accurate than geostrophic balance is the so-called cyclostrophic or gradient-wind balance, which retains a centrifugal term in the momentum equation. Thus, we omit only the material derivative in the meridional momentum equation (2.50b) and obtain

$$2u\Omega\sin\vartheta + \frac{u^2}{a}\tan\vartheta \approx -\frac{\partial \phi}{\partial y} = -\frac{1}{a}\frac{\partial \phi}{\partial \vartheta}. \tag{2.207}$$

For large-scale flow this only differs significantly from geostrophic balance very close to the equator. Combining cyclostrophic and hydrostatic balance gives a modified thermal wind relation, and this takes a simple form when expressed in terms of angular momentum, namely

$$\frac{\partial m^2}{\partial z} \approx -\frac{a^3\cos^3\vartheta}{\sin\vartheta}\frac{\partial b}{\partial y}. \tag{2.208}$$

If the density or buoyancy is constant then there is no shear and (2.205) or (2.208) give the Taylor–Proudman result. But suppose that the temperature falls in the poleward direction. Then thermal wind balance implies that the (eastward) wind will increase with height — just as is observed in the atmosphere! In general, a vertical shear of the horizontal wind is associated with a horizontal temperature gradient, and this is one of the most simple and far-reaching effects in geophysical fluid dynamics. The underlying physical mechanism is illustrated in Fig. 2.6.

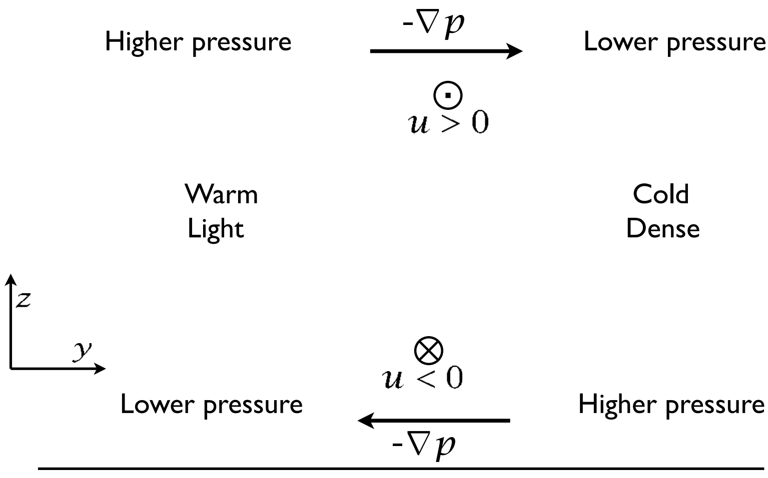

Fig. 2.6 The mechanism of thermal wind. A cold fluid is denser than a warm fluid, so by hydrostasy the vertical pressure gradient is greater where the fluid is cold. Thus, the pressure gradients form as shown, where 'higher' and 'lower' mean relative to the average at that height. The horizontal pressure gradients are balanced by the Coriolis force, producing (for $f > 0$) the horizontal winds shown (⊗ into the paper, and ⊙ out of the paper). Only the wind *shear* is given by the thermal wind.

Pressure coordinates

In pressure coordinates geostrophic balance is just

$$\boldsymbol{f} \times \boldsymbol{u}_g = -\nabla_p \Phi, \tag{2.209}$$

where Φ is the geopotential and ∇_p is the gradient operator taken at constant pressure. If f if constant, it follows from (2.209) that the geostrophic wind is non-divergent on pressure surfaces. Taking the vertical derivative of (2.209) (that is, its derivative with respect to p) and using the hydrostatic equation, $\partial\Phi/\partial p = -\alpha$, gives the thermal wind equation

$$\boldsymbol{f} \times \frac{\partial \boldsymbol{u}_g}{\partial p} = \nabla_p \alpha = \frac{R}{p}\nabla_p T, \tag{2.210}$$

where the last equality follows using the ideal gas equation and because the horizontal derivative is at constant pressure. In component form this is

$$-f\frac{\partial v_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial x}, \qquad f\frac{\partial u_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y}. \tag{2.211}$$

In log-pressure coordinates, with $Z = -H\ln(p/p_R)$, thermal wind is

$$\boldsymbol{f} \times \frac{\partial \boldsymbol{u}_g}{\partial Z} = -\frac{R}{H}\nabla_Z T. \tag{2.212}$$

The physical meaning in all these cases is the same: a horizontal temperature gradient, or a temperature gradient along an isobaric surface, is accompanied by a vertical shear of the horizontal wind.

2.8.5 * Effects of rotation on hydrostatic balance

Because rotation inhibits vertical motion, we might expect it to affect the requirements for hydrostasy. The simplest setting in which to see this is the rotating Boussinesq equations, (2.167). Let us non-dimensionalize these by writing

$$
\begin{gathered}
(x, y) = L(\widehat{x}, \widehat{y}), \qquad z = H\widehat{z}, \qquad \boldsymbol{u} = U\widehat{\boldsymbol{u}}, \qquad t = T\widehat{t} = \frac{U}{L}\widehat{t}, \qquad \boldsymbol{f} = f_0\widehat{\boldsymbol{f}}, \\
w = \frac{\beta H U}{f_0}\widehat{w} = \widehat{\beta}\frac{HU}{L}\widehat{w}, \qquad \phi = \Phi\widehat{\phi} = f_0 U L\widehat{\phi}, \qquad b = B\widehat{b} = \frac{f_0 u L}{H}\widehat{b},
\end{gathered}
\tag{2.213}
$$

where $\widehat{\beta} \equiv \beta L/f_0$. (If $\boldsymbol{f}$ is constant, then $\widehat{\boldsymbol{f}}$ is a unit vector in the vertical direction.) These relations are the same as (2.173), except for the scaling for w, which is suggested by (2.197), and the scaling for ϕ and b', which are suggested by geostrophic and thermal wind balance.

Substituting into (2.167) we obtain the following scaled momentum equations:

$$
\boxed{Ro\,\frac{\mathrm{D}\widehat{\boldsymbol{u}}}{\mathrm{D}\widehat{t}} + \widehat{\boldsymbol{f}} \times \widehat{\boldsymbol{u}} = -\nabla\widehat{\phi}, \qquad Ro\,\widehat{\beta}\alpha^2\frac{\mathrm{D}\widehat{w}}{\mathrm{D}\widehat{t}} = -\frac{\partial\widehat{\phi}}{\partial z} - \widehat{b}}\,. \tag{2.214a,b}
$$

Here, $\mathrm{D}/\mathrm{D}\widehat{t} = \partial/\partial\widehat{t} + \widehat{\boldsymbol{u}}\cdot\nabla_z + \widehat{\beta}\partial/\partial\widehat{z}$ and $Ro = U/(f_0 L)$. There are two notable aspects to these equations. First and most obviously, when $Ro \ll 1$, (2.214a) reduces to geostrophic balance, $\boldsymbol{f}\times\boldsymbol{u} = -\nabla\widehat{\phi}$. Second, the material derivative in (2.214b) is multiplied by three non-dimensional parameters, and we can understand the appearance of each as follows.

(i) The aspect ratio dependence (α^2) arises in the same way as for non-rotating flows — that is, because of the presence of w and z in the vertical momentum equation as opposed to (u, v) and (x, y) in the horizontal equations.

(ii) The Rossby number dependence (*Ro*) arises because in rotating flow the pressure gradient is balanced by the Coriolis force, which is Rossby number larger than the advective terms.

(iii) The factor $\widehat{\beta}$ arises because in rotating flow w is smaller than u by $\widehat{\beta}$ times the aspect ratio.

The factor $Ro\,\widehat{\beta}\alpha^2$ is very small for large-scale flow; the reader is invited to calculate representative values. Evidently, a rapidly rotating fluid is more likely to be in hydrostatic balance than a non-rotating fluid, other conditions being equal. The combined effects of rotation and stratification are, not surprisingly, quite subtle and we leave that topic for chapter 5.

2.9 STATIC INSTABILITY AND THE PARCEL METHOD

In this and the next couple of sections we consider how a fluid might oscillate if it were perturbed away from a resting state. Our focus is on vertical displacements, and the restoring force is gravity, and we will neglect the effects of rotation, and indeed initially we will neglect horizontal motion entirely. Given that, the simplest and most direct way to approach the problem is to consider from first principles the pressure and gravitational forces on a displaced parcel. To this end, consider a fluid initially at rest in a constant gravitational field, and therefore in hydrostatic balance. Suppose that a small parcel of the fluid is adiabatically displaced upwards by the small distance δz, without altering the overall pressure field; that is, the fluid parcel instantly assumes the pressure of its environment. If after the displacement the parcel is lighter than its environment, it will accelerate upwards, because the upward pressure gradient force is now greater than the downward gravity force on the parcel; that is, the parcel is *buoyant* (a manifestation of Archimedes' principle) and the fluid is *statically unstable.* If on the other hand the fluid parcel finds itself heavier than its surroundings, the downward gravitational force will be greater than the upward pressure force and the fluid will sink back towards its original position and an oscillatory motion will develop. Such an equilibrium is *statically stable.* Using such simple 'parcel' arguments we will now develop criteria for the stability of the environmental profile.

2.9.1 A simple special case: a density-conserving fluid

Consider first the simple case of an incompressible fluid in which the density of the displaced parcel is conserved, that is $\mathrm{D}\rho/\mathrm{D}t = 0$ (and refer to Fig. 2.7 setting $\rho_\theta = \rho$). If the environmental profile is $\widetilde{\rho}(z)$ and the density of the parcel is ρ then a parcel displaced from a level z [where its density is $\widetilde{\rho}(z)$] to a level $z + \delta z$ [where the density of the parcel is still $\widetilde{\rho}(z)$] will find that its density then differs from its surroundings by the amount

$$\delta\rho = \rho(z + \delta z) - \widetilde{\rho}(z + \delta z) = \widetilde{\rho}(z) - \widetilde{\rho}(z + \delta z) = -\frac{\partial \widetilde{\rho}}{\partial z}\delta z. \tag{2.215}$$

The parcel will be heavier than its surroundings, and therefore the parcel displacement will be stable, if $\partial\widetilde{\rho}/\partial z < 0$. Similarly, it will be unstable if $\partial\widetilde{\rho}/\partial z > 0$. The upward force (per unit volume) on the displaced parcel is given by

$$F = -g\delta\rho = g\frac{\partial \widetilde{\rho}}{\partial z}\delta z, \tag{2.216}$$

and thus Newton's second law implies that the motion of the parcel is determined by

$$\rho(z)\frac{\partial^2 \delta z}{\partial t^2} = g\frac{\partial \widetilde{\rho}}{\partial z}\delta z, \tag{2.217}$$

or

$$\frac{\partial^2 \delta z}{\partial t^2} = \frac{g}{\widetilde{\rho}}\frac{\partial \widetilde{\rho}}{\partial z}\delta z = -N^2\delta z, \tag{2.218}$$

where

$$N^2 = -\frac{g}{\widetilde{\rho}}\frac{\partial \widetilde{\rho}}{\partial z} \tag{2.219}$$

is the *buoyancy frequency*, or the *Brunt–Väisälä frequency,* for this problem. If $N^2 > 0$ then a parcel displaced upward is heavier than its surroundings, and thus experiences a restoring force; the density profile is said to be stable and N is the frequency at which the fluid parcel oscillates. If $N^2 < 0$, the density profile is unstable and the parcel continues to ascend and convection ensues. In liquids it is often a good approximation to replace $\widetilde{\rho}$ by ρ_0 in the denominator of (2.219).

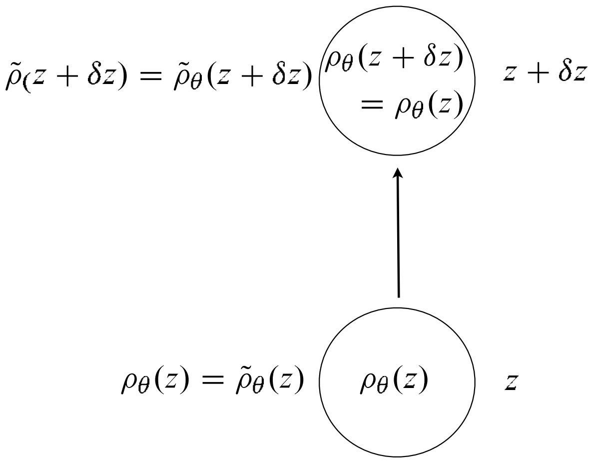

Fig. 2.7 A parcel is adiabatically displaced upward from level z to $z + \delta z$, preserving its potential density, which it takes from the environment at level z. If $z + \delta z$ is the reference level, the potential density there is equal to the actual density. The parcel's stability is determined by the difference between its density and the environmental density [see (2.220)]; if the difference is positive the displacement is stable, and conversely.

2.9.2 The general case: using potential density

More generally, in an adiabatic displacement it is *potential density,* ρ_θ, and not density itself that is materially conserved. Consider a parcel that is displaced adiabatically a vertical distance from z to $z + \delta z$; the parcel preserves its potential density, and let us use the pressure at level $z + \delta z$ as the reference level. The *in situ* density of the parcel at $z + \delta z$, namely $\rho(z + \delta z)$, is then equal to its potential density $\rho_\theta(z + \delta z)$ and, because ρ_θ is conserved, this is equal to the potential density of the environment at z, $\tilde{\rho}_\theta(z)$. The difference in *in situ* density between the parcel and the environment at $z + \delta z$, $\delta\rho$, is thus equal to the difference between the potential density of the environment at z and at $z + \delta z$. Putting this together (and see Fig. 2.7) we have

$$
\begin{aligned}
\delta\rho &= \rho(z + \delta z) - \tilde{\rho}(z + \delta z) = \rho_\theta(z + \delta z) - \tilde{\rho}_\theta(z + \delta z) \\
&= \rho_\theta(z) - \tilde{\rho}_\theta(z + \delta z) = \tilde{\rho}_\theta(z) - \tilde{\rho}_\theta(z + \delta z),
\end{aligned}
\tag{2.220}
$$

and therefore

$$
\delta\rho = -\frac{\partial \tilde{\rho}_\theta}{\partial z}\delta z, \tag{2.221}
$$

where the derivative on the right-hand side is the environmental gradient of potential density. If the right-hand side is positive, the parcel is heavier than its surroundings and the displacement is stable. Thus, the conditions for stability are:

$$
\boxed{
\begin{aligned}
&\text{stability}: && \frac{\partial \tilde{\rho}_\theta}{\partial z} < 0 \\
&\text{instability}: && \frac{\partial \tilde{\rho}_\theta}{\partial z} > 0
\end{aligned}
}\,. \tag{2.222a,b}
$$

That is, *the stability of a parcel of fluid is determined by the gradient of the locally-referenced potential density.* The equation of motion of the fluid parcel is

$$
\frac{\partial^2 \delta z}{\partial t^2} = \frac{g}{\rho}\left(\frac{\partial \tilde{\rho}_\theta}{\partial z}\right)\delta z = -N^2 \delta z, \tag{2.223}
$$

where, noting that $\rho(z) = \widetilde{\rho}_\theta(z)$ to within $O(\delta z)$,

$$\boxed{N^2 = -\frac{g}{\widetilde{\rho}_\theta}\left(\frac{\partial \widetilde{\rho}_\theta}{\partial z}\right)}. \tag{2.224}$$

This is a general expression for the buoyancy frequency, true in both liquids and gases. It is important to realize that the quantity $\widetilde{\rho}_\theta$ is the *locally-referenced* potential density of the environment, as will become more clear below.

An ideal gas

In the atmosphere potential density is related to potential temperature by $\rho_\theta = p_R/(\theta R)$. Using this in (2.224) gives

$$\boxed{N^2 = \frac{g}{\widetilde{\theta}}\left(\frac{\partial \widetilde{\theta}}{\partial z}\right)}, \tag{2.225}$$

where $\widetilde{\theta}$ refers to the environmental profile of potential temperature. The reference value p_R does not appear, and we are free to choose this value arbitrarily — the surface pressure is a common choice. The conditions for stability, (2.222), then correspond to $N^2 > 0$ for stability and $N^2 < 0$ for instability. In the troposphere (the lowest several kilometres of the atmosphere) the average N is about $0.01\ \mathrm{s}^{-1}$, with a corresponding period, $(2\pi/N)$, of about 10 minutes. In the stratosphere (which lies above the troposphere) N^2 is a few times higher than this.

A liquid ocean

No simple, accurate, analytic expression is available for computing static stability in the ocean. If the ocean had no salt, then the potential density referenced to the surface would generally be a measure of the sign of stability of a fluid column, if not of the buoyancy frequency. However, in the presence of salinity, the surface-referenced potential density is not necessarily even a measure of the sign of stability, because the coefficients of compressibility β_T and β_S vary in different ways with pressure. To see this, suppose two neighbouring fluid elements at the surface have the same potential density, but different salinities and temperatures, and displace them both adiabatically to the deep ocean. Although their potential densities (referenced to the surface) are still equal, we can say little about their actual densities, and hence their stability relative to each other, without doing a detailed calculation because they will each have been compressed by different amounts. It is the profile of the *locally-referenced* potential density that determines the stability.

An approximate expression for stability that is sometimes useful arises by noting that in an adiabatic displacement

$$\delta\rho_\theta = \delta\rho - \frac{1}{c_s^2}\delta p = 0. \tag{2.226}$$

If the fluid is hydrostatic $\delta p = -\rho g \delta z$ so that if a parcel is displaced adiabatically its density changes according to

$$\left(\frac{\partial \rho}{\partial z}\right)_{\rho_\theta} = -\frac{\rho g}{c_s^2}. \tag{2.227}$$

If a parcel is displaced a distance δz upwards then the density difference between it and its new surroundings is

$$\delta\rho = -\left[\left(\frac{\partial\rho}{\partial z}\right)_{\rho_\theta} - \left(\frac{\partial\widetilde{\rho}}{\partial z}\right)\right]\delta z = \left[\frac{\rho g}{c_s^2} + \left(\frac{\partial\widetilde{\rho}}{\partial z}\right)\right]\delta z, \tag{2.228}$$

where the tilde again denotes the environmental field. It follows that the stratification is given by

$$\boxed{N^2 = -g\left[\frac{g}{c_s^2} + \frac{1}{\widetilde{\rho}}\left(\frac{\partial\widetilde{\rho}}{\partial z}\right)\right]}. \tag{2.229}$$

This expression holds for both liquids and gases, and for ideal gases it is precisely the same as (2.225) (problem 2.9). In liquids, a good approximation is to use a reference value ρ_0 for the undifferentiated density in the denominator, whence (2.229) becomes equal to the Boussinesq expression (2.107). Typical values of N in the upper ocean where the density is changing most rapidly (i.e., in the pycnocline — 'pycno' for density, 'cline' for changing) are about $0.01\,\mathrm{s}^{-1}$, falling to $0.001\,\mathrm{s}^{-1}$ in the more homogeneous abyssal ocean. These frequencies correspond to periods of about 10 and 100 minutes, respectively.

** Cabbeling*

Cabbeling is an instability that arises because of the nonlinear equation of state of seawater. From Fig. 1.3 we see that the contours are slightly convex, bowing upwards, especially in the plot at sea level. Suppose we mix two parcels of water, each with the same density ($\sigma_\theta = 28$, say), but with different initial values of temperature and salinity. Then the resulting parcel of water will have a temperature and a salinity equal to the average of the two parcels, but its density will be *higher* than either of the two original parcels. In the appropriate circumstances such mixing may thus lead to a convective instability; this may, for example, be an important source of 'bottom water' formation in the Weddell Sea, off Antarctica.[10]

2.9.3 Lapse rates in dry and moist atmospheres

A dry ideal gas

The negative of the rate of change of the temperature in the vertical is known as the *temperature lapse rate*, or often just the lapse rate, and the lapse rate corresponding to $\partial\theta/\partial z = 0$ is called the *dry adiabatic lapse rate* and denoted Γ_d. Using $\theta = T(p_0/p)^{R/c_p}$ and $\partial p/\partial z = -\rho g$ we find that the lapse rate and the potential temperature lapse rate are related by

$$\frac{\partial T}{\partial z} = \frac{T}{\theta}\frac{\partial\theta}{\partial z} - \frac{g}{c_p}, \tag{2.230}$$

so that the dry adiabatic lapse rate is given by

$$\Gamma_d = \frac{g}{c_p}, \tag{2.231}$$

as in fact we derived in (1.134). (We use the subscript d, for dry, to differentiate it from the moist lapse rate considered below.) The conditions for static stability corresponding to

(2.222) are thus:

$$\boxed{\begin{array}{llll} \text{stability}: & \dfrac{\partial\tilde{\theta}}{\partial z} > 0; & \text{or} & -\dfrac{\partial\tilde{T}}{\partial z} < \Gamma_d \\ \text{instability}: & \dfrac{\partial\tilde{\theta}}{\partial z} < 0; & \text{or} & -\dfrac{\partial\tilde{T}}{\partial z} > \Gamma_d \end{array}}, \qquad (2.232\text{a,b})$$

where a tilde indicates that the values are those of the environment. The atmosphere is in fact generally stable by this criterion: the observed lapse rate, corresponding to an observed buoyancy frequency of about $10^{-2}\,\mathrm{s}^{-1}$, is often about $7\,\mathrm{K\,km}^{-1}$, whereas a dry adiabatic lapse rate is about $10\,\mathrm{K\,km}^{-1}$. Why the discrepancy? One reason, particularly important in the tropics, is that the atmosphere contains water vapour.

* *Effects of water vapour on the lapse rate of an ideal gas*

The amount of water vapour that can be contained in a given volume is an increasing function of temperature (with the presence or otherwise of dry air in that volume being largely irrelevant). Thus, if a parcel of water vapour is cooled, it will eventually become saturated and water vapour will condense into liquid water. A measure of the amount of water vapour in a unit volume is its partial pressure, and the partial pressure of water vapour at saturation, e_s, is given by the Clausius–Clapeyron equation,

$$\frac{\mathrm{d}e_s}{\mathrm{d}T} = \frac{L_c e_s}{R_v T^2}, \qquad (2.233)$$

where L_c is the latent heat of condensation or vapourization (per unit mass) and R_v is the gas constant for water vapour. If a parcel rises adiabatically it will cool, and at some height (known as the 'lifting condensation level', a function of its initial temperature and humidity only) the parcel will become saturated and any further ascent will cause the water vapour to condense. The ensuing condensational heating causes the temperature and buoyancy of the parcel to increase; the parcel thus rises further, causing more water vapour to condense, and so on, and the consequence of this is that an environmental profile that is stable if the air is dry may be unstable if saturated. Let us now derive an expression for the lapse rate of a saturated parcel that is ascending adiabatically apart from the affects of condensation.

Let w denote the mass of water vapour per unit mass of dry air, the mixing ratio, and let w_s be the saturation mixing ratio. ($w_s = \alpha e_s/(p - e_s) \approx \alpha_w e_s/p$ where $\alpha_w = 0.62$, the ratio of the mass of a water molecule to one of dry air.) The diabatic heating associated with condensation is then given by

$$Q_{cond} = -L_c \frac{\mathrm{D}w_s}{\mathrm{D}t}, \qquad (2.234)$$

so that the thermodynamic equation is

$$c_p \frac{\mathrm{D}\ln\theta}{\mathrm{D}t} = -\frac{L_c}{T}\frac{\mathrm{D}w_s}{\mathrm{D}t}, \qquad (2.235)$$

or, in terms of p and and T

$$c_p \frac{\mathrm{D}\ln T}{\mathrm{D}t} - R\frac{\mathrm{D}\ln P}{\mathrm{D}t} = -\frac{L_c}{T}\frac{\mathrm{D}w_s}{\mathrm{D}t}. \qquad (2.236)$$

If these material derivatives are due to the parcel ascent then

$$\frac{\mathrm{d}\ln T}{\mathrm{d}z} - \frac{R}{c_p}\frac{\mathrm{d}\ln p}{\mathrm{d}z} = -\frac{L_c}{Tc_p}\frac{\mathrm{d}w_s}{\mathrm{d}z}, \tag{2.237}$$

and using the hydrostatic relationship and the fact that w_s is a function of T and p we obtain

$$\frac{\mathrm{d}T}{\mathrm{d}z} + \frac{g}{c_p} = -\frac{L_c}{c_p}\left[\left(\frac{\partial w_s}{\partial T}\right)_p \frac{\mathrm{d}T}{\mathrm{d}z} - \left(\frac{\partial w_s}{\partial p}\right)_T \rho g\right]. \tag{2.238}$$

Solving for $\mathrm{d}T/\mathrm{d}z$, the lapse rate, Γ_s, of an ascending saturated parcel is given by

$$\Gamma_s = -\frac{\mathrm{d}T}{\mathrm{d}z} = \frac{g}{c_p}\frac{1-\rho L_c(\partial w_s/\partial p)_T}{1+(L_c/c_p)(\partial w_s/\partial T)_p} \approx \frac{g}{c_p}\frac{1+L_c w_s/(RT)}{1+L_c^2 w_s/(c_p R T^2)}. \tag{2.239}$$

where the last near equality follows with use of the Clausius–Clapeyron relation. The quantity Γ_s is variously called the *pseudoadiabatic* or *moist adiabatic* or *saturated adiabatic* lapse rate. Because g/c_p is the dry adiabatic lapse rate Γ_d, $\Gamma_s < \Gamma_d$, and values of Γ_s are typically around $6\,\mathrm{K\,km^{-1}}$ in the lower atmosphere; however, $\mathrm{d}w_s/\mathrm{d}T$ is an increasing function of T so that Γ_s decreases with increasing temperature and can be as low as $3.5\,\mathrm{K\,km^{-1}}$. For a saturated parcel, the stability conditions analogous to (2.232) are

$$\text{stability:} \qquad -\frac{\partial \widetilde{T}}{\partial z} < \Gamma_s, \tag{2.240a}$$

$$\text{instability:} \qquad -\frac{\partial \widetilde{T}}{\partial z} > \Gamma_s. \tag{2.240b}$$

where $\widetilde{T}$ is the environmental temperature. The observed environmental profile in convecting situations is often a combination of the dry adiabatic and moist adiabatic profiles: an unsaturated parcel that is is unstable by the dry criterion will rise and cool following a dry adiabat, Γ_d, until it becomes saturated at the lifting condensation level, above which it will rise following a saturation adiabat, Γ_s. Such convection will proceed until the atmospheric column is stable and, especially in low latitudes, the lapse rate of the atmosphere is largely determined by such convective processes.

* *Equivalent potential temperature*

Suppose that all the moisture in a parcel of air condenses, and that all the heat released goes into heating the parcel. The *equivalent potential temperature*, θ_{eq} is the potential temperature that the parcel then achieves. We may obtain an approximate analytic expression for it by noting that the first law of thermodynamics, $đQ = T\,\mathrm{d}\eta$, then implies, by definition of potential temperature,

$$-L_c\,\mathrm{d}w = c_p T\,\mathrm{d}\ln\theta, \tag{2.241}$$

where $\mathrm{d}w$ is the change in water vapour mixing ratio, so that a reduction of w via condensation leads to heating. Integrating gives, by definition of equivalent potential temperature,

$$-\int_w^0 \frac{L_c w}{c_p T}\,\mathrm{d}w = \int_\theta^{\theta_{eq}} \mathrm{d}\ln\theta, \tag{2.242}$$

and so, if T and L_c are assumed to be constant,

$$\theta_{eq} = \theta \exp\left(\frac{L_c w}{c_p T}\right). \tag{2.243}$$

The equivalent potential temperature so defined is approximately conserved during condensation, the approximation arising going from (2.242) to (2.243). It is a useful expression for diagnostic purposes, and in constructing theories of convection, but it is not accurate enough to use as a prognostic variable in a putatively realistic numerical model. The 'equivalent temperature' may be defined in terms of the equivalent potential temperature by

$$T_{eq} \equiv \theta_{eq}\left(\frac{p}{p_R}\right)^{\kappa}. \tag{2.244}$$

2.10 GRAVITY WAVES

The parcel approach to oscillations and stability, while simple and direct, is divorced from the fluid-dynamical equations of motion, making it hard to include other effects such as rotation, or to explore the effects of possible differences between the hydrostatic and non-hydrostatic cases. To remedy this, we now use the equations of motion to analyse the motion resulting from a small disturbance.

2.10.1 Gravity waves and convection in a Boussinesq fluid

Let us consider a Boussinesq fluid, initially at rest, in which the buoyancy varies linearly with height and the buoyancy frequency, N, is a constant. Linearizing the equations of motion about this basic state gives the linear momentum equations,

$$\frac{\partial u'}{\partial t} = -\frac{\partial \phi'}{\partial x}, \qquad \frac{\partial w'}{\partial t} = -\frac{\partial \phi'}{\partial z} + b', \tag{2.245a,b}$$

the mass continuity and thermodynamic equations,

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0, \qquad \frac{\partial b'}{\partial t} + w' N^2 = 0, \tag{2.246a,b}$$

where for simplicity we assume that the flow is a function only of x and z. A little algebra gives a single equation for w',

$$\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial^2}{\partial t^2} + N^2 \frac{\partial^2}{\partial x^2}\right] w' = 0. \tag{2.247}$$

Seeking solutions of the form $w' = \mathrm{Re}\, W \exp[\mathrm{i}(kx + mz - \omega t)]$ (where Re denotes the real part) yields the dispersion relationship for gravity waves:

$$\boxed{\omega^2 = \frac{k^2 N^2}{k^2 + m^2}}. \tag{2.248}$$

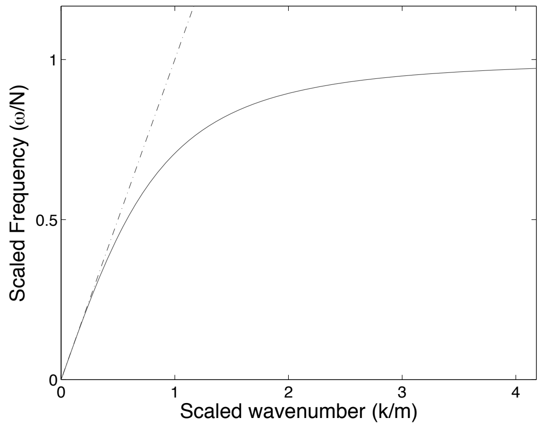

Fig. 2.8 Scaled frequency, ω/N, plotted as a function of scaled horizontal wavenumber, k/m, using the full dispersion relation of (2.248) (solid line, asymptoting to unit value for large k/m) and with the hydrostatic dispersion relation (2.252) (dashed line, tending to ∞ for large k/m).

The frequency (see Fig. 2.8) is thus always less than N, approaching N for small horizontal scales, $k \gg m$. If we neglect pressure perturbations, as in the parcel argument, then the two equations,

$$\frac{\partial w'}{\partial t} = b', \qquad \frac{\partial b'}{\partial t} + w'N^2 = 0, \tag{2.249}$$

form a closed set, and give $\omega^2 = N^2$.

If the basic state density increases with height then $N^2 < 0$ and we expect this state to be unstable. Indeed, the disturbance grows exponentially according to $\exp(\sigma t)$ where

$$\sigma = \mathrm{i}\omega = \frac{\pm k\widetilde{N}}{(k^2 + m^2)^{1/2}}, \tag{2.250}$$

where $\widetilde{N}^2 = -N^2$. Most convective activity in the ocean and atmosphere is, ultimately, related to an instability of this form, although of course there are many complicating issues — water vapour in the atmosphere, salt in the ocean, the effects of rotation and so forth.

Hydrostatic gravity waves and convection

Let us now suppose that the fluid satisfies the hydrostatic Boussinesq equations. The linearized two-dimensional equations of motion become

$$\frac{\partial u'}{\partial t} = -\frac{\partial \phi'}{\partial x}, \qquad 0 = -\frac{\partial \phi'}{\partial z} + b', \tag{2.251a}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0, \qquad \frac{\partial b'}{\partial t} + w'N^2 = 0, \tag{2.251b}$$

where these are the horizontal and vertical momentum equations, the mass continuity equation and the thermodynamic equation respectively. A little algebra gives the dispersion relation,

$$\omega^2 = \frac{k^2 N^2}{m^2}. \tag{2.252}$$

The frequency and, if N^2 is negative the growth rate, is unbounded for as $k/m \to \infty$, and the hydrostatic approximation thus has quite unphysical behaviour for small horizontal scales (see also problem 2.11).[11]

2.11 * ACOUSTIC-GRAVITY WAVES IN AN IDEAL GAS

We now consider wave motion in a stratified, compressible fluid such as the Earth's atmosphere. The complete problem is complicated and uninformative; we will specialize to the case of an isothermal, stationary atmosphere and ignore the effects of rotation and sphericity, but otherwise we will make few approximations. In this section we will denote the unperturbed state with a subscript 0 and the perturbed state with a prime ($'$); we will also omit many algebraic details. Because it is at rest, the basic state is in hydrostatic balance,

$$\frac{\partial p_0}{\partial z} = -\rho_0(z)g. \tag{2.253}$$

Ignoring variations in the y-direction for algebraic simplicity, the linearized equations of motion are:

u momentum: $$\rho_0 \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x} \tag{2.254a}$$

w momentum: $$\rho_0 \frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z} - \rho' g \tag{2.254b}$$

mass conservation: $$\frac{\partial \rho'}{\partial t} + w' \frac{\partial \rho_0}{\partial z} = -\rho_0 \left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} \right) \tag{2.254c}$$

thermodynamic: $$\frac{\partial \theta'}{\partial t} + w' \frac{\partial \theta_0}{\partial z} = 0 \tag{2.254d}$$

equation of state: $$\frac{\theta'}{\theta_0} + \frac{\rho'}{\rho_0} = \frac{1}{\gamma} \frac{p'}{p_0}. \tag{2.254e}$$

For an isothermal basic state we have $p_0 = \rho_0 R T_0$ where T_0 is a constant, so that $\rho_0 = \rho_s e^{-z/H}$ and $p_0 = p_s e^{-z/H}$ where $H = RT_0/g$. Further, using $\theta = T(p_s/p)^\kappa$ where $\kappa = R/c_p$, we have $\theta_0 = T_0 e^{\kappa z/H}$ and so $N^2 = \kappa g/H$. It is also convenient to use (1.99) on page 23 to rewrite the linear thermodynamic equation in the form

$$\frac{\partial p'}{\partial t} - w' \frac{p_0}{H} = -\gamma p_0 \left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} \right). \tag{2.254f}$$

Differentiating (2.254a) with respect to time and using (2.254f) leads to

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \frac{\partial^2}{\partial x^2} \right) u' = c_s^2 \left(\frac{\partial}{\partial z} - \frac{1}{\gamma H} \right) \frac{\partial}{\partial x} w'. \tag{2.255a}$$

where $c_s^2 = (\partial p / \partial / \rho)_\eta = \gamma R T_0 = \gamma p_0 / \rho_0$ is the square of the speed of sound, and $\gamma = c_p/c_v = 1/(1-\kappa)$. Similarly, differentiating (2.254b) with respect to time and using (2.254c) and (2.254f) leads to

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \left[\frac{\partial^2}{\partial z^2} - \frac{1}{H} \frac{\partial}{\partial z} \right] \right) w' = c_s^2 \left(\frac{\partial}{\partial z} - \frac{\kappa}{H} \right) \frac{\partial u'}{\partial x}, \tag{2.255b}$$

Equations (2.255a) and (2.255b) combine to give, after some cancellation,

$$\frac{\partial^4 w'}{\partial t^4} - c_s^2 \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{H} \frac{\partial}{\partial z} \right) w' - c_s^2 \frac{\kappa g}{H} \frac{\partial^2 w'}{\partial x^2} = 0. \tag{2.256}$$

If we set $w' = W(x,z,t)\mathrm{e}^{z/(2H)}$, so that $W = (\rho_0/\rho_s)^{1/2} w$, then the term with the single z-derivative is eliminated, giving

$$\frac{\partial^4 W}{\partial t^4} - c_s^2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{4H^2}\right) W - c_s^2 \frac{\kappa g}{H}\frac{\partial^2 W}{\partial x^2} = 0. \tag{2.257}$$

Although superficially complicated, this equation has constant coefficients and we may seek wave-like solutions of the form

$$W = \mathrm{Re}\,\widetilde{W}\mathrm{e}^{\mathrm{i}(kx+mz-\omega t)}, \tag{2.258}$$

where $\widetilde{W}$ is the complex wave amplitude. Using (2.258) in (2.257) leads to the dispersion relation for acoustic-gravity waves, namely

$$\omega^4 - c_s^2\omega^2\left(k^2 + m^2 + \frac{1}{4H^2}\right) + c_s^2 N^2 k^2 = 0, \tag{2.259}$$

with solution

$$\boxed{\omega^2 = \frac{1}{2}c_s^2 K^2\left[1 \pm \left(1 - \frac{4N^2k^2}{c_s^2 K^4}\right)^{1/2}\right]}, \tag{2.260}$$

where $K^2 = k^2 + m^2 + 1/(4H^2)$. (The factor $[1 - 4N^2k^2/(c_s^2K^4)]$ is always positive — see problem 2.26.) For an isothermal, ideal-gas atmosphere $4N^2H^2/c_s^2 \approx 0.8$ and so this may be written

$$\frac{\omega^2}{N^2} \approx 2.5\widehat{K}^2\left[1 \pm \left(1 - \frac{0.8\widehat{k}^2}{\widehat{K}^4}\right)^{1/2}\right], \tag{2.261}$$

where $\widehat{K}^2 = \widehat{k}^2 + \widehat{m}^2 + 1/4$, and $(\widehat{k},\widehat{m}) = (kH, mH)$.

2.11.1 Interpretation

Acoustic and gravity waves

There are two branches of roots in (2.260), corresponding to acoustic waves (using the plus sign in the dispersion relation) and internal gravity waves (using the minus sign). These (and the Lamb wave, described below) are plotted in Fig. 2.9. If $4N^2k^2/c_s^2K^4 \ll 1$ then the two sets of waves are well separated. From (2.261) this is satisfied when

$$\frac{4\kappa}{\gamma}(kH)^2 \approx 0.8(kH)^2 \ll \left[(kH)^2 + (mH)^2 + \frac{1}{4}\right]^2; \tag{2.262}$$

that is, when *either* $mH \gg 1$ or $kH \gg 1$. The two roots of the dispersion relation are then

$$\omega_a^2 \approx c_s^2 K^2 = c_s^2\left(k^2 + m^2 + \frac{1}{4H^2}\right) \tag{2.263}$$

and

$$\omega_g^2 \approx \frac{N^2k^2}{k^2 + m^2 + 1/(4H^2)}, \tag{2.264}$$

corresponding to acoustic and gravity waves, respectively. The acoustic waves owe their existence to the presence of compressibility in the fluid, and they have no counterpart in the

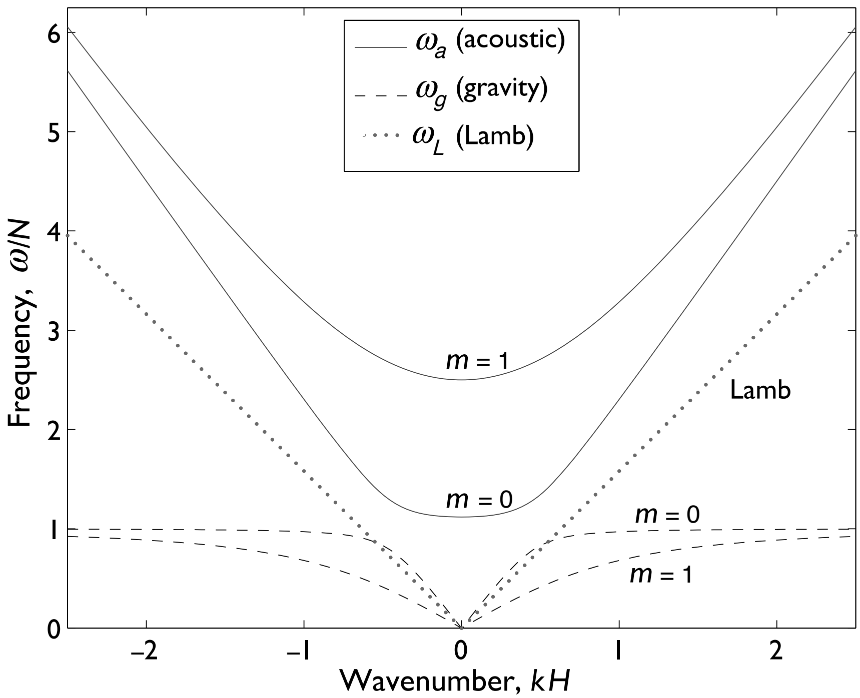

Fig. 2.9 Dispersion diagram for acoustic gravity waves in an isothermal atmosphere, calculated using (2.261). The frequency is given in units of the the buoyancy frequency N, and the wavenumbers are non-dimensionalized by the inverse of the scale height, H. The solid curves indicate acoustic waves, whose frequency is always higher than that of the corresponding Lamb wave at the same wavenumber (i.e., ck), and of the base acoustic frequency $\approx 1.12N$. The dashed curves indicate internal gravity waves, whose frequency asymptotes to N at small horizontal scales.

Boussinesq system. On the other hand, the internal gravity waves are just modified forms of those found in the Boussinesq system, and if we take the limit $(kH, mH) \to \infty$ then the gravity wave branch reduces to $\omega_g^2 = N^2k^2/(k^2 + m^2)$, which is the dispersion relationship for gravity waves in the Boussinesq approximation. We may consider this to be the limit of infinite scale height or (equivalently) the case in which wavelengths of the internal waves are sufficiently small that the fluid is essentially incompressible.

Vertical structure

Recall that $w' = W(x, z, t)\mathrm{e}^{z/(2H)}$ and, by inspection of (2.255), u' has the same vertical structure. That is,

$$w' \propto \mathrm{e}^{z/(2H)}, \qquad u' \propto \mathrm{e}^{z/(2H)}, \tag{2.265}$$

and the amplitude of the velocity field of the internal waves increases with height. The pressure and density perturbation amplitudes fall off with height, varying like

$$p' \propto \mathrm{e}^{-z/(2H)}, \qquad \rho' \propto \mathrm{e}^{-z/(2H)}. \tag{2.266}$$

The kinetic energy of the perturbation, $\rho_0(u'^2 + w'^2)$ is *constant* with height, because $\rho_0 = \rho_s e^{-z/H}$.

Hydrostatic approximation and Lamb waves

Equations (2.255) also admit to a solution with $w' = 0$. We then have

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \frac{\partial^2}{\partial x^2}\right) u' = 0 \qquad \text{and} \qquad \left(\frac{\partial}{\partial z} - \frac{\kappa}{H}\right) \frac{\partial u'}{\partial x} = 0, \tag{2.267}$$

and these have solutions of the form

$$u' = \mathrm{Re}\, \widetilde{U} \mathrm{e}^{\kappa z/H} \mathrm{e}^{\mathrm{i}(kx-\omega t)}, \qquad \omega = ck, \tag{2.268}$$

where $\widetilde{U}$ is the wave amplitude. These are horizontally propagating sound waves, known as *Lamb waves* after the hydrodynamicist Horace Lamb. Their velocity perturbation amplitude increases with height, but the pressure perturbation falls with height; that is

$$u' \propto \mathrm{e}^{\kappa z/H} \approx \mathrm{e}^{2z/(7H)}, \qquad p' \propto \mathrm{e}^{(\kappa-1)z/H} \approx \mathrm{e}^{-5z/(7H)}. \tag{2.269}$$

Their kinetic energy density, $\rho_0 u'^2$, varies as

$$KE \propto \mathrm{e}^{-z/H+2\kappa z/H} = \mathrm{e}^{(2R-c_p)z/(c_pH)]} = \mathrm{e}^{(R-c_v)z/(c_pH)} \approx \mathrm{e}^{-3z/(7H)} \tag{2.270}$$

for an ideal gas. (In a simple ideal gas, $c_v = nR/2$ where n is the number of excited degrees of freedom, 5 for a diatomic molecule.) The kinetic energy density thus falls away exponentially from the surface, and in this sense Lamb waves are an example of edge waves or surface-trapped waves.

Now consider the case in which we make the hydrostatic approximation ab initio, but do not restrict the perturbation to have $w' = 0$. The linearized equations are identical to (2.254), except that (2.254b) is replaced by

$$\frac{\partial p'}{\partial z} = -\rho' g. \tag{2.271}$$

The consequence of this is that first term $(\partial^2 w'/\partial t^2)$ in (2.255b) disappears, as do the first two terms in (2.256) [the terms $\partial^4 w'/\partial t^4 - c^2(\partial^2/\partial t^2)(\partial^2 w'/\partial x^2)$]. It is a simple matter to show that the dispersion relation is then

$$\omega^2 = \frac{N^2 k^2}{m^2 + 1/(4H^2)}. \tag{2.272}$$

These are long gravity waves, and may be compared with the corresponding Boussinesq result (2.252). Again, the frequency increases without bound as the horizontal wavelength diminishes. The Lamb wave, of course, still exists in the hydrostatic model, because (2.267) is still a valid solution. Thus, horizontally propagating sound waves still exist in hydrostatic (primitive equation) models, but vertically propagating sound waves do not — essentially because the term $\partial w/\partial t$ is absent from the vertical momentum equation.

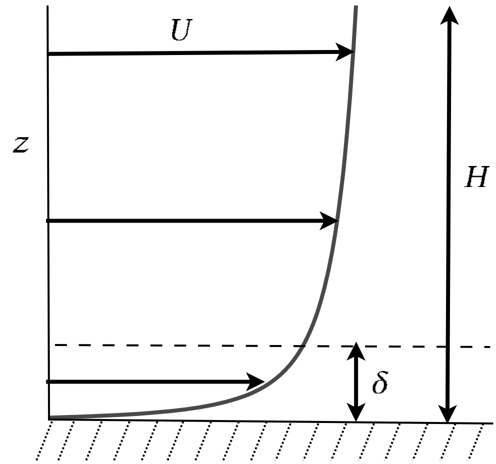

Fig. 2.10 An idealized boundary layer. The values of a field, such as velocity, U, may vary rapidly in a boundary in order to satisfy the boundary conditions at a rigid surface. The parameter δ is a measure of the boundary layer thickness, and H is a typical scale of variation away from the boundary.

2.12 THE EKMAN LAYER

In the final topic of this chapter, we return to geostrophic flow and consider the effects of friction. The fluid fields in the interior of a domain are often set by different physical processes than those occurring at a boundary, and consequently often change rapidly in a thin *boundary layer,* as in Fig. 2.10. Such boundary layers nearly always involve one or both of viscosity and diffusion, because these appear in the terms of highest differential order in the equations of motion, and so are responsible for the number and type of boundary conditions that the equations must satisfy — for example, the presence of molecular viscosity leads to the condition that the tangential flow (as well as the normal flow) must vanish at a rigid surface.

In many boundary layers in non-rotating flow the dominant balance in the momentum equation is between the advective and viscous terms. In some contrast, in large-scale atmospheric and oceanic flow the effects of rotation are large, and this results in a boundary layer, known as the *Ekman layer,* in which the dominant balance is between Coriolis and frictional or stress terms.[12] Now, the direct effects of molecular viscosity and diffusion are nearly always negligible at distances more than a few millimetres away from a solid boundary, but it is inconceivable that the entire boundary layer between the free atmosphere (or free ocean) and the surface is only a few millimetres thick. Rather, in practice a balance occurs between the Coriolis terms and the forces due to the stress generated by small-scale turbulent motion, and this gives rise to a boundary layer that has a typical depth of a few tens to several hundreds of metres. Because the stress arises from the turbulence we cannot with confidence determine its precise form; thus, we should try to determine what general properties Ekman layers may have that are *independent* of the precise form of the friction.

The atmospheric Ekman layer occurs near the ground, and the stress at the ground itself is due to the surface wind (and its vertical variation). In the ocean the main Ekman layer is near the surface, and the stress at ocean surface is largely due to the presence of the overlying wind. There is also a weak Ekman layer at the bottom of the ocean, analogous to the atmospheric Ekman layer. To analyse all these layers, let us assume the following.

⋆ The Ekman layer is Boussinesq. This is a very good assumption for the ocean, and a reasonable one for the atmosphere if the boundary layer is not too deep.

- ⋆ The Ekman layer has a finite depth that is less than the total depth of the fluid, this depth being given by the level at which the frictional stresses essentially vanish. Within the Ekman layer, frictional terms are important, whereas geostrophic balance holds beyond it.
- ⋆ The nonlinear and time-dependent terms in the equations of motion are negligible, hydrostatic balance holds in the vertical, and buoyancy is constant, not varying in the horizontal.
- ⋆ As needed, we shall assume that friction can be parameterized by a viscous term of the form $\rho_0^{-1}\partial\boldsymbol{\tau}/\partial z = A\,\partial^2\boldsymbol{u}/\partial z^2$, where A is constant and $\boldsymbol{\tau}$ is the stress. [In general, stress is a tensor, τ_{ij}, with an associated force given by $F_i = \partial\tau_{ij}/\partial x_j$, summing over the repeated index. It is common in geophysical fluid dynamics that the vertical derivative dominates, and in this case the force is $\boldsymbol{F} = \partial\boldsymbol{\tau}/\partial z$. We still use the word stress for $\boldsymbol{\tau}$, but it now refers to a vector whose derivative in a particular direction (z in this case) is the force on a fluid.] In laboratory settings A may be the molecular viscosity, whereas in the atmosphere and ocean it is a so-called *eddy viscosity.* (In turbulent flows momentum is transferred by the near-random motion of small parcels of fluid and, by analogy with the motion of molecules that produces a molecular viscosity, the associated stress is approximately represented, or parameterized, using a turbulent or eddy viscosity that may be orders of magnitude larger than the molecular one.)

2.12.1 Equations of motion and scaling

Frictional–geostrophic balance in the horizontal momentum equation is:

$$\boldsymbol{f} \times \boldsymbol{u} = -\nabla_z \phi + \frac{\partial \widetilde{\boldsymbol{\tau}}}{\partial z}. \tag{2.273}$$

where $\widetilde{\boldsymbol{\tau}} \equiv \boldsymbol{\tau}/\rho_0$ is the kinematic stress and $\boldsymbol{f} = f\mathbf{k}$, where the Coriolis parameter f is allowed to vary with latitude. If we model the stress with an eddy viscosity, (2.273) becomes

$$\boldsymbol{f} \times \boldsymbol{u} = -\nabla_z \phi + A\frac{\partial^2 \boldsymbol{u}}{\partial z^2}. \tag{2.274}$$

The vertical momentum equation is $\partial\phi/\partial z = b$, i.e., hydrostatic balance, and, because buoyancy is constant, we may without loss of generality write this as

$$\frac{\partial \phi}{\partial z} = 0. \tag{2.275}$$

The equation set is completed by the mass continuity equation, $\nabla \cdot \boldsymbol{v} = 0$.

The Ekman number

We non-dimensionalize the equations by setting

$$(u, v) = U(\hat{u}, \hat{v}), \quad (x, y) = L(\hat{x}, \hat{y}), \quad f = f_0 \hat{f}, \quad z = H\hat{z}, \quad \phi = \Phi\hat{\phi}, \tag{2.276}$$

where hatted variables are non-dimensional. H is a scaling for the height, and at this stage we will suppose it to be some height scale in the free atmosphere or ocean, not the height of the Ekman layer itself. Geostrophic balance suggests that $\Phi = f_0 UL$. Substituting (2.276) into (2.274) we obtain

$$\widehat{\boldsymbol{f}} \times \widehat{\boldsymbol{u}} = -\widehat{\nabla}\widehat{\phi} + Ek\frac{\partial^2 \widehat{\boldsymbol{u}}}{\partial \widehat{z}^2}, \tag{2.277}$$

where the parameter

$$\boxed{Ek \equiv \left(\frac{A}{f_0 H^2}\right)}, \tag{2.278}$$

is the *Ekman number*, and it determines the importance of frictional terms in the horizontal momentum equation. If $Ek \ll 1$ then the friction is small in the flow interior where $\widehat{z} = \mathcal{O}(1)$. However, the friction term cannot necessarily be neglected in the boundary layer because it is of the highest differential order in the equation, and so determines the boundary conditions; if Ek is small the vertical scales become small and the second term on the right-hand side of (2.277) remains finite. The case when this term is simply omitted from the equation is therefore a *singular limit,* meaning that it differs from the case with $Ek \to 0$. If $Ek \geq 1$ friction is important everywhere, but it is usually the case that Ek is small for atmospheric and oceanic large-scale flow, and the interior flow is very nearly geostrophic. (In part this is because A itself is only large near a rigid surface where the presence of a shear creates turbulence and a significant eddy viscosity.)

Momentum balance in the Ekman layer

For definiteness, suppose the fluid lies above a rigid surface at $z = 0$. Sufficiently far away from the boundary the velocity field is known, and we suppose this flow to be in geostrophic balance. We then write the velocity field and the pressure field as the sum of the interior geostrophic part, plus a boundary layer correction:

$$\widehat{\boldsymbol{u}} = \widehat{\boldsymbol{u}}_g + \widehat{\boldsymbol{u}}_E, \qquad \widehat{\phi} = \widehat{\phi}_g + \widehat{\phi}_E, \tag{2.279}$$

where the Ekman layer corrections, denoted with a subscript E, are negligible away from the boundary layer. Now, in the fluid interior we have, by hydrostatic balance, $\partial\widehat{\phi}_g/\partial\widehat{z} = 0$. In the boundary layer we still have $\partial\widehat{\phi}_g/\partial\widehat{z} = 0$ so that, to satisfy hydrostasy, $\partial\widehat{\phi}_E/\partial\widehat{z} = 0$. But because $\widehat{\phi}_E$ vanishes away from the boundary we have $\widehat{\phi}_E = 0$ everywhere. Thus, *there is no boundary layer in the pressure field.* Note that this is a much stronger result than saying that pressure is continuous, which is nearly always true in fluids; rather, it is a special result for Ekman layers.

Using (2.279) with $\widehat{\phi}_E = 0$, the dimensional horizontal momentum equation (2.273) becomes, in the Ekman layer,

$$\boldsymbol{f} \times \boldsymbol{u}_E = \frac{\partial \widetilde{\boldsymbol{\tau}}}{\partial z}. \tag{2.280}$$

The dominant force balance in the Ekman layer is thus between the Coriolis force and the friction. We can determine the thickness of the Ekman layer if we model the stress with an eddy viscosity so that

$$\boldsymbol{f} \times \boldsymbol{u}_E = A\frac{\partial^2 \boldsymbol{u}_E}{\partial z^2}, \tag{2.281}$$

or, non-dimensionally,

$$\widehat{\boldsymbol{f}} \times \widehat{\boldsymbol{u}}_E = Ek\frac{\partial^2 \widehat{\boldsymbol{u}}_E}{\partial \widehat{z}^2}. \tag{2.282}$$

It is evident this equation can only be satisfied if $\hat{z} \neq \mathcal{O}(1)$, implying that H is not a proper scaling for z in the boundary layer. Rather, if the vertical scale in the Ekman layer is $\hat{\delta}$ (meaning $\hat{z} \sim \hat{\delta}$) we must have $\hat{\delta} \sim Ek^{1/2}$. In dimensional terms this means the thickness of the Ekman layer is

$$\delta = H\hat{\delta} = HEk^{1/2} \tag{2.283}$$

or

$$\boxed{\delta = \left(\frac{A}{f_0}\right)^{1/2}}. \tag{2.284}$$

[This estimate also emerges directly from (2.281).] Note that (2.283) can be written as

$$Ek = \left(\frac{\delta}{H}\right)^2. \tag{2.285}$$

That is, the Ekman number is equal to the square of the ratio of the depth of the Ekman layer to an interior depth scale of the fluid motion. In laboratory flows where A is the molecular viscosity we can thus estimate the Ekman layer thickness, and if we know the eddy viscosity of the ocean or atmosphere we can estimate their respective Ekman layer thicknesses. We can invert this argument and obtain an estimate of A if we know the Ekman layer depth. In the atmosphere, deviations from geostrophic balance are very small in the atmosphere above 1 km, and using this gives $A \approx 10^2\,\mathrm{m^2\,s^{-1}}$. In the ocean Ekman depths are often 50 m or less, and eddy viscosities are about $0.1\,\mathrm{m^2\,s^{-1}}$.

2.12.2 Integral properties of the Ekman layer

What can we deduce about the Ekman layer without specifying the detailed form of the frictional term? Using dimensional notation we recall frictional–geostrophic balance,

$$\boldsymbol{f} \times \boldsymbol{u} = -\nabla\phi + \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z}, \tag{2.286}$$

where $\boldsymbol{\tau}$ is zero at the edge of the Ekman layer. In the Ekman layer itself we have

$$\boldsymbol{f} \times \boldsymbol{u}_E = \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z}. \tag{2.287}$$

Consider either a top or bottom Ekman layer, and integrate over its thickness. From (2.287) we obtain

$$\boldsymbol{f} \times \boldsymbol{M}_E = \boldsymbol{\tau}_T - \boldsymbol{\tau}_B, \tag{2.288}$$

where

$$\boldsymbol{M}_E = \int_{Ek} \rho_0 \boldsymbol{u}_E \,\mathrm{d}z \tag{2.289}$$

is the ageostrophic mass transport in the Ekman layer, and $\boldsymbol{\tau}_T$ and $\boldsymbol{\tau}_B$ are the respective stresses at the top and the bottom of the Ekman layer at hand. The stress at the top (bottom) will be zero in a bottom (top) Ekman layer and therefore, from (2.288),

$$\boxed{\begin{array}{ll} \text{top Ekman layer:} & \boldsymbol{M}_E = -\dfrac{1}{f}\mathbf{k} \times \boldsymbol{\tau}_T \\ \text{bottom Ekman layer:} & \boldsymbol{M}_E = \dfrac{1}{f}\mathbf{k} \times \boldsymbol{\tau}_B \end{array}}. \tag{2.290a,b}$$

The transport is thus at right angles to the stress at the surface, and proportional to the magnitude of the stress. These properties have a simple physical explanation: integrated over the depth of the Ekman layer the surface stress must be balanced by the Coriolis force, which in turn acts at right angles to the mass transport. A consequence of (2.290) is that the mass transports in adjacent oceanic and atmospheric Ekman layers are equal and opposite, because the stress is continuous across the ocean–atmosphere interface. Equation (2.290a) is particularly useful in the ocean, where the stress at the surface is primarily due to the wind, and is largely independent of the interior oceanic flow. In the atmosphere, the surface stress mainly arises as a result of the interior atmospheric flow, and to calculate it we need to parameterize the stress in terms of the flow.

Finally, we obtain an expression for the vertical velocity induced by an Ekman layer. The mass conservation equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2.291}$$

Integrating this over an Ekman layer gives

$$\frac{1}{\rho_0}\nabla \cdot \boldsymbol{M}_{To} = -(w_T - w_B), \tag{2.292}$$

where $\boldsymbol{M}_{To}$ is the total (Ekman plus geostrophic) mass transport in the Ekman layer,

$$\boldsymbol{M}_{To} = \int_{Ek} \rho_0 \boldsymbol{u}\,\mathrm{d}z = \int_{Ek} \rho_0(\boldsymbol{u}_g + \boldsymbol{u}_E)\,\mathrm{d}z \equiv \boldsymbol{M}_g + \boldsymbol{M}_E, \tag{2.293}$$

and w_T and w_B are the vertical velocities at the top and bottom of the Ekman layer; the former (latter) is zero in a top (bottom) Ekman layer. Equations (2.293) and (2.288) give

$$\mathbf{k} \times (\boldsymbol{M}_{To} - \boldsymbol{M}_g) = \frac{1}{f}(\boldsymbol{\tau}_T - \boldsymbol{\tau}_B). \tag{2.294}$$

Taking the curl of this (i.e., cross-differentiating) gives

$$\nabla \cdot (\boldsymbol{M}_{To} - \boldsymbol{M}_g) = \mathrm{curl}_z[(\boldsymbol{\tau}_T - \boldsymbol{\tau}_B)/f], \tag{2.295}$$

where the curl_z operator on a vector $\boldsymbol{A}$ is defined by $\mathrm{curl}_z\boldsymbol{A} \equiv \partial_x A_y - \partial_y A_x$. Using (2.292) we obtain, for top and bottom Ekman layers respectively,

$$\boxed{w_B = \frac{1}{\rho_0}\left(\mathrm{curl}_z \frac{\boldsymbol{\tau}_T}{f} + \nabla \cdot \boldsymbol{M}_g\right), \qquad w_T = \frac{1}{\rho_0}\left(\mathrm{curl}_z \frac{\boldsymbol{\tau}_B}{f} - \nabla \cdot \boldsymbol{M}_g\right)}, \tag{2.296a,b}$$

where $\nabla \cdot \boldsymbol{M}_g = -(\beta/f)\boldsymbol{M}_g \cdot \mathbf{j}$ is the divergence of the geostrophic transport in the Ekman layer, and this is often small compared to the other terms in these equations. Thus, friction induces a vertical velocity at the edge of the Ekman layer, proportional to the curl of the stress at the surface, and this is perhaps the most used result in Ekman layer theory. Numerical models sometimes do not have the vertical resolution to explicitly resolve an Ekman layer, and (2.296) provides a means of *parameterizing* the Ekman layer in terms of resolved or known fields. It is particularly useful for the top Ekman layer in the ocean, where the stress can be regarded as a given function of the overlying wind.

2.12.3 Explicit solutions. I: a bottom boundary layer

We now assume that the frictional terms can be parameterized as an eddy viscosity and calculate the explicit form of the solution in the boundary layer. The frictional–geostrophic balance may be written as

$$\boldsymbol{f} \times (\boldsymbol{u} - \boldsymbol{u}_g) = A\frac{\partial^2 \boldsymbol{u}}{\partial z^2}, \tag{2.297a}$$

where

$$f(u_g, v_g) = \left(-\frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial x}\right). \tag{2.297b}$$

We continue to assume there are no horizontal gradients of temperature, so that, via thermal wind, $\partial u_g/\partial z = \partial v_g/\partial z = 0$.

Boundary conditions and solution

Appropriate boundary conditions for a bottom Ekman layer are

$$\text{at } z = 0: \qquad u = 0, \quad v = 0 \qquad \text{(the no slip condition)} \tag{2.298a}$$

$$\text{as } z \to \infty: \qquad u = u_g, \quad v = v_g \qquad \text{(a geostrophic interior).} \tag{2.298b}$$

Let us seek solutions to (2.297a) of the form

$$u = u_g + A_0 e^{\alpha z}, \qquad v = v_g + B_0 e^{\alpha z}, \tag{2.299}$$

where A_0 and B_0 are constants. Substituting into (2.297a) gives two homogeneous algebraic equations

$$A_0 f - B_0 A \alpha^2 = 0, \qquad -A_0 A \alpha^2 - B_0 f = 0. \tag{2.300a,b}$$

For non-trivial solutions the solvability condition $\alpha^4 = -f^2/A^2$ must hold, from which we find $\alpha = \pm(1 \pm \mathrm{i})\sqrt{f/2A}$. Using the boundary conditions we then obtain the solution

$$u = u_g - e^{-z/d}\left[u_g \cos(z/d) + v_g \sin(z/d)\right] \tag{2.301a}$$

$$v = v_g + e^{-z/d}\left[u_g \sin(z/d) - v_g \cos(z/d)\right], \tag{2.301b}$$

where $d = \sqrt{2A/f}$ is, within a constant factor, the depth of the Ekman layer obtained from scaling considerations. The solution decays exponentially from the surface with this e-folding scale, so that d is a good measure of the Ekman layer thickness. Note that the boundary layer correction depends on the interior flow, since the boundary layer serves to bring the flow to zero at the surface.

To illustrate the solution, suppose that the pressure force is directed in the y-direction (northwards), so that the geostrophic current is eastwards. Then the solution, the now-famous *Ekman spiral*, is plotted in Figs. 2.11 and 2.12. The wind falls to zero at the surface, and its direction just above the surface is northeastwards; that is, it is rotated by 45° to the left of its direction in the free atmosphere. Although this result is independent of the value of the frictional coefficients, it is dependent on the form of the friction chosen. The force balance in the Ekman layer is between the Coriolis force, the stress, and the pressure force. At the surface the Coriolis force is zero, and the balance is entirely between the northward pressure force and the southward stress force.

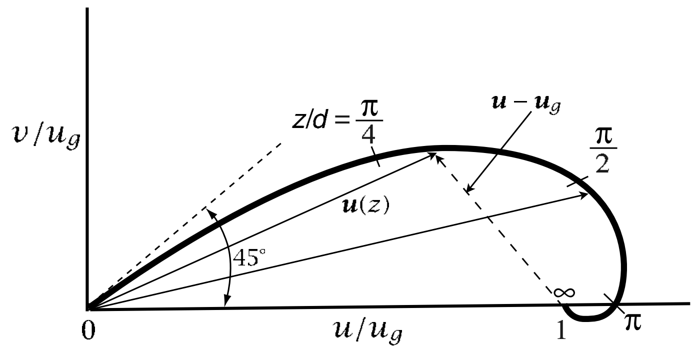

Fig. 2.11 The idealized Ekman layer solution in the lower atmosphere, plotted as a hodograph of the wind components: the arrows show the velocity vectors at a particular heights, and the curve traces out the continuous variation of the velocity. The values on the curve are of the non-dimensional variable z/d, where $d = (2A/f)^{1/2}$, and v_g is chosen to be zero.

Transport, force balance and vertical velocity

The cross-isobaric flow is given by (for $v_g = 0$)

$$V = \int_0^\infty v \, \mathrm{d}z = \int_0^\infty u_g \mathrm{e}^{-z/d} \sin(z/d) \, \mathrm{d}z = \frac{u_g d}{2}. \tag{2.302}$$

For positive f, this is to the left of the geostrophic flow — that is, down the pressure gradient. In the general case ($v_g \neq 0$) we obtain

$$V = \int_0^\infty (v - v_g) \, \mathrm{d}z = \frac{d}{2}(u_g - v_g). \tag{2.303}$$

Similarly, the additional zonal transport produced by frictional effects are, for $v_g = 0$,

$$U = \int_0^\infty (u - u_g) \, \mathrm{d}z = -\int_0^\infty \mathrm{e}^{-z/d} \sin(z/d) \, \mathrm{d}z = -\frac{u_g d}{2}, \tag{2.304}$$

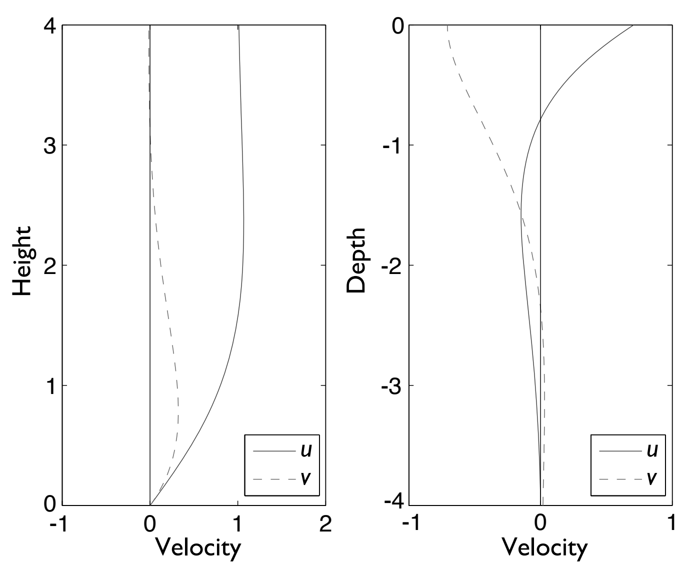

Fig. 2.12 Solutions for a bottom Ekman layer with a given flow in the fluid interior (left), and for a top Ekman layer with a given surface stress (right), both with $d = 1$. On the left we have $u_g = 1$, $v_g = 0$. On the right we have $u_g = v_g = 0$, $\tilde{\tau}_y = 0$ and $\sqrt{2}\tilde{\tau}_x/(fd) = 1$.

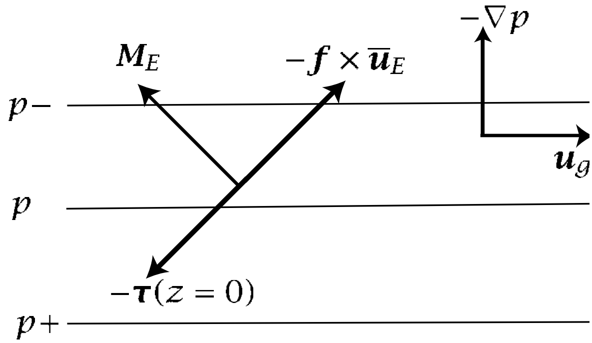

Fig. 2.13 A bottom Ekman layer, generated from an eastward geostrophic flow above it. An overbar denotes a vertical integral over the Ekman layer, so that $-\boldsymbol{f} \times \overline{\boldsymbol{u}}_E$ is the Coriolis force on the vertically integrated Ekman velocity. $\boldsymbol{M}_E$ is the frictionally induced boundary layer transport, and $\boldsymbol{\tau}$ is the stress.

and in the general case

$$U = \int_0^\infty (u - u_g)\,\mathrm{d}z = -\frac{d}{2}(u_g + v_g). \tag{2.305}$$

Thus, the total transport caused by frictional forces is

$$\boldsymbol{M}_E = \frac{\rho_0 d}{2}\left[-\mathbf{i}(u_g + v_g) + \mathbf{j}(u_g - v_g)\right]. \tag{2.306}$$

The total stress at the bottom surface $z = 0$ induced by frictional forces is

$$\widetilde{\boldsymbol{\tau}}_B = A\frac{\partial \boldsymbol{u}}{\partial z}\bigg|_{z=0} = \frac{A}{d}\left[\mathbf{i}(u_g - v_g) + \mathbf{j}(u_g + v_g)\right], \tag{2.307}$$

using the solution (2.301). Thus, using (2.306), (2.307) and $d^2 = 2A/f$, we see that the total frictionally induced transport in the Ekman layer is related to the stress at the surface by $\boldsymbol{M}_E = (\mathbf{k} \times \boldsymbol{\tau}_B)/f$, reprising the result of the previous more general analysis, (2.296). From (2.307), the stress is at an angle of 45° to the left of the velocity at the surface. (However, this result is not generally true for all forms of stress.) These properties are illustrated in Fig. 2.13.

The vertical velocity at the top of the Ekman layer, w_E, is obtained using (2.306) or (2.307). If f if constant we obtain

$$w_E = -\frac{1}{\rho_0}\nabla \cdot \boldsymbol{M}_E = \frac{1}{f_0}\mathrm{curl}_z\widetilde{\boldsymbol{\tau}}_B = \frac{d}{2}\zeta_g, \tag{2.308}$$

where ζ_g is the vorticity of the geostrophic flow. Thus, the vertical velocity at the top of the Ekman layer, which arises because of the frictionally-induced divergence of the cross-isobaric flow in the Ekman layer, is proportional to the geostrophic vorticity in the free fluid and is proportional to the Ekman layer height $\sqrt{2A/f_0}$.

Another bottom boundary condition

In the analysis above we assumed a *no slip* condition at the surface, namely that the velocity tangential to the surface vanishes. This is certainly appropriate if A is a molecular velocity, but in a turbulent flow, where A is interpreted as an eddy viscosity, the flow very close to the surface may be far from zero. Then, unless we wish to explicitly calculate the flow in an additional very thin viscous boundary layer the no-slip condition may be inappropriate. An alternative, slightly more general boundary condition is to suppose that the stress at the surface is given by

$$\boldsymbol{\tau} = \rho_0 C \boldsymbol{u}, \tag{2.309}$$

where C is a constant. The surface boundary condition is then

$$A\frac{\partial \boldsymbol{u}}{\partial z} = C\boldsymbol{u}. \tag{2.310}$$

If C is infinite we recover the no-slip condition. If $C = 0$, we have a condition of no stress at the surface, also known as a *free slip* condition. For intermediate values of C the boundary condition is known as a 'mixed condition'. Evaluating the solution in these cases is left as an exercise for the reader (problem 2.28).

2.12.4 Explicit solutions. II: the upper ocean

Boundary conditions and solution

The wind provides a stress on the upper ocean, and the Ekman layer serves to communicate this to the oceanic interior. Appropriate boundary conditions are thus:

$$\text{at } z = 0: \qquad A\frac{\partial u}{\partial z} = \tilde{\tau}^x, \quad A\frac{\partial v}{\partial z} = \tilde{\tau}^y \qquad \text{(a given surface stress)} \tag{2.311a}$$

$$\text{as } z \to -\infty: \qquad u = u_g, \quad v = v_g \qquad \text{(a geostrophic interior)} \tag{2.311b}$$

where $\tilde{\boldsymbol{\tau}}$ is the given (kinematic) wind stress at the surface. Solutions to (2.297a) with (2.311) are found by the same methods as before, and are

$$u = u_g + \frac{\sqrt{2}}{fd}\mathrm{e}^{z/d}\left[\tilde{\tau}^x \cos(z/d - \pi/4) - \tilde{\tau}^y \sin(z/d - \pi/4)\right], \tag{2.312}$$

and

$$v = v_g + \frac{\sqrt{2}}{fd}\mathrm{e}^{z/d}\left[\tilde{\tau}^x \sin(z/d - \pi/4) + \tilde{\tau}^y \cos(z/d - \pi/4)\right]. \tag{2.313}$$

Note that the boundary layer correction depends only on the imposed surface stress, and not the interior flow itself. This is a consequence of the type of boundary conditions chosen, for in the absence of an imposed stress the boundary layer correction is zero — the interior flow already satisfies the gradient boundary condition at the top surface. Similar to the bottom boundary layer, the velocity vectors of the solution trace a diminishing spiral as they descend into the interior (Fig. 2.14, which is drawn for the Southern Hemisphere).

Transport, surface flow and vertical velocity

The transport induced by the surface stress is obtained by integrating (2.312) and (2.313) from the surface to $-\infty$. We explicitly find

$$U = \int_{-\infty}^{0} (u - u_g)\,\mathrm{d}z = \frac{\tilde{\tau}^y}{f}, \qquad V = \int_{-\infty}^{0} (v - v_g)\,\mathrm{d}z = -\frac{\tilde{\tau}^x}{f}, \tag{2.314}$$

which indicates that the ageostrophic transport is perpendicular to the wind stress, as noted previously from more general considerations. Suppose that the surface wind is eastward; in this case $\tilde{\tau}^y = 0$ and the solutions immediately give

$$u(0) - u_g = \tilde{\tau}^x/fd, \qquad v(0) - v_g = -\tilde{\tau}^x/fd. \tag{2.315}$$

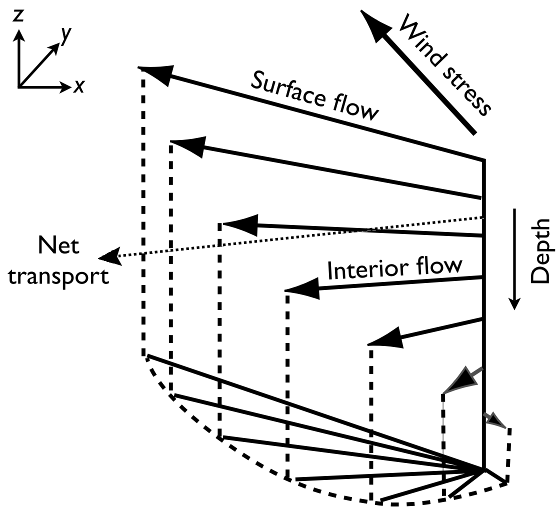

Fig. 2.14 An idealized Ekman spiral in a southern hemisphere ocean, driven by an imposed wind stress. A northern hemisphere spiral would be the reflection of this about the vertical axis. Such a clean spiral is rarely observed in the real ocean. The net transport is at right angles to the wind, independent of the detailed form of the friction. The angle of the surface flow is 45° to the wind only for a Newtonian viscosity.

Therefore the magnitudes of the frictional flow in the x and y directions are equal to each other, and the ageostrophic flow is 45° to the right (for $f > 0$) of the wind. This result depends on the form of the frictional parameterization, but not on the size of the viscosity.

At the edge of the Ekman layer the vertical velocity is given by (2.296), and so is proportional to the curl of the wind stress. (The second term on the right-hand side of (2.296) is the vertical velocity due to the divergence of the geostrophic flow, and is usually much smaller than the first term.) The production of a vertical velocity at the edge of the Ekman layer is one of most important effects of the layer, especially with regard to the large-scale circulation, for it provides an efficient means whereby surface fluxes are communicated to the interior flow (see Fig. 2.15).

2.12.5 Observations of the Ekman layer

Ekman layers — and in particular the Ekman spiral — are generally quite hard to observe, in either the ocean or atmosphere, both because of the presence of phenomena that are not included in the theory, and because of the technical difficulties of actually measuring the vector velocity profile, especially in the ocean. Ekman-layer theory does not take into account the effects of stratification or of inertial and gravity waves (section 2.10), nor does it account for the effects of convection or buoyancy-driven turbulence. If gravity waves are present, the instantaneous flow will be non-geostrophic and so time-averaging will be required to extract the geostrophic flow. If strong convection is present, the simple eddy-viscosity parameterizations used to derive the Ekman spiral will be rendered invalid, and the spiral Ekman profile cannot be expected to be observed in either atmosphere or ocean.

In the atmosphere, in convectively neutral cases, the Ekman profile can sometimes be qualitatively discerned. In convectively unstable situations the Ekman profile is generally not observed, but the flow is nevertheless cross-isobaric, from high pressure to low, consistent with the theory. (For most purposes, it is in any case the integral properties of the Ekman layer that is most important.) In the ocean, from about 1980 onwards improved instruments have made it possible to observe the vector current with depth, and to average that current

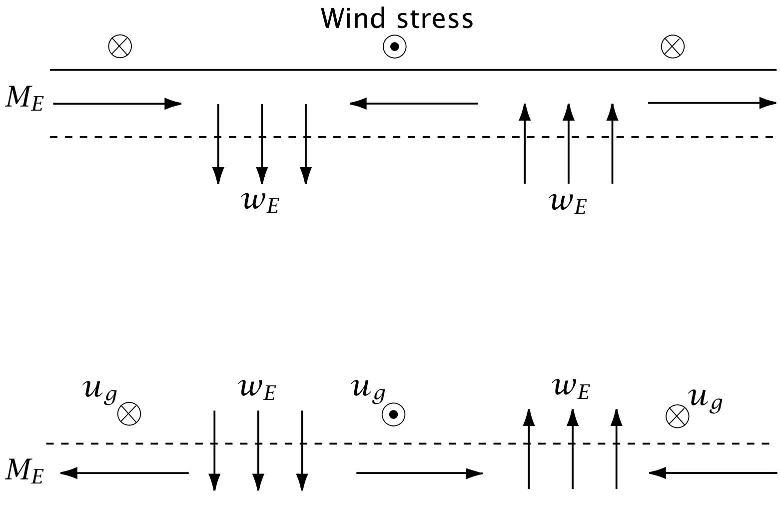

Fig. 2.15 Upper and lower Ekman layers. The upper Ekman layer in the ocean is primarily driven by an imposed wind stress, whereas the lower Ekman layer in the atmosphere or ocean largely results from the interaction of interior geostrophic velocity and a rigid lower surface. The upper part of figure shows the vertical Ekman 'pumping' velocities that result from the given wind stress, and the lower part of the figure shows the Ekman pumping velocities given the interior geostrophic flow.

and correlate it with the overlying wind, and a number of observations generally consistent with Ekman dynamics have emerged.[13] There are some differences between observations and theory, and these can be ascribed to the effects of stratification (which causes a shallowing and flattening of the spiral), and to the interaction of the Ekman spiral with turbulence (and the inadequacy of the eddy-diffusivity parameterization). In spite of these differences, Ekman layer theory remains a remarkable and enduring foundation of geophysical fluid dynamics.

2.12.6 * Frictional parameterization of the Ekman layer

[Some readers will be reading these sections on Ekman layers after having been introduced to quasi-geostrophic theory; this section is for them. Other readers may return to this section after reading chapter 5, or take (2.316) on faith.]

Suppose that the free atmosphere is described by the quasi-geostrophic vorticity equation,

$$\frac{D\zeta_g}{Dt} = f_0 \frac{\partial w}{\partial z}, \tag{2.316}$$

where ζ_g is the geostrophic relative vorticity. Let us further model the atmosphere as a single homogeneous layer of thickness H lying above an Ekman layer of thickness $d \ll H$. If the vertical velocity is negligible at the top of the layer (at $z = H + d$) the equation of motion becomes

$$\frac{D\zeta_g}{Dt} = \frac{f_0[w(H+d) - w(d)]}{H} = -\frac{f_0 d}{2H}\zeta_g \tag{2.317}$$

using (2.308). This equation shows that the Ekman layer acts as a *linear drag* on the interior flow, with a drag coefficient r equal to $f_0 d/2H$ and with associated time scale T_{Ek} given by

$$T_{Ek} = \frac{2H}{f_0 d} = \frac{2H}{\sqrt{2 f_0 A}}. \tag{2.318}$$

In the oceanic case the corresponding vorticity equation for the interior flow is

$$\frac{D\zeta_g}{Dt} = \frac{1}{H}\mathrm{curl}_z \boldsymbol{\tau}_s, \tag{2.319}$$

where $\boldsymbol{\tau}_s$ is the surface stress. The surface stress thus acts as if it were a body force on the interior flow, and neither the Coriolis parameter nor the depth of the Ekman layer explicitly appear in this formula.

The Ekman layer is a very efficient way of communicating surface stresses to the interior. To see this, suppose that eddy mixing were the sole mechanism of transferring stress from the surface to the fluid interior, and there were no Ekman layer. Then the time scale of spindown of the fluid would be given by using

$$\frac{d\zeta}{dt} = A\frac{\partial^2\zeta}{\partial z^2}, \tag{2.320}$$

implying a turbulent spin-down time, T_{turb}, of

$$T_{turb} \sim \frac{H^2}{A}, \tag{2.321}$$

where H is the depth over which we require a spin-down. This is much longer than the spin-down of a fluid that has an Ekman layer, for we have

$$\frac{T_{turb}}{T_{Ek}} = \frac{(H^2/A)}{(2H/f_0 d)} = \frac{H}{d} \gg 1, \tag{2.322}$$

using $d = \sqrt{2A/f_0}$. The effects of friction are enhanced because of the presence of a secondary circulation confined to the Ekman layers (as in Fig. 2.15) in which the vertical scales are much smaller than those in the fluid interior and so where viscous effects become significant; these frictional stresses are then communicated to the fluid interior via the induced vertical velocities at the edge of the Ekman layers.

Notes

1 The distinction between Coriolis force and acceleration is not always made in the literature. For a fluid in geostrophic balance, one might either say that there is a balance between the pressure force and the Coriolis force, with no net acceleration, or that the pressure force produces a Coriolis acceleration. The descriptions are equivalent, because of Newton's second law, but should not be conflated.

The Coriolis forces is named after Gaspard Gustave de Coriolis (1792–1843), who introduced the force in the context of rotating mechanical systems (Coriolis 1832, 1835). See Persson (1998) for a historical account and interpretation.

2 Phillips (1973). There is also a related discussion in Stommel & Moore (1989).

3 Phillips (1966). See White (2003) for a review. In the early days of numerical modelling these equations were the most primitive — i.e., the least filtered — equations that could practically be integrated numerically. Associated with increasing computer power there is a tendency for comprehensive numerical models to use non-hydrostatic equations of motion that do not make the shallow-fluid or traditional approximations, and it is conceivable that the meaning of the word 'primitive' may evolve to accommodate them.

4 The Boussinesq approximation is named for Boussinesq (1903), although similar approximations were used earlier by Oberbeck (1879, 1888). Spiegel & Veronis (1960) give a physically based derivation for an ideal gas, and Mihaljan (1962) provides a somewhat more general asymptotic derivation. Mahrt (1986) discusses applicability to the atmosphere.

5 As pointed out to me by W. R. Young.

6 Various versions of anelastic equations exist — see Batchelor (1953a), Ogura & Phillips (1962), Gough (1969), Gilman & Glatzmaier (1981), Lipps & Hemler (1982), and Durran (1989), although not all have potential vorticity and energy conservation laws (Bannon 1995, 1996; Scinocca & Shepherd 1992). The system we derive is most similar to that of Ogura & Phillips (1962) and unpublished notes by J. S. A. Green. The connection between the Boussinesq and anelastic equations is discussed by, among others, Lilly (1996) and Ingersoll (2005).

7 A numerical model that includes sound waves must take very small timesteps in order to maintain numerical stability, in particular to satisfy the Courant–Friedrichs–Lewy (CFL) criterion. An alternative is to use an implicit timestepping scheme that effectively lets the numerics do the filtering of the sound waves, and this approach is favoured by many numerical modellers. If we make the hydrostatic approximation then all sound waves except those that propagate horizontally are eliminated, and there is little need, vis-à-vis the numerics, to also make the anelastic approximation.

8 The Rossby number is named for C.-G. Rossby (see endnote on page 241), but it was also used by Kibel (1940) and is sometimes called the Kibel or Rossby-Kibel number. The notion of geostrophic balance and so, implicitly, that of a small Rossby number, predates either Rossby or Kibel.

9 After Taylor (1921b) and Proudman (1916). The Taylor–Proudman effect is sometimes called the Taylor–Proudman 'theorem', but it is more usefully thought of as a physical effect, with manifestations even when the conditions for its satisfaction are not precisely met.

10 Foster (1972).

11 Many numerical models of the large-scale circulation in the atmosphere and ocean do make the hydrostatic approximation. In these models convection must be *parameterized;* otherwise, it would simply occur at the smallest scale available, namely the size of the numerical grid, and this type of unphysical behaviour should be avoided. Of course in non-hydrostatic models convection must also be parameterized if the horizontal resolution of the model is too coarse to properly resolve the convective scales. See also problem 2.11.

12 After Ekman (1905). The problem was posed to V. W. Ekman (1874-1954), a student of Vilhelm Bjerknes, by Fridtjof Nansen, the polar explorer and statesman, who wanted to understand the motion of pack ice and of his ship, the *Fram*, embedded in the ice.

13 For oceanic observations see Davis *et al.* (1981), Price *et al.* (1987), Rudnick & Weller (1993). For the atmosphere see, e.g., Nicholls (1985).

Further reading

Cushman-Roisin, B., 1994. *An Introduction to Geophysical Fluid Dynamics.*
Provides a compact introduction to a variety of topics in GFD.

Gill, A. E., 1982. *Atmosphere–Ocean Dynamics.*
A rich book, especially strong on equatorial dynamics and gravity wave motion.

Holton, J. R., 1992. *An Introduction to Dynamical Meteorology.*
A deservedly well-known textbook at the upper-division undergraduate/beginning graduate level.

Pedlosky, J., 1987. *Geophysical Fluid Dynamics.*
A primary reference for flow at low Rossby number. Although the book requires some effort, there is a handsome pay-off for those who study it closely.

White (2002) provides a clear and thorough summary of the equations of motion for meteorology, including the non-hydrostatic and primitive equations.

Zdunkowski, W. & Bott, A., 2003. *Dynamics of the Atmosphere: A Course in Theoretical Meteorology.*
Concentrates on the mathematical tools and equations of motion of dynamical meteorology.

Problems

2.1 Show that for an ideal gas in hydrostatic balance, changes in dry static energy ($M = c_p T + gz$) and potential temperature (θ) are related by $\delta M = c_p(T/\theta)\delta\theta$. (The quantity $c_p T/\theta$ is known as the 'Exner function', and is denoted Π.)

2.2 For an ideal gas in hydrostatic balance, show that:

(a) The integral of the potential plus internal energy from the surface to the top of the atmosphere $[\int (P + I)\,\mathrm{d}p]$ is is equal to its enthalpy;

(b) $\mathrm{d}\sigma/\mathrm{d}z = c_p(T/\theta)\,\mathrm{d}\theta/\mathrm{d}z$, where $\sigma = I + p\alpha + \Phi$ is the dry static energy;

(c) The following expressions for the pressure gradient force are all equal (even without hydrostatic balance):

$$-\frac{1}{\rho}\nabla p = -\theta\,\nabla\Pi = -\frac{c_s^2}{\rho\theta}\nabla(\rho\theta), \tag{P2.1}$$

where $\Pi = c_p T/\theta$ is the Exner function.

(d) Show that item *(a)* also holds for a gas with an arbitrary equation of state.

2.3 Show that, without approximation, the unforced, inviscid momentum equation may be written in the forms

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = T\nabla\eta - \nabla(p\alpha + I) \tag{P2.2}$$

and

$$\frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{\omega}\times\boldsymbol{v} = T\nabla\eta - \nabla B \tag{P2.3}$$

where $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$, η is the specific entropy ($\mathrm{d}\eta = c_p\,\mathrm{d}\ln\theta$) and $B = I + \boldsymbol{v}^2/2 + p\alpha$ where I is the internal energy per unit mass.
Hint: First show that $T\nabla\eta = \nabla I + p\nabla\alpha$, and note also the vector identity $\boldsymbol{v}\times(\nabla\times\boldsymbol{v}) = \frac{1}{2}\nabla(\boldsymbol{v}\cdot\boldsymbol{v}) - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$.

2.4 Consider two-dimensional fluid flow in a rotating frame of reference on the f-plane. Linearize the equations about a state of rest.

(a) Ignore the pressure term and determine the general solution to the resulting equations. Show that the speed of fluid parcels is constant. Show that the trajectory of the fluid parcels is a circle with radius $|U|/f$, where $|U|$ is the fluid speed.

(b) What is the period of oscillation of a fluid parcel?

(c) ♦ If parcels travel in straight lines in inertial frames, why is the answer to (b) not the same as the period of rotation of the frame of reference? [To answer this fully you need to understand the dynamics underlying inertial oscillations and inertia circles. See Durran (1993), Egger (1999) and Phillips (2000).]

2.5 A fluid at rest evidently satisfies the hydrostatic relation, which says that the pressure at the surface is given by the weight of the fluid above it. Now consider a *deep* atmosphere on a spherical planet. A unit cross-sectional area at the planet's surface lies beneath a column of fluid whose cross-section increases with height, because the total area of the atmosphere increases with distance away from the centre of the planet. Is the pressure at the surface still given by the hydrostatic relation, or is it greater than this because of the increased mass of fluid in the column? If it is still given by the hydrostatic relation, then the pressure at the surface, integrated over the entire area of the planet, is less than the total weight of the fluid; resolve this paradox. But if the pressure at the surface is greater than that implied by hydrostatic balance, explain how the hydrostatic relation fails.

2.6 In a self-gravitating spherical fluid, like a star, hydrostatic balance may be written

$$\frac{\partial p}{\partial r} = -\frac{GM(r)}{r^2}\rho, \tag{P2.4}$$

where $M(r)$ is the mass interior to a sphere of radius r, and G is a constant. Obtain an expression for the pressure as a function of radius when the fluid (a) has constant density, and (b) is an isothermal ideal gas (if possible). The star is of radius a.

2.7 At what latitude is the angle between the direction of Newtonian gravity (due solely to the mass of the Earth) and that of effective gravity (Newtonian gravity plus centrifugal terms) the largest? At what latitudes, if any, is this angle zero?

2.8 ♦ Write the momentum equations in true spherical coordinates, including the centrifugal and gravitational terms. Show that for reasonable values of the wind, the dominant balance in the meridional component of this equation involves a balance between centrifugal and pressure gradient terms. Can this balance be subtracted out of the equations in a sensible way, so leaving a useful horizontal momentum equation that involves the Coriolis and acceleration terms? If so, obtain a closed set of equations for the flow this way. Discuss the pros and cons of this approach versus the geometric approximation discussed in section 2.2.1.

2.9 For an ideal gas show that the expressions (2.225) and (2.229) are equivalent.

2.10 Consider an ocean at rest with known vertical profiles of potential temperature and salinity, $\theta(z)$ and $S(z)$. Suppose we also know the equation of state in the form $\rho = \rho(\theta, S, p)$. Obtain an expression for the buoyancy frequency. Check your expression by substituting the equation of state for an ideal gas and recovering a known expression for the buoyancy frequency.

2.11 *Convection and its parameterization*

(a) Consider a Boussinesq system in which the vertical momentum equation is modified by the parameter α to read

$$\alpha^2 \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial \phi}{\partial z} + b, \tag{P2.5}$$

and the other equations are unchanged. (If $\alpha = 0$ the system is hydrostatic, and if $\alpha = 1$ the system is the original one.) Linearize these equations about a state of rest and of constant stratification (as in section 2.10.1) and obtain the dispersion relation for the system, and plot it for various values of α, including 0 and 1. Show that for $\alpha > 1$ the system approaches its limiting frequency more rapidly than with $\alpha = 1$.

(b) ♦ Argue that if $N^2 < 0$, convection in a system with $\alpha > 1$ generally occurs at a larger scale than with $\alpha = 1$. Show this explicitly by adding some diffusion or friction to the

right-hand sides of the equations of motion and obtaining the dispersion relation. You may do this approximately.

2.12 *(a)* The *geopotential height* is the height of a given pressure level. Show that in an atmosphere with a uniform lapse rate (i.e., $\mathrm{d}T/\mathrm{d}z = \Gamma =$ constant) the geopotential height at a pressure p is given by

$$z = \frac{T_0}{\Gamma}\left[1 - \left(\frac{p_0}{p}\right)^{-R\Gamma/g}\right] \tag{P2.6}$$

where T_0 is the temperature at $z = 0$.

(b) In an isothermal atmosphere, obtain an expression for the geopotential height as function of pressure, and show that this is consistent with the expression (P2.6) in the appropriate limit.

2.13 Show that the inviscid, adiabatic, hydrostatic primitive equations for a compressible fluid conserve a form of energy (kinetic plus potential plus internal), and that the kinetic energy has no contribution from the vertical velocity. You may assume Cartesian geometry and a uniform gravitational field in the vertical direction.

2.14 Consider the simple Boussinesq equations, $\mathrm{D}\boldsymbol{v}/\mathrm{D}t = -\nabla p + \mathbf{k}b + \nu\nabla^2\boldsymbol{v}$, $\nabla\cdot\boldsymbol{v} = 0$, $\mathrm{D}b/\mathrm{D}t = Q + \kappa\nabla^2 b$. Obtain an energy equation similar to (2.112), but now with the terms on the right-hand side that represent viscous and diabatic effects. Over a closed volume, show that the dissipation of kinetic energy is balanced by a buoyancy source. Also show that, in a statistically steady state, the heating must occur at a lower level than the cooling if a kinetic-energy dissipating circulation is to be maintained.

2.15 ♦ Suppose a fluid is contained in a closed container, with insulating sidewalls, and heated from below and cooled from above. The heating and cooling are adjusted so that there is no net energy flux into the fluid. Let us also suppose that any viscous dissipation of kinetic energy is returned as heating, so the total energy of the fluid is exactly constant. Suppose the fluid starts out at rest and at a uniform temperature, and the heating and cooling are then turned on. A very short time afterwards, the fluid is lighter at the bottom and heavier at the top; that is, its potential energy has increased. Where has this energy come from? Discuss this paradox for both a compressible fluid (e.g., an ideal gas) and for a simple Boussinesq fluid.

2.16 Consider a rapidly rotating (i.e., in near geostrophic balance) Boussinesq fluid on the f-plane.

(a) Show that the pressure divided by the density scales as $\phi \sim fUL$.

(b) Show that the horizontal divergence of the geostrophic wind vanishes. Thus, argue that the scaling $W \sim UH/L$ is an *overestimate* for the magnitude of the vertical velocity. (Optional extra: obtain a scaling estimate for the magnitude of vertical velocity in rapidly rotating flow.)

(c) Using these results, or otherwise, discuss whether hydrostatic balance is more or less likely to hold in a rotating flow than in non-rotating flow.

2.17 Estimate the size of the zonal wind 5 km above the surface in the mid-latitude atmosphere in summer and winter using (approximate) values for the meridional temperature gradient in the atmosphere. Also estimate the shear corresponding to the pole–equator temperature gradient in the ocean.

2.18 Using approximate but realistic values for the observed stratification, what is the buoyancy period for (a) the mid-latitude troposphere, (b) the stratosphere, (c) the oceanic thermocline, (d) the oceanic abyss?

2.19 Consider a dry, hydrostatic, ideal-gas atmosphere whose lapse rate is one of constant potential temperature. What is its vertical extent? That is, at what height does the density vanish? Is this a problem for the anelastic approximation discussed in the text?

2.20 Show that for an ideal gas, the expressions (2.229), (2.224), (2.225) are all equivalent, and express N^2 terms of the temperature lapse rate, $\partial T/\partial z$.

2.21 ♦ Calculate an approximate but reasonably accurate expression for the buoyancy equation for seawater. (From notes by R. de Szoeke)
Solution (i): the buoyancy frequency is given by

$$N^2 = -\frac{g}{\rho}\left(\frac{\partial \rho_{pot}}{\partial z}\right)_{\text{env}} = \frac{g}{\alpha}\left(\frac{\partial \alpha_{pot}}{\partial z}\right)_{\text{env}} = -\frac{g^2}{\alpha^2}\left(\frac{\partial \alpha_{pot}}{\partial p}\right)_{\text{env}} \tag{P2.7}$$

where $\alpha_{pot} = \alpha(\theta, S, p_R)$ is the potential density, and p_R a reference pressure. From (1.155)

$$\alpha_{pot} = \alpha_0\left[1 - \frac{\alpha_0}{c_0^2}p_R + \beta_T(1+\gamma^* p_R)\theta' + \frac{1}{2}\beta_T^*\theta'^2 - \beta_S(S - S_0)\right]. \tag{P2.8}$$

Using this and (P2.7) we obtain the buoyancy frequency,

$$N^2 = -\frac{g^2}{\alpha^2}\alpha_0\left[\beta_T\left(1 + \gamma p_R + \frac{\beta_T^*}{\beta_T}\theta\right)\left(\frac{\partial \theta}{\partial p}\right)_{\text{env}} - \beta_S\left(\frac{\partial S}{\partial p}\right)_{\text{env}}\right], \tag{P2.9}$$

although we must substitute local pressure for the reference pressure p_R. (Why?)
Solution (ii): the sound speed is given by

$$c_s^{-2} = -\frac{1}{\alpha^2}\left(\frac{\partial \alpha}{\partial p}\right)_{\theta,S} = \frac{1}{\alpha^2}\left(\frac{\alpha_0^2}{c_0^2} - \gamma\alpha_1\theta\right) \tag{P2.10}$$

and, using (P2.7) and (2.229) the square of the buoyancy frequency may be written

$$N^2 = \frac{g}{\alpha}\left(\frac{\partial \alpha}{\partial z}\right)_{\text{env}} - \frac{g^2}{c_s^2} = -\frac{g^2}{\alpha^2}\left[\left(\frac{\partial \alpha}{\partial p}\right)_{\text{env}} + \frac{\alpha^2}{c_s^2}\right]. \tag{P2.11}$$

Using (1.155), (P2.10) and (P2.11) we recover (P2.9), although now with p explicitly in place of p_R.

2.22 *(a)* Use the chain rule to show that the horizontal gradients of a field in height coordinates and in ξ coordinates are related by

$$\nabla_z\Psi = \nabla_\xi\Psi - (\partial\Psi/\partial\xi)(\partial\xi/\partial z)\nabla_\xi z. \tag{P2.12}$$

(b) Show that w, the vertical velocity in height coordinates, may be expressed in ξ coordinates as

$$w = \mathrm{D}z/\mathrm{D}t = (\partial z/\partial t)_\xi + \boldsymbol{u}\cdot\nabla_\xi z + \dot{\xi}\partial z/\partial\xi. \tag{P2.13}$$

(c) Use the above expressions to verify (2.143), the expression for the material derivative in ξ coordinates.

2.23 Begin with the mass conservation in height coordinates, namely $\mathrm{D}\rho/\mathrm{D}t + \rho\nabla\cdot\boldsymbol{v} = 0$. Transform this into pressure coordinates using the chain rule (or otherwise) and derive the mass conservation equation in the form $\nabla_p\cdot\boldsymbol{u} + \partial\omega/\partial p = 0$.

2.24 ♦ Starting with the primitive equations in pressure coordinates, derive the form of the primitive equations of motion in sigma-pressure coordinates. In particular, show that the prognostic equation for surface pressure is,

$$\frac{\partial p_s}{\partial t} + \nabla\cdot(p_s\boldsymbol{u}) + p_s\frac{\partial\dot{\sigma}}{\partial\sigma} = 0 \tag{P2.14}$$

and that hydrostatic balance may be written $\partial\Phi/\partial\sigma = -RT/\sigma$.

2.25 Starting with the primitive equations in pressure coordinates, derive the form of the primitive equations of motion in log-pressure coordinates in which $Z = -H \ln(p/p_r)$ is the vertical coordinate. Here, H is a reference height (e.g., a scale height RT_r/g where T_r is a typical or an average temperature) and p_r is a reference pressure (e.g., 1000 mb). In particular, show that if the 'vertical velocity' is $W = \mathrm{D}Z/\mathrm{D}t$ then $W = -H\omega/p$ and that

$$\frac{\partial \omega}{\partial p} = -\frac{\partial}{\partial p}\left(\frac{pW}{H}\right) = \frac{\partial W}{\partial Z} - \frac{W}{H}. \tag{P2.15}$$

and obtain the mass conservation equation (2.158b). Show that this can be written in the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_s}\frac{\partial}{\partial Z}(\rho_s W) = 0, \tag{P2.16}$$

where $\rho_s = \rho_r \exp(-Z/H)$.

2.26 *(a)* Prove that the argument of the square root in (2.260) is always positive.
Solution: The largest value or the argument occurs when $m = 0$ and $k^2 = 1/(4H^2)$. The argument is then $1 - 4H^2N^2/c_s^2$. But $c_s^2 = \gamma RT_0 = \gamma gH$ and $N^2 = g\kappa/H$ so that $4N^2H^2/c_s^2 = 4\kappa/\gamma \approx 0.8$.

(b) ♦ This argument seems to depend on the parameters in the ideal gas equation of state. Is it more general than this? Is a natural system possible for which the argument is negative, and if so what physical interpretation could one ascribe to the situation?

2.27 Consider a wind stress imposed by a mesoscale cyclonic storm (in the atmosphere) given by

$$\boldsymbol{\tau} = -A\mathrm{e}^{-(r/\lambda)^2}(y\,\mathbf{i} - x\,\mathbf{j}) \tag{P2.17}$$

where $r^2 = x^2 + y^2$, and A and λ are constants. Also assume constant Coriolis gradient $\beta = \partial f/\partial y$ and constant ocean depth H. In the ocean, find (a) the Ekman transport, (b) the vertical velocity $w_E(x, y, z)$ below the Ekman layer, (c) the northward velocity $v(x, y, z)$ below the Ekman layer and (d) indicate how you would find the westward velocity $u(x, y, z)$ below the Ekman layer.

2.28 ♦ In an atmospheric Ekman layer on the f-plane let us write the momentum equation as

$$\boldsymbol{f} \times \boldsymbol{u} = -\nabla\phi + \frac{1}{\rho_a}\frac{\partial \boldsymbol{\tau}}{\partial z}, \tag{P2.18}$$

where $\boldsymbol{\tau} = A\rho_a \partial \boldsymbol{u}/\partial z$ and A is a constant eddy viscosity coefficient. An *independent* formula for the stress at the ground is $\boldsymbol{\tau} = C\rho_a \boldsymbol{u}$, where C is a constant. Let us take $\rho_a = 1$, and assume that in the free atmosphere the wind is geostrophic and zonal, with $\boldsymbol{u}_g = U\mathbf{i}$.

(a) Find an expression for the wind vector at the ground. Discuss the limits $C = 0$ and $C = \infty$. Show that when $C = 0$ the frictionally-induced vertical velocity at the top of the Ekman layer is zero.

(b) Find the vertically integrated horizontal mass flux caused by the boundary layer.

(c) When the stress on the atmosphere is $\boldsymbol{\tau}$, the stress on the ocean beneath is also $\boldsymbol{\tau}$. Why? Show how this consistent with Newton's third law.

(d) Determine the direction and strength of the surface current, and the mass flux in the oceanic Ekman layer, in terms of the geostrophic wind in the atmosphere, the oceanic Ekman depth and the ratio ρ_a/ρ_o, where ρ_o is the density of the seawater. Include a figure showing the directions of the various winds and currents. How does the boundary-layer mass flux in the ocean compare to that in the atmosphere? (Assume, as needed, that the stress in the ocean may be parameterized with an eddy viscosity.)

Partial solution for (a): A useful trick in Ekman layer problems is to write the velocity as a complex number, $\hat{u} = u + \mathrm{i}v$ and $\hat{u}_g = u_g + \mathrm{i}v_g$. The Ekman layer equation, (2.297a), may then be written as

$$A\frac{\partial^2 \widehat{U}}{\partial z^2} = \mathrm{i}f\widehat{U}, \tag{P2.19}$$

where $\widehat{U} = \hat{u} - \hat{u}_g$. The solution to this is

$$\hat{u} - \hat{u}_g = [\hat{u}(0) - \hat{u}_g]\,\mathrm{exp}\left[-\frac{(1+\mathrm{i})z}{d}\right], \tag{P2.20}$$

where $d = \sqrt{2A/f}$ and the boundary condition of finiteness at infinity eliminates the exponentially growing solution. The boundary condition at $z = 0$ is $\partial\hat{u}/\partial z = (C/A)\hat{u}$; applying this gives $[\hat{u}(0) - \hat{u}_g]\,\mathrm{exp}(\mathrm{i}\pi/4) = -Cd\hat{u}(0)/(\sqrt{2}A)$, from which we obtain $\hat{u}(0)$, and the rest of the solution follows. We may also obtain a solution using the same method that was used to obtain (2.301).

2.29 *The logarithmic boundary layer*

Close to ground rotational effects are unimportant and small-scale turbulence generates a *mixed layer.* In this layer, assume that the stress is constant and that it can be parameterized by an eddy diffusivity the size of which is proportional to the distance from the surface. Show that the velocity then varies logarithmically with height.

Solution: Write the stress as $\tau = \rho_0 u^{*2}$ where the constant u^* is called the friction velocity. Using the eddy diffusivity hypothesis this stress is given by

$$\tau = \rho_0 u^{*2} = \rho_0 A\frac{\partial u}{\partial z} \quad \text{where} \quad A = u^* k z, \tag{P2.21}$$

where k is von Karman's ('universal') constant (approximately equal to 0.4). From (P2.21) we have $\partial u/\partial z = u^*/(kz)$ which integrates to give $u = (u^*/k)\ln(z/z_0)$. The parameter z_0 is known as the roughness length.