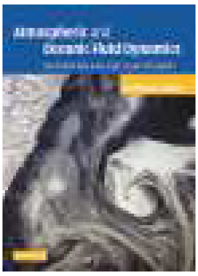

Atmospheric and Oceanic Fluid Dynamics

Fundamentals and Large-scale Circulation

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Chapter

11 - The Overturning Circulation: Hadley and Ferrel Cells pp. 451-484

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*I think the causes of the general trade-winds have not been fully explained by any of those who have wrote on that subject... That the action of the Sun is the original cause of these Winds, I think all are agreed.*

George Hadley, *Concerning the Cause of the General Trade Winds,* 1735.

CHAPTER

ELEVEN

# The Overturning Circulation: Hadley and Ferrel Cells

IN THIS CHAPTER AND THE TWO FOLLOWING we discuss the large-scale circulation, and in particular the *general circulation,* of the atmosphere, this being the mean flow on scales from the synoptic eddy scale — about 1000 km — to the global scale. In this chapter we focus on the dynamics of the Hadley Cell and then, rather descriptively, on the mid-latitude overturning cell or the Ferrel Cell. The latter provides a starting point for chapter 12 which discusses the dynamics of the extratropical zonally averaged circulation. Finally, in chapter 13, we consider the deviations from zonal symmetry, or more specifically the stationary wave pattern, and the stratosphere. In these three chapters we will use many of the tools developed in the previous chapters, but those readers who already have some acquaintance with geophysical fluid dynamics may simply wish to jump in here.

The atmosphere is a terribly complex system, and we cannot hope to fully explain its motion as the analytic solution to a small set of equations. Rather, a full understanding of the atmosphere requires describing it in a consistent way on many levels simultaneously. One of these levels involves simulating the flow by numerically solving the governing equations of motion as completely as possible, for example by using a comprehensive General Circulation Model (GCM). However, such a simulation brings problems of its own, including the problem of understanding the simulation, and discerning whether it is a good representation of reality. Thus, in this chapter and the two following we concentrate on simpler, more conceptual models. We begin this chapter with a brief observational overview of some of the pre-eminent large-scale features of the atmosphere, concentrating on the zonally averaged fields.[1]

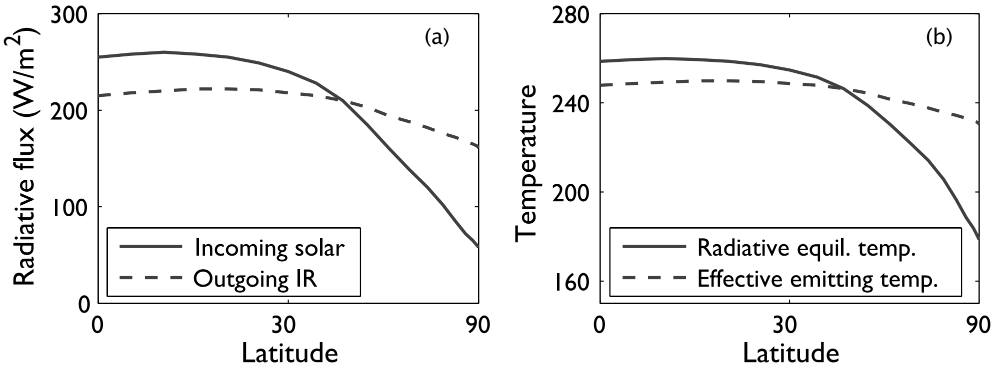


**Fig. 11.1** (a) The (approximate) observed net average incoming solar radiation and outgoing infrared radiation at the top of the atmosphere, as a function of latitude (plotted on a sine scale). (b) The temperatures associated with these fluxes, calculated using $T = (R/\sigma)^{1/4}$, where $R$ is the solar flux for the radiative equilibrium temperature and $R$ is the infrared flux for the effective emitting temperature. Thus, the solid line is an approximate radiative equilibrium temperature

## 11.1 BASIC FEATURES OF THE ATMOSPHERE

### 11.1.1 The radiative equilibrium distribution

A gross but informative measure characterizing the atmosphere, and the effects that dynamics have on it, is the pole-to-equator temperature distribution. The *radiative equilibrium* temperature is the hypothetical, three-dimensional, temperature field that would obtain if there were no atmospheric or oceanic motion, given the composition and radiative properties of the atmosphere and surface. The field is a function only of the incoming solar radiation at the top of the atmosphere, although to evaluate it entails a complicated calculation, especially as the radiative properties of the atmosphere depend on the amount of water vapour and cloudiness in the atmosphere. (The distribution of absorbers is usually taken to be that which obtains in the observed, moving, atmosphere, in order that the differences between the calculated radiative equilibrium temperature and the observed temperature are due to fluid motion.)

A much simpler calculation that illustrates the essence of the situation is to first note that at the top of the atmosphere the globally averaged incoming solar radiation is balanced by the outgoing infrared radiation. If there is no lateral transport of energy in the atmosphere or ocean then *at each latitude* the incoming solar radiation will be balanced by the outgoing infrared radiation, and if we parameterize the latter using a single latitudinally-dependent temperature we will obtain a crude radiative-equilibrium temperature for the atmospheric column at each latitude. Specifically, a black body subject to a net incoming radiation of $S$ (watts per square metre) has a radiative-equilibrium temperature $T_{rad}$ given by $\sigma T_{rad}^4 = S$, this being Stefan's law with Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$. Thus, for the Earth, we have, at each latitude,

$$\sigma T_{rad}^4 = S(\vartheta)(1 - \alpha), \tag{11.1}$$

where $\alpha$ is the albedo of the Earth and $S(\vartheta)$ is the incoming solar radiation at the top of the atmosphere, and its solution is shown in Fig. 11.1. The solid lines in the two panels show the net solar radiation and the solution to (11.1), $T_{rad}$; the dashed lines show the observed outgoing infrared radiative flux, $I$, and the effective emitting temperature associated with it, $(I/\sigma)^{1/4}$. The emitting temperature does not quantitatively characterize that temperature at the Earth's surface, nor at any single level in the atmosphere, because the atmosphere is not a black body and the outgoing radiation originates from multiple levels. Nevertheless, the qualitative point is evident: the radiative equilibrium temperature has a much stronger pole-to-equator gradient than does the effective emitting temperature, indicating that there is a poleward transport of heat in the atmosphere–ocean system. More detailed calculations indicate that the atmosphere is further from its radiative equilibrium in winter than summer, indicating a larger heat transport. The transport occurs because polewards moving air tends to have a higher static energy ($c_p T + gz$ for dry air; in addition there is some energy transport associated with water vapour evaporation and condensation) than the equatorwards moving air, most of this movement being associated with the large-scale circulation. The radiative forcing thus seeks to maintain a pole-to-equator temperature gradient, and the ensuing circulation seeks to reduce this gradient.

### 11.1.2 Observed wind and temperature fields

The observed zonally averaged temperature and zonal wind fields are illustrated in Fig. 11.2. The vertical coordinate is log pressure, multiplied by a constant factor $H = RT_0/g = 7.5\,\mathrm{km}$, so that the ordinate is similar to height in kilometres. [In an isothermal hydrostatic atmosphere $(RT_0/g)\,\mathrm{d}\ln p = -\,\mathrm{d}z$, and the value of $H$ chosen corresponds to $T_0 = 256\,\mathrm{K}$.] To a good approximation temperature and zonal wind are related by thermal wind balance, which in pressure coordinates is

$$f\frac{\partial u}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y}. \tag{11.2}$$

In the lowest several kilometres of the atmosphere temperature falls almost monotonically with latitude and height, and this region is called the *troposphere*. The temperature in the lower troposphere in fact varies more rapidly with latitude than does the effective emitting temperature, $T_E$, the latter being more characteristic of the temperature in the mid-to-upper troposphere. The meridional temperature gradient is much larger in winter than summer, because in winter high latitudes receive virtually no direct heating from the Sun. It is also strongest at the edge of the subtropics, and here it is associated with a zonal jet, particularly strong in winter. There is no need to 'drive' this wind with any kind of convergent momentum fluxes: given the temperature, the flow is a consequence of thermal wind balance, and to the extent that the upper troposphere is relatively frictionless there is no need to maintain it against dissipation. Of course just as the radiative-equilibrium temperature gradient is much larger than that observed, so the zonal wind shear associated with it is much larger than that observed. Thus, the overall effect of the atmospheric and oceanic circulation, and in particular of the turbulent circulation of the mid-latitude atmosphere, is to *reduce* the amplitude of the vertical shear of the eastward flow by way of a poleward heat transport. Observations indicate that about two-thirds of this transport is effected by the atmosphere, and about a third by the ocean, more in low latitudes.[2]

Above the troposphere is the *stratosphere,* and here temperature typically increases with height. The boundary between the two regions is called the *tropopause,* and this varies in

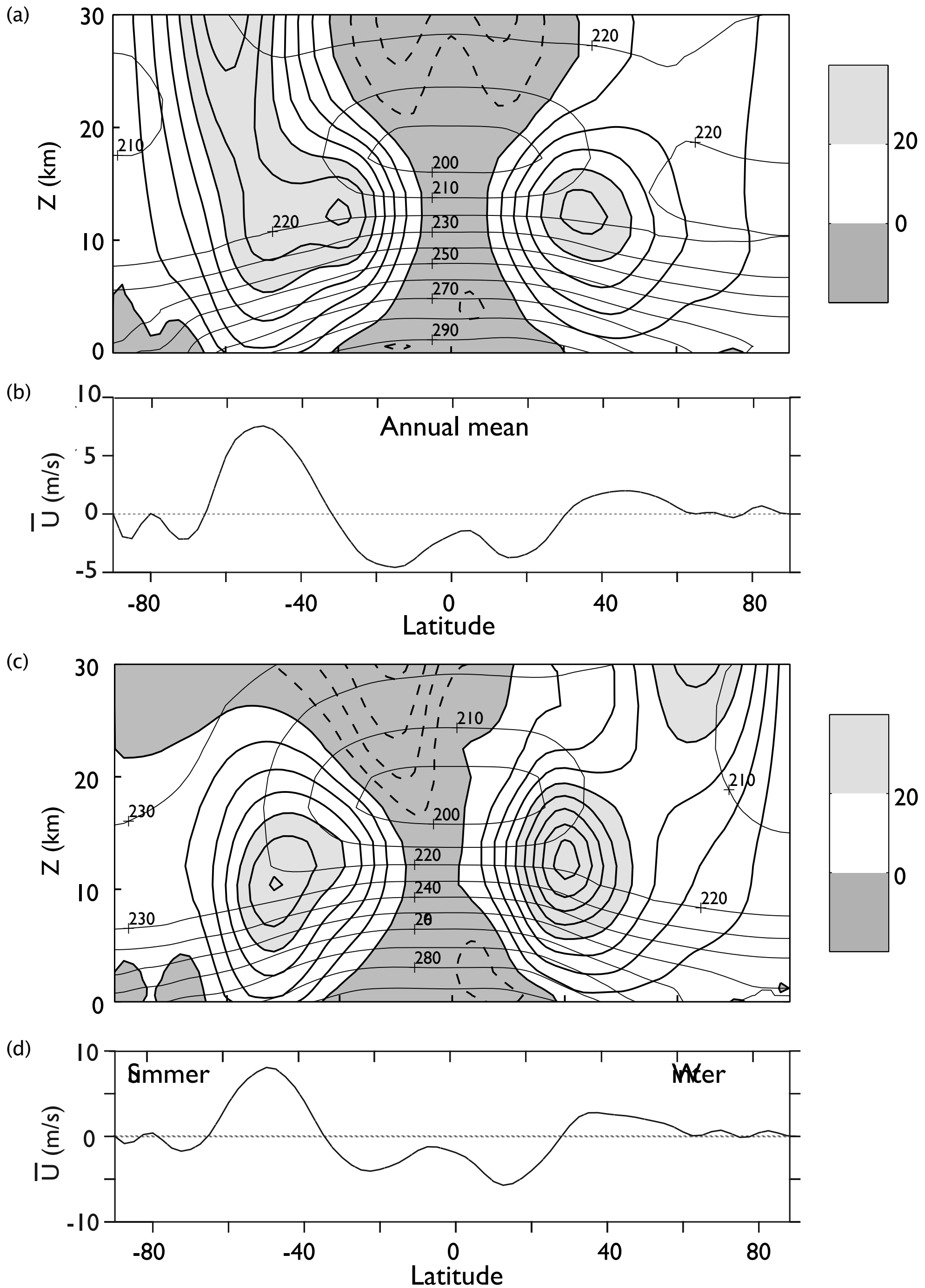


**Fig. 11.2** (a) Annual mean, zonally averaged zonal wind (heavy contours and shading) and the zonally averaged temperature (lighter contours). (b) Annual mean, zonally averaged zonal winds at the surface. (c) and (d) Same as (a) and (b), except for northern hemisphere winter (DJF). The wind contours are at intervals of 5 m s$^{-1}$ with shading for eastward winds above 20 m s$^{-1}$ and for all westward winds, and the temperature contours are labelled. The ordinate of (a) and (c) is $Z = -H \log(p/p_R)$, where $p_R$ is a constant, with scale height $H = 7.5$ km.

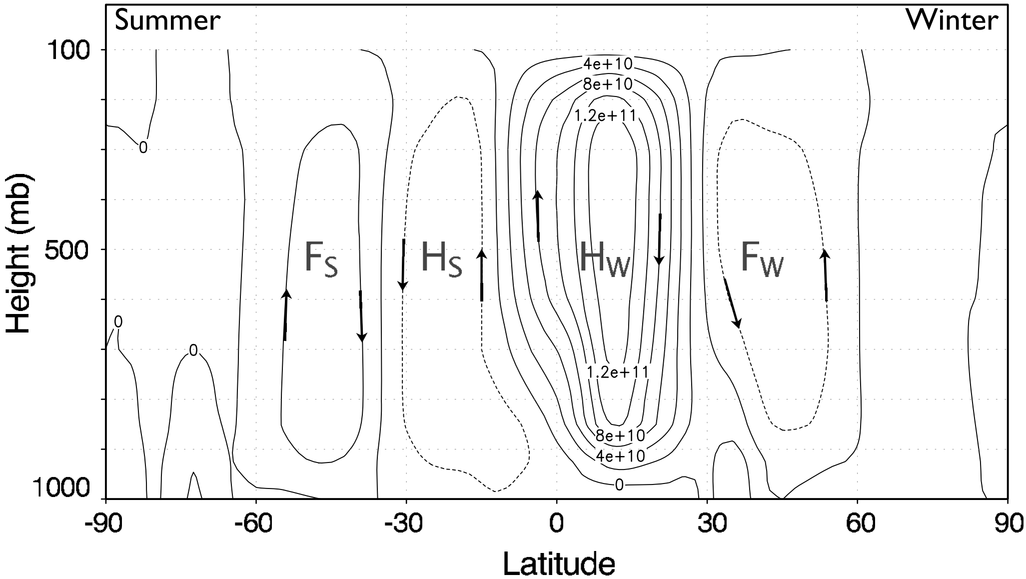


**Fig. 11.3** The observed, zonally averaged, meridional overturning circulation of the atmosphere, in units of kg s$^{-1}$, averaged over December–January–February (DJF). In each hemisphere note the presence of a direct *Hadley Cell* ($H_W$ and $H_S$ in winter and summer) with rising motion near the equator, descending motion in the subtropics, and an indirect *Ferrel Cell* ($F_W$ and $F_S$) at mid-latitudes. There are also hints of a weak direct cell at high latitudes. The winter Hadley Cell is far stronger than the summer one.

height from about 16 km in the tropics to about 8 km in polar regions. We consider the maintenance of this stratification in section 12.5.

The surface winds typically have, going from the equator to the pole, an E–W–E (easterly–westerly–easterly) pattern, although the polar easterlies are weak and barely present in the Northern Hemisphere. (Meteorologists use 'westerly' to denote winds from the west, that is eastward winds; similarly 'easterlies' are westward winds. We will use both 'westerly' and 'eastward', and both 'easterly' and 'westward', and the reader should be comfortable with all these terms.) In a given hemisphere, the surface winds are stronger in winter than summer, and they are also consistently stronger in the Southern Hemisphere than in the Northern Hemisphere, because in the former the surface drag is weaker because of the relative lack of continental land masses and topography. The surface winds are *not* explained by thermal wind balance. Indeed, unlike the upper level winds, they must be maintained against the dissipating effects of friction, and this implies a momentum convergence into regions of surface westerlies and a divergence into regions of surface easterlies. Typically, the maxima in the eastward surface winds are in mid-latitudes and somewhat polewards of the subtropical maxima in the upper-level westerlies and at latitudes where the zonal flow is a little more constant with height. The mechanisms of the momentum transport in the mid-latitudes and the maintenance of the surface westerly winds are the topics of section 12.1.

**Some Features of the Large-scale Atmospheric Circulation**

From Figures 11.1–11.3 we see or infer the following.

- ⋆ A pole–equator temperature gradient that is much smaller than the radiative equilibrium gradient.
- ⋆ A troposphere, in which temperature generally falls with height, above which lies the stratosphere, in which temperature increases with height. The two regions are separated by a tropopause, which varies in height from about 16 km at the equator to about 6 km at the pole.
- ⋆ A monotonically decreasing temperature from equator to pole in the troposphere, but a weakening and sometimes reversal of this above the tropopause.
- ⋆ A westerly (i.e., eastward) tropospheric jet. The time and zonally averaged jet is a maximum at the edge or just polewards of the subtropics, where it is associated with a strong meridional temperature gradient. In mid-latitudes the jet has a stronger barotropic component.
- ⋆ An E–W–E (easterlies–westerlies–easterlies) surface wind distribution. The latitude of the maximum in the surface westerlies is in mid-latitudes, where the zonally averaged flow is more barotropic.

### 11.1.3 Meridional overturning circulation

The observed (Eulerian) zonally averaged meridional overturning circulation is illustrated in Fig. 11.3. The figure shows a streamfunction, $\Psi$ for the vertical and meridional velocities such that, in the pressure coordinates used in the figure,

$$\frac{\partial \Psi}{\partial y} = \overline{\omega}, \qquad \frac{\partial \Psi}{\partial p} = -\overline{v}. \tag{11.3}$$

where the overbar indicates a zonal average. In each hemisphere there is rising motion near the equator and sinking in the subtropics, and this circulation is known as the *Hadley Cell*.[3] The Hadley Cell is a thermally direct cell (i.e., the warmer fluid rises, the colder fluid sinks), is much stronger in the winter hemisphere, and extends to about 30°. In mid-latitudes the sense of the overturning circulation is apparently reversed, with rising motion in the high-mid-latitudes, at around 60° and sinking in the subtropics, and this is known as the *Ferrel Cell*. However, as with most pictures of averaged streamlines in unsteady flow, this gives a misleading impression as to the actual material flow of parcels of air because of the presence of eddying motion, and we discuss this in the next chapter. At low latitudes the circulation is more nearly zonally symmetric and the picture does give a qualitatively correct representation of the actual flow. At high latitudes there is again a thermally direct cell (although it is weak and not always present), and thus the atmosphere is often referred to as having a three-celled structure.

### 11.1.4 Summary

Some of the main features of the zonally averaged circulation are summarized in the shaded box on the preceding page. We emphasize that the zonally averaged circulation is not synonymous with a zonally symmetric circulation, and the mid-latitude circulation is highly asymmetric. Any model of the mid-latitudes that did not take into account the zonal asymmetries in the circulation — of which the weather is the main manifestation — would be seriously in error. This was first explicitly realized in the 1920s, and taking into account such asymmetries is the main task of the dynamical meteorology of the mid-latitudes, and is the subject of the next chapter. On the other hand, the large-scale tropical circulation of the atmosphere is to a large degree zonally symmetric or nearly so, and although monsoonal circulations and the Walker circulation (a cell with rising air in the Eastern Pacific and descending motion in the Western Pacific) are zonally asymmetric, they are also relatively weaker than typical mid-latitude weather systems. Indeed the boundary between the tropics and mid-latitude may be usefully defined by the latitude at which such zonal asymmetries become dynamically important on the large scale and this boundary, at about 30° on average, roughly coincides with the latitude at which the mean meridional overturning circulation vanishes. We begin our dynamical description with a study of the low-latitude zonally symmetric atmospheric circulation.

## 11.2 A STEADY MODEL OF THE HADLEY CELL

*Ceci n'est pas une pipe.*
René Magritte (1898–1967), title of painting.

### 11.2.1 Assumptions

Let us try to construct a zonally symmetric model of the Hadley Cell,[4] recognizing that such a model is likely applicable mainly to the tropical atmosphere, this being more zonally symmetric than the mid-latitudes. We will suppose that heating is maximum at the equator, and our intuitive picture, drawing on the observed flow of Fig. 11.3, is of air rising at the equator and moving polewards at some height $H$, descending at some latitude $\vartheta_H$, and returning equatorwards near the surface. We will make three major assumptions:

*(i)* that the circulation is steady;

*(ii)* that the polewards moving air conserves its axial angular momentum, whereas the zonal flow associated with the near-surface, equatorwards moving flow is frictionally retarded and is weak;

*(iii)* that the circulation is in thermal wind balance.

We also assume the model is symmetric about the equator (an assumption we relax in section 11.4). These are all reasonable assumptions, but they cannot be rigorously justified; in other words, we are constructing a *model* of the Hadley Cell, schematically illustrated in Fig. 11.4. The model defines a limiting case — steady, inviscid, zonally-symmetric flow — that cannot be expected to describe the atmosphere quantitatively, but that can be analysed fairly completely. Another limiting case, in which eddies play a significant role, is described in section 11.5. The real atmosphere may defy such simple characterizations, but the two limits provide invaluable benchmarks of understanding.

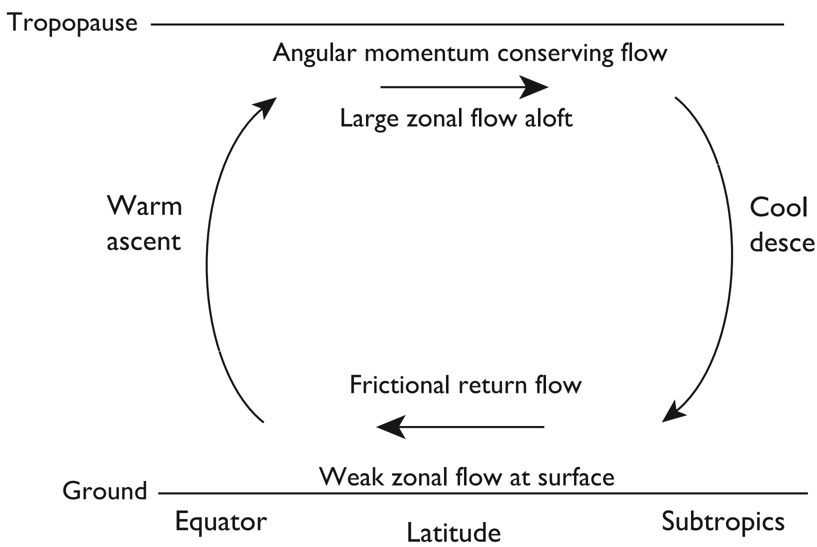


**Fig. 11.4** A simple model of the Hadley Cell. Rising air near the equator moves polewards near the tropopause, descending in the subtropics and returning near the surface. The polewards moving air conserves its axial angular momentum, leading to a zonal flow that increases away from the equator. By the thermal wind relation the temperature of the air falls as it moves polewards, and to satisfy the thermodynamic budget it sinks in the subtropics. The return flow at the surface is frictionally retarded and small.

### 11.2.2 Dynamics

We now try to determine the strength and poleward extent of the Hadley circulation in our steady model. For simplicity we will work with a Boussinesq atmosphere, but this is not an essential aspect. We will first derive the conditions under which conservation on angular momentum will hold, and then determine the consequences of that.

The zonally averaged zonal momentum equation may be easily derived from (2.50a) and/or (2.62) and in the absence of friction it is

$$\frac{\partial \overline{u}}{\partial t} - (f + \overline{\zeta})\overline{v} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -\frac{1}{a\cos^2\vartheta}\frac{\partial}{\partial \vartheta}(\cos^2\vartheta\,\overline{u'v'}) - \frac{\partial \overline{u'w'}}{\partial z}, \tag{11.4}$$

where $\overline{\zeta} = -(a\cos\vartheta)^{-1}\partial_\vartheta(\overline{u}\cos\vartheta)$ and the overbars represent zonal averages. If we neglect the vertical advection and the eddy terms on the right-hand side, then a steady solution, if it exists, obeys

$$(f + \overline{\zeta})\overline{v} = 0. \tag{11.5}$$

Presuming that the meridional flow $\overline{v}$ is non-zero (an issue we address in section 11.2.8) then $f + \overline{\zeta} = 0$, or equivalently

$$2\Omega\sin\vartheta = \frac{1}{a}\frac{\partial \overline{u}}{\partial \vartheta} - \frac{\overline{u}\tan\vartheta}{a}. \tag{11.6}$$

At the equator we shall assume that $\overline{u} = 0$, because here parcels have risen from the surface

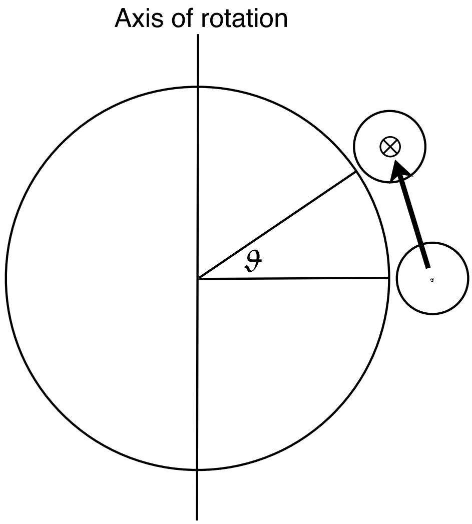


**Fig. 11.5** If a ring of air at the equator moves polewards it moves closer to the axis of rotation. If the parcels in the ring conserve their angular momentum their zonal velocity must increase; thus, if $m = (\overline{u} + \Omega a \cos\vartheta) a \cos\vartheta$ is preserved and $\overline{u} = 0$ at $\vartheta = 0$ we recover (11.7).

where, by assumption, the flow is weak. Equation (11.6) then has a solution of

$$\boxed{\overline{u} = \Omega a \frac{\sin^2\vartheta}{\cos\vartheta} \equiv U_M}. \tag{11.7}$$

This gives the zonal velocity of the polewards moving air in the upper branch of the (model) Hadley Cell, above the frictional boundary layer. We can derive (11.7) directly from the conservation of axial angular momentum, $m$, of a parcel of air at a latitude $\vartheta$. In the shallow atmosphere approximation we have [cf. (2.64) and equations following]

$$\overline{m} = (\overline{u} + \Omega a \cos\vartheta) a \cos\vartheta, \tag{11.8}$$

and if $\overline{u} = 0$ at $\vartheta = 0$ and if $\overline{m}$ is conserved on a polewards moving parcel, then (11.8) leads to (11.7). It also may be directly checked that

$$f + \overline{\zeta} = -\frac{1}{a^2\cos\vartheta}\frac{\partial \overline{m}}{\partial\vartheta}. \tag{11.9}$$

We have thus shown that, if eddy fluxes and frictional effects are negligible, the poleward flow will conserve its angular momentum, the result of which, by (11.7), is that magnitude of the zonal flow in the Earth's rotating frame will increase with latitude (see Fig. 11.5). (Also, given the absence of eddies our model is zonally symmetric and we shall drop the overbars over the variables.)

If (11.7) gives the zonal velocity in the upper branch of the Hadley Cell, and that in the lower branch is close to zero, then the thermal wind equation can be used to infer the vertically averaged temperature. Although the geostrophic wind relation is not valid at the equator (a more accurate balance is the so-called cyclostrophic balance, $fu + u^2\tan\vartheta/a = -a^{-1}\partial\phi/\partial\vartheta$) the zonal wind is in fact geostrophically balanced until very close to the equator, and at the equator itself the horizontal temperature gradient in our model vanishes, because of the assumed interhemispheric symmetry. Thus, conventional thermal wind balance suffices for our purposes, and this is

$$2\Omega \sin\vartheta \frac{\partial u}{\partial z} = -\frac{1}{a}\frac{\partial b}{\partial \vartheta}, \tag{11.10}$$

where $b = g\,\delta\theta/\theta_0$ is the buoyancy and $\delta\theta$ is the deviation of potential temperature from a constant reference value $\theta_0$. (Be reminded that $\theta$ is potential temperature, whereas $\vartheta$ is latitude.) Vertically integrating from the ground to the height $H$ where the outflow occurs and substituting (11.7) for $u$ yields

$$\frac{1}{a\theta_0}\frac{\partial \theta}{\partial \vartheta} = -\frac{2\Omega^2 a}{gH}\frac{\sin^3\vartheta}{\cos\vartheta}, \tag{11.11}$$

where $\theta = H^{-1}\int_0^H \delta\theta\,\mathrm{d}z$ is the vertically averaged potential temperature. If the latitudinal extent of the Hadley Cell is not too great we can make the small-angle approximation, and replace $\sin\vartheta$ by $\vartheta$ and $\cos\vartheta$ by one, then integrating (11.11) gives

$$\boxed{\theta = \theta(0) - \frac{\theta_0 \Omega^2 y^4}{2gHa^2}}, \tag{11.12}$$

where $y = a\vartheta$ and $\theta(0)$ is the potential temperature at the equator, as yet unknown. Away from the equator, the zonal velocity given by (11.7) increases rapidly polewards and the temperature correspondingly drops. How far polewards is this solution valid? And what determines the value of the integration constant $\theta(0)$? To answer these questions we turn to thermodynamics.

### 11.2.3 Thermodynamics

In the above discussion, the temperature field is slaved to the momentum field in that it seems to follow passively from the dynamics of the momentum equation. Nevertheless, the thermodynamic equation must still be satisfied. Let us assume that the thermodynamic forcing can be represented by a Newtonian cooling to some specified radiative equilibrium temperature, $\theta_E$; this is a severe simplification, especially in equatorial regions where the release of heat by condensation is important. The thermodynamic equation is then

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{\theta_E - \theta}{\tau}, \tag{11.13}$$

where $\tau$ is a relaxation time scale, perhaps a few weeks. Let us suppose that $\theta_E$ falls monotonically from the equator to the pole, and that it increases linearly with height, and a simple representation of this is

$$\frac{\theta_E(\vartheta, z)}{\theta_0} = 1 - \frac{2}{3}\Delta_H P_2(\sin\vartheta) + \Delta_V\left(\frac{z}{H} - \frac{1}{2}\right), \tag{11.14}$$

where $\Delta_H$ and $\Delta_V$ are non-dimensional constants that determine the fractional temperature difference between the equator and the pole, and the ground and the top of the fluid, respectively. $P_2$ is the second Legendre polynomial, and it is usually the leading term in the Taylor expansion of symmetric functions (symmetric around the equator) that decrease from pole to equator; it also integrates to zero over the sphere. $P_2(y) = (3y^2 - 1)/2$, so that in the small-angle approximation and at $z = H/2$, or for the vertically averaged field, we have

$$\frac{\theta_E}{\theta_0} = 1 + \frac{1}{3}\Delta_H - \Delta_H \left(\frac{y}{a}\right)^2 \tag{11.15}$$

or

$$\theta_E = \theta_{E0} - \Delta\theta \left(\frac{y}{a}\right)^2, \tag{11.16}$$

where $\theta_{E0}$ is the equilibrium temperature at the equator, $\Delta\theta$ determines the equator–pole radiative-equilibrium temperature difference, and

$$\theta_{E0} = \theta_0(1 + \Delta_H/3), \qquad \Delta\theta = \theta_0 \Delta_H. \tag{11.17}$$

Now, let us suppose that the solution (11.12) is valid between the equator and a latitude $\vartheta_H$ where $v = 0$, so that within this region the system is essentially closed. Conservation of potential temperature then requires that the solution (11.12) must satisfy

$$\int_0^{Y_H} \theta \, \mathrm{d}y = \int_0^{Y_H} \theta_E \, \mathrm{d}y, \tag{11.18}$$

where $Y_H = a\vartheta_H$ is as yet undetermined. Polewards of this, the solution is just $\theta = \theta_E$. Now, we may demand that the solution be continuous at $y = Y_H$ (without temperature continuity the thermal wind would be infinite) and so

$$\theta(Y_H) = \theta_E(Y_H). \tag{11.19}$$

The constraints (11.18) and (11.19) determine the values of the unknowns $\theta(0)$ and $Y_H$. A little algebra (problem 11.1) gives

$$Y_H = \left(\frac{5\Delta\theta g H}{3\Omega^2 \theta_0}\right)^{1/2}, \tag{11.20}$$

and

$$\theta(0) = \theta_{E0} - \left(\frac{5\Delta\theta^2 g H}{18 a^2 \Omega^2 \theta_0}\right). \tag{11.21}$$

A useful non-dimensional number that parameterizes these solutions is

$$R \equiv \frac{gH\Delta\theta}{\theta_0 \Omega^2 a^2} = \frac{gH\Delta_H}{\Omega^2 a^2}, \tag{11.22}$$

which is the square of the ratio of the speed of shallow water waves to the rotational velocity of the Earth, multiplied by the fractional temperature difference from equator to pole. Typical values for the Earth's atmosphere are a little less than 0.1. In terms of $R$ we have

$$\boxed{Y_H = a\left(\frac{5}{3}R\right)^{1/2}}, \tag{11.23}$$

and

$$\boxed{\theta(0) = \theta_{E0} - \left(\frac{5}{18}R\right)\Delta\theta}. \tag{11.24}$$

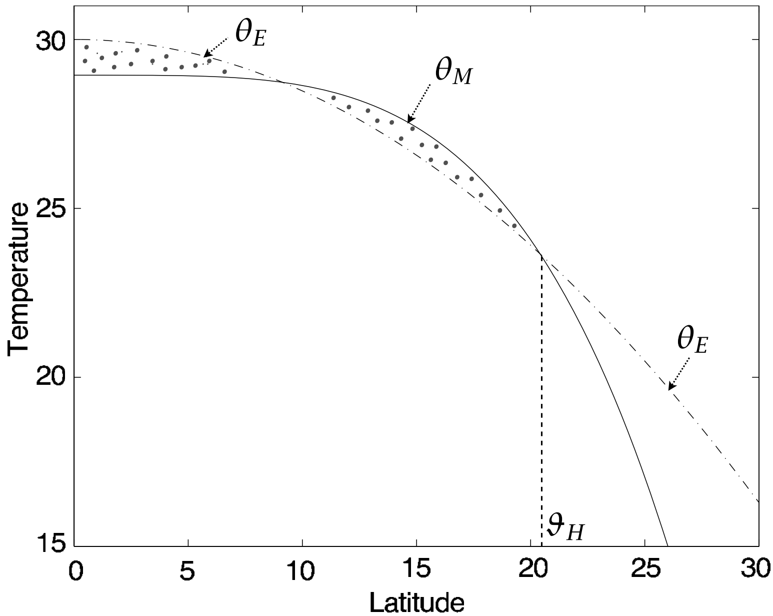


**Fig. 11.6** The radiative equilibrium temperature ($\theta_E$, dashed line) and the angular-momentum-conserving solution ($\theta_M$, solid line) as a function of latitude. The two dotted regions have equal areas. The parameters are: $\theta_{EO} = 303\,\mathrm{K}$, $\Delta\theta = 50\,\mathrm{K}$, $\theta_0 = 300\,\mathrm{K}$, $\Omega = 7.272 \times 10^{-5}\,\mathrm{s}^{-1}$, $g = 9.81\,\mathrm{m\,s}^{-2}$, $H = 10\,\mathrm{km}$. These give $R = 0.076$ and $Y_H/a = 0.356$, corresponding to $\vartheta_H = 20.4°$.

The solution, (11.12) with $\theta(0)$ given by (11.24) is plotted in Fig. 11.6. Perhaps the single most important aspect of the model is that it predicts that the Hadley Cell has a *finite* meridional extent, *even for an atmosphere that is completely zonally symmetric.* The baroclinic instability that does occur in mid-latitudes is not necessary for the Hadley Cell to terminate in the subtropics, although it may be an important factor, or even the determining factor, in the real world.

### 11.2.4 Zonal wind

The angular-momentum-conserving zonal wind is given by (11.7), which in the small-angle approximation becomes

$$U_M = \Omega \frac{y^2}{a}. \tag{11.25}$$

This relation holds for $y < Y_H$. The zonal wind corresponding to the radiative-equilibrium solution is given using thermal wind balance and (11.16), which leads to

$$U_E = \Omega a R. \tag{11.26}$$

That the radiative-equilibrium zonal wind is a constant follows from our choice of the second Legendre function for the radiative equilibrium temperature and is not a fundamental result; nonetheless, for most reasonable choices of $\theta_E$ the corresponding zonal wind will vary

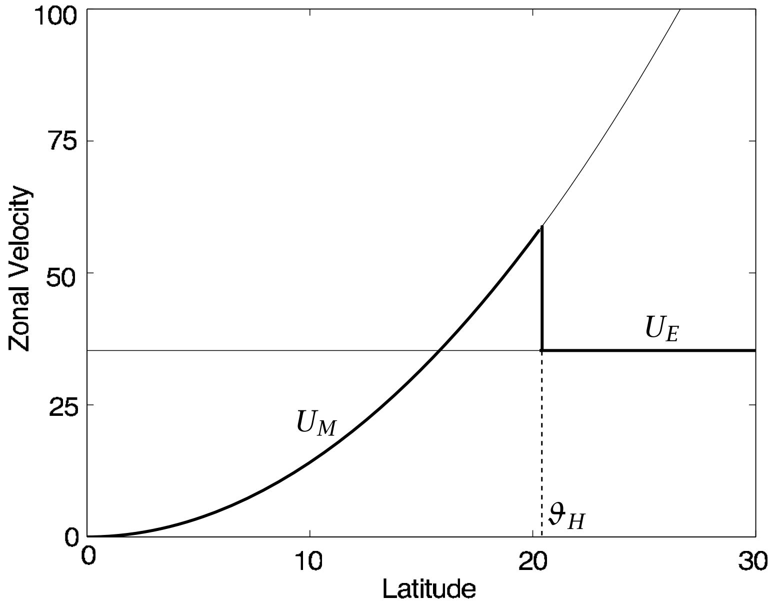


**Fig. 11.7** The zonal wind corresponding to the radiative equilibrium temperature ($U_E$,) and the angular-momentum-conserving solution ($U_M$) as a function of latitude, given (11.25) and (11.26) respectively. The parameters are the same as those of Fig. 11.6, and the radiative equilibrium wind, $U_E$ is a constant, $\Omega aR$. The actual zonal wind (in the model) follows the thick solid line: $u = U_m$ for $\vartheta < \vartheta_H$ $(y < Y_H)$, and $u = U_E$ for $\vartheta > \vartheta_H$ $(y > Y_H)$.

much less than the angular-momentum-conserving wind (11.25). The winds are illustrated in Fig. 11.7. There is a discontinuity in the zonal wind at the edge of the Hadley Cell, and of the meridional temperature gradient, but not of the temperature itself.

### 11.2.5 Properties of solution

From (11.23) we can see that the model predicts that the latitudinal extent of the Hadley Cell is:

- ⋆ proportional to the square root of the meridional radiative equilibrium temperature gradient: the stronger the gradient, the farther the circulation must extend to achieve thermodynamic balance via the equal-area construction in Fig. 11.6;
- ⋆ proportional to the square root of the height of the outward flowing branch: the higher the outward flowing branch, the weaker the ensuing temperature gradient of the solution (via thermal wind balance), and so the further polewards the circulation must go;
- ⋆ inversely proportional to the rotation rate $\Omega$: the stronger the rotation rate, the stronger the angular-momentum-conserving wind, the stronger the ensuing temperature gradient and so the more compact the circulation.

These precise dependencies on particular powers of parameters are not especially significant in themselves, nor are they robust to changes in parameters. For example, were we to chose a meridional distribution of radiative equilibrium temperature different from (11.14) we might find different exponents in some of the solutions, although we would expect the same qualitative dependencies. However, the dependencies do provide predictions that may be tested with a numerical model. Also, as we have already noted, a key property of the model is that it predicts that the Hadley Cell has a finite meridional extent, even in the absence of mid-latitude baroclinic instability.

Another interesting property of the solutions is a discontinuity in the zonal wind. For tropical latitudes (i.e., $y < Y_H$), then $\overline{u} = U_M$ (the constant angular momentum solution), whereas for $y > Y_H$, $\overline{u} = U_E$ (the thermal wind associated with radiative equilibrium temperature $\theta_E$). There is therefore a discontinuity of $\overline{u}$ at $y = Y_H$, because $u$ is related to the meridional gradient of $\theta$ which changes discontinuously, even though $\theta$ itself is continuous. No such discontinuity is observed in the real world, although one may observe a baroclinic jet at the edge of the Hadley Cell.

### 11.2.6 Strength of the circulation

We can make an estimate of the strength of the Hadley Cell by consideration of the thermodynamic equation at the equator, namely

$$w \frac{\partial \theta}{\partial z} \approx \frac{\theta_{E0} - \theta}{\tau}, \tag{11.27}$$

this being a balance between adiabatic cooling and radiative heating. If the static stability is determined largely by the forcing, and not by the meridional circulation itself, then

$$\frac{1}{\theta_0} \frac{\partial \theta}{\partial z} \approx \frac{\Delta_V}{H}, \tag{11.28}$$

and (11.27) gives

$$w \approx \frac{H}{\theta_0 \Delta_V} \frac{\theta_{E0} - \theta}{\tau}. \tag{11.29}$$

Thus, the strength of the circulation is proportional to the distance of the solution from the radiative equilibrium temperature. The right-hand side of (11.27) can be evaluated from the solution itself, and from (11.24) we have

$$\frac{\theta_{E0} - \theta}{\tau} = \frac{5R\Delta\theta}{18\tau}. \tag{11.30}$$

The vertical velocity is then given by

$$w \approx \frac{5R\Delta\theta H}{18\tau\Delta_V \theta_0} = \frac{5R\Delta_H H}{18\tau\Delta_V}. \tag{11.31}$$

Using mass continuity we can transform this into an estimate for the meridional velocity. Thus, if we let $(v/Y_H) \sim (w/H)$ and use (11.23), we obtain

$$v \sim \frac{R^{3/2} a \Delta_H}{\tau \Delta_V} \propto \frac{\Delta_H^{5/2}}{\Delta_V}, \tag{11.32}$$

and the mass flux, or the meridional overturning stream function $\Psi$, of the circulation scales as

$$\Psi \sim vH \sim \frac{R^{3/2} a H \Delta_H}{\tau \Delta_V} \propto (\Delta\theta)^{5/2}. \tag{11.33}$$

This evidently increases fairly rapidly as the gradient of the radiative equilibrium temperature increases. The characteristic overturning time of the circulation, $\tau_d$ is then

$$\tau_d = \frac{H}{w} \sim \frac{\tau \Delta_V}{R \Delta_H}. \tag{11.34}$$

We require $\tau_d/\tau \gg 1$ for the effects of the circulation on the static stability to be small and therefore $\Delta_V/(R\Delta_H) \gg 1$, or equivalently, using (11.17),

$$\theta_0 \Delta_V \gg R(\theta_{E0} - \theta_0). \tag{11.35}$$

If the converse were true, and $\tau \gg \tau_d$, then the potential temperature would be nearly conserved as a parcel ascended in the rising branch of the Hadley Cell, and the static stability would be nearly neutral.

### 11.2.7 † Effects of moisture

Suppose now that moisture is present, but that the Hadley Cell remains a self-contained system; that is, it neither imports nor exports moisture. We envision that water vapour joins the circulation by way of evaporation from a saturated surface into the equatorward, lower branch of the Hadley Cell, and that this water vapour then condenses in and near the upward branch of the cell. The latent heat released by condensation is exactly equal to the heat required to evaporate moisture from the surface, and no heat is lost or gained to the system. However, the heating *distribution* is changed from the dry case, becoming a strong function of the solution itself and likely to have a sharp maximum near the equator. Even if we were to try to parameterize the latent heat release by simply choosing a flow dependent radiative equilibrium temperature, the resulting problem would still be quite nonlinear and a general analytic solution seems out of our reach.[5]

Nevertheless, we may see quite easily the qualitative features of moisture, at least within the context of this model. The meridional distribution of temperature is still given by way of thermal wind balance with an angular-momentum-conserving zonal wind, and so is still given by (11.12). We may also assume that that the meridional extent of the Hadley Cell is unaltered; that is, a solution exists with circulation confined to $\vartheta < \vartheta_H$ (although it may not be the unique solution). Then, if $\theta_E^*$ is the effective radiative equilibrium temperature of the moist solution, we have that $\theta_E^*(Y_H) = \theta_E(Y_H)$ and, in the small-angle approximation,

$$\int_0^{Y_H} \theta \, \mathrm{d}y = \int_0^{Y_H} \theta_E^* \, \mathrm{d}y = \int_0^{Y_H} \theta_E \, \mathrm{d}y, \tag{11.36}$$

where the first equality holds because it defines the solution, and the second equality holds because moisture provides no net energy source. Because condensation will occur mainly in the upward branch of the Hadley Cell, $\theta_E^*$ will be peaked near the equator, as schematically sketched in Fig. 11.8. This construction makes it clear that the main difference between the dry and moist solutions is that the latter has a more intense overturning circulation,

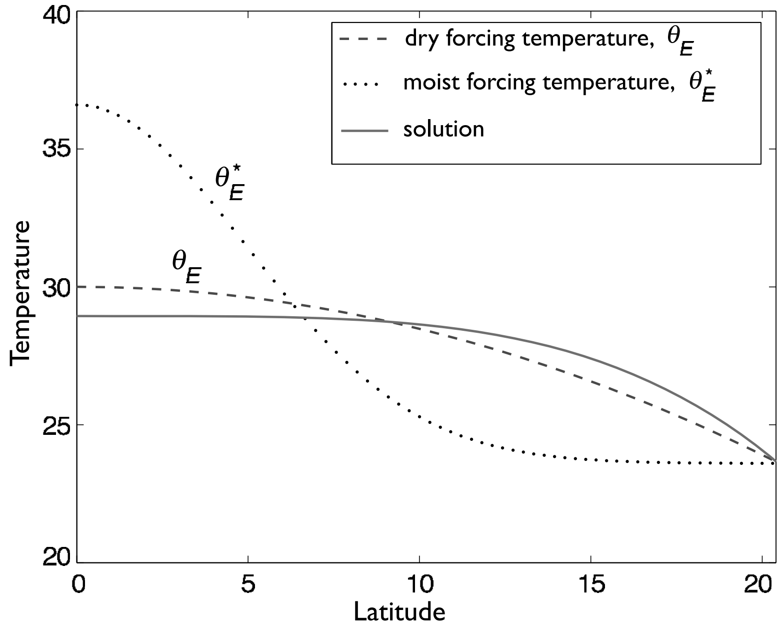


**Fig. 11.8** Schema of the effects of moisture on a model of the Hadley Cell. The temperature of the solution (solid line) is the same as that of a dry model, because this is determined from the angular-momentum-conserving wind. The heating distribution (as parameterized by a forcing temperature) is peaked near the equator in the moist case, leading to a more vigorous overturning circulation.

because, from (11.27), the circulation increases with the temperature difference between the solution and the forcing temperature. Concomitantly, our intuition suggests that the upward branch of the moist Hadley circulation will become much narrower and more intense than the downward branch because of the enhanced efficiency of moist convection, and these expectations are generally confirmed by numerical integrations of the moist equations of motion.

### 11.2.8 The radiative equilibrium solution

Instead of a solution given by (11.12), could the temperature not simply be in radiative equilibrium everywhere? Such a state would have no meridional overturning circulation and the zonal velocity would be determined by thermal wind balance; that is,

$$v = 0, \qquad \theta = \theta_E, \qquad f\frac{u}{H} = -g\frac{\partial}{\partial y}\left(\frac{\theta_E}{\theta_0}\right). \tag{11.37}$$

To answer this question we consider the steady zonally symmetric zonal angular momentum equation with viscosity; that is, the zonally averaged, viscous, steady, shallow atmosphere version of (2.68), namely

$$\frac{1}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}(vm\cos\vartheta) + \frac{\partial(mw)}{\partial z} = \frac{\nu}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}\left(\cos^2\vartheta\frac{\partial}{\partial\vartheta}\frac{u}{\cos\vartheta}\right) + \nu a\cos\vartheta\frac{\partial^2 u}{\partial z^2}, \tag{11.38}$$

where the variables vary only in the $\vartheta$–$z$, or $y$–$z$, plane. The viscous term on the right-hand side arises from the expansion in spherical coordinates of the Laplacian. Note that it is angular velocity, not the angular momentum, that is diffused, because there is no diffusion of the angular momentum due to the Earth's rotation. However, to a very good approximation, the viscous term will be dominated by vertical derivatives and we may then write (11.38) as

$$\nabla_x \cdot (\boldsymbol{v} m) = \nu \frac{\partial^2 m}{\partial z^2}. \tag{11.39}$$

where $\nabla_x \cdot$ is the divergence in the meridional plane. The right-hand side now has a diffusive form, and in section 10.5.1 we showed that variables obeying equations like this can have no extrema within the fluid. Thus, there can be no maximum or minimum of angular momentum in the interior of the fluid, a result sometimes called Hide's theorem.[6] In effect, diffusion always acts to smooth away an isolated extremum, and this cannot be counterbalanced by advection. The result also implies that there cannot be any interior extrema in a statistically steady state if there is any zonally asymmetric eddy motion that transports angular momentum downgradient.

If the viscosity were so large that the viscous term were dominant in (11.38), then the fluid would evolve toward a state of solid body rotation, this being the fluid state with no internal stresses. In that case, there would be a maximum of angular momentum at the equator — a state of 'super-rotation'. (Related mechanisms have been proposed for the maintenance of super-rotation on Venus.[7])

A maxima of $m$ can, however, occur at the surface, even with a viscous term like that of (11.39). Suppose we add a surface stress to the right-hand side of (11.39), and that this stress acts in the opposite direction to that of the zonal wind. Then, in a region of surface easterlies the surface stress contribution would be positive, acting as a source of positive angular momentum; there then exists the possibility that this will exactly balance the diffusion term allowing a maximum of $m$ to occur. However, such a surface stress does not allow this maximum to be a region of surface westerlies at the equator, because the stress would then act in the same way as the diffusion and reduce the angular momentum.

Returning now to the question posed at the head of this section, suppose that the radiative equilibrium solution does hold. Then a radiative equilibrium temperature decreasing away from the equator more rapidly than the angular-momentum-conserving solution $\theta_M$ implies, using thermal wind balance, a maximum of $m$ at the equator and above the surface, in violation of the no-extremum principle. Of course, we have derived the angular-momentum-conserving solution in the inviscid limit, in which the no-extrema principle does not apply. But any small viscosity will make the radiative equilibrium solution completely invalid, but potentially have only a small effect on the angular-momentum-conserving solution; that is, in the *limit of small viscosity* the angular-momentum-conserving solution can conceivably hold approximately, at least in the absence of boundary layers, whereas the radiative equilibrium solution cannot.

However, if the radiative equilibrium temperature varies more slowly with latitude than the temperature corresponding to the angular momentum conserving solution then a radiative equilibrium solution *can* pertain, without violating Hide's theorem. In particular, this is the case if $\theta_E \propto P_4(\sin\vartheta)$, where $P_4$ is the fourth Legendre polynomial, and so the possibility exists of two equilibrium solutions for the same forcing; however, $P_4$ is an unrealistically flat radiative equilibrium temperature for the Earth's atmosphere.

## 11.3 A SHALLOW WATER MODEL OF THE HADLEY CELL

Although expressed in the notation of the primitive equations, the model described above takes no account of any vertical structure in its stratification and is, *de facto,* a shallow water model. Furthermore, the geometric aspects of sphericity play no essential role. Thus, we may transparently express the essence of the model by:

*(i)* explicitly using the shallow water equations instead of the stratified equations;

*(ii)* using the equatorial $\beta$-plane, with $f = f_0 + \beta y$ and $f_0 = 0$.

Let us therefore, as an exercise, construct a reduced-gravity model with an active upper layer overlying a stationary lower layer.

### 11.3.1 Momentum balance

The inviscid zonal momentum equation of the upper layer is

$$\frac{\mathrm{D}u}{\mathrm{D}t} - \beta y v = 0 \tag{11.40}$$

or

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(u - \frac{\beta y^2}{2}\right) = 0, \tag{11.41}$$

which is the $\beta$-plane analogue of the conservation of axial angular momentum. (In this section, all variables are zonally averaged, but we omit any explicit notation denoting this.) From (11.41) we obtain the zonal wind as a function of latitude,

$$u = \frac{1}{2}\beta y^2 + A, \tag{11.42}$$

where $A$ is a constant, which is zero if $u = 0$ at the 'equator', $y = 0$. The flow given by (11.42) is then analogous to the angular momentum conserving flow in the spherical model, (11.7). Because the lower layer is stationary, the analogue of thermal wind balance in the stratified model is just geostrophic balance, namely

$$fu = -g'\frac{\partial h}{\partial y}, \tag{11.43}$$

where $h$ is the thickness of the active upper layer. Using (11.43) and $f = \beta y$ we obtain

$$g'\frac{\partial h}{\partial y} = -\frac{1}{2}\beta^2 y^3 \tag{11.44}$$

giving

$$h = -\frac{1}{8g'}\beta^2 y^4 + h(0) \tag{11.45}$$

where $h(0)$ is the value of $h$ at $y = 0$.

### 11.3.2 Thermodynamic balance

The thermodynamic equation in the shallow water equations is just the mass conservation equation, which we write as

$$\frac{\mathrm{D}h}{\mathrm{D}t} = -\frac{1}{\tau}(h - h^*), \tag{11.46}$$

where the right-hand side represents heating — $h^*$ is the field to which the height relaxes on a time scale $\tau$. For illustrative purposes we will choose

$$h^* = h_0(1 - \alpha|y|). \tag{11.47}$$

(If we chose the more realistic quadratic dependence on $y$, the model would be more similar to that of the previous section.) To be in thermodynamic equilibrium we require that the right-hand side integrates to zero over the Hadley Cell; that is

$$\int_0^Y (h - h^*)\, \mathrm{d}y = 0 \tag{11.48}$$

where $Y$ is the latitude of the poleward extent of the Hadley Cell, thus far unknown. Polewards of this, the height field is simply in equilibrium with the forcing — there is no meridional motion and $h = h^*$. Since the height field must be continuous, we require that

$$h(Y) = h^*(Y). \tag{11.49}$$

The two constraints (11.48) and (11.49) provide values of the unknowns $h(0)$ and $Y$, and give

$$Y = \left(\frac{5h_0\alpha g'}{\beta^2}\right)^{1/3}, \tag{11.50}$$

which is analogous to (11.20), as well as an expression for $h(0)$ that we leave as a problem for the reader. The qualitative dependence on the parameters is similar to that of the full model, although the latitudinal extent of the Hadley Cell is proportional to the cube root of the meridional thickness gradient $\alpha$.

## 11.4 † ASYMMETRY AROUND THE EQUATOR

The Sun is overhead at the equator but two days out of the year, and in this section we investigate the effects that asymmetric heating has on the Hadley circulation. Observations indicate except for the brief periods around the equinoxes, the circulation is dominated by a single cell with rising motion centred in the summer hemisphere, but extending well into the winter hemisphere. That is, as seen in Fig. 11.3, the 'winter cell' is broader and stronger than the 'summer cell', and it behooves us to try to explain this. We will stay in the framework of the inviscid angular-momentum model of section 11.2, changing only the forcing field to represent the asymmetry and being a little more attentive to the details of spherical geometry.[8]

To represent an asymmetric heating we may choose a radiative equilibrium temperature of the form

$$\begin{aligned}\frac{\theta_E(\vartheta, z)}{\theta_0} &= 1 - \frac{2}{3}\Delta_H P_2(\sin\vartheta - \sin\vartheta_0) + \Delta_V\left(\frac{z}{H} - \frac{1}{2}\right)\\ &= 1 + \frac{\Delta_H}{3}\left[1 - 3(\sin\vartheta - \sin\vartheta_0)^2\right] + \Delta_V\left(\frac{z}{H} - \frac{1}{2}\right).\end{aligned} \tag{11.51}$$

This is similar to (11.14), but now the forcing temperature falls monotonically from a specified latitude $\vartheta_0$. If $\vartheta_0 = 0$ the model is identical to the earlier one, but if not we envision

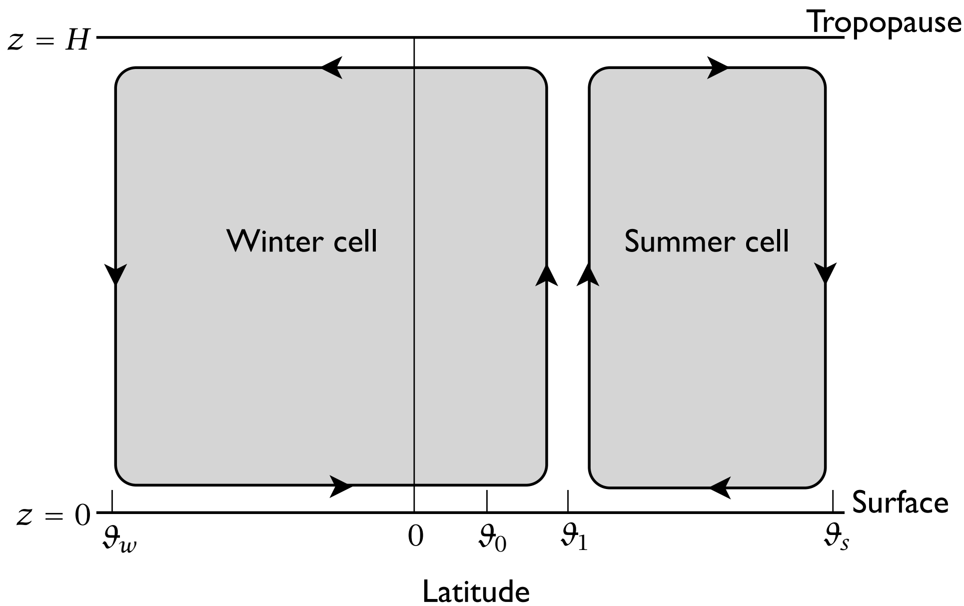


**Fig. 11.9** Schematic of a Hadley circulation model when the heating is centred off the equator, at a latitude $\vartheta_0$. The lower level convergence occurs at a latitude $\vartheta_1$ that is not in general equal to $\vartheta_0$. The resulting winter Hadley Cell is stronger and wider than the summer cell.

a circulation as qualitatively sketched in Fig. 11.9, with rising motion off the equator at some latitude $\vartheta_1$, extending into the winter hemisphere to a latitude $\vartheta_w$, and into the summer hemisphere to $\vartheta_s$. We will discover that, in general, $\vartheta_1 \neq \vartheta_0$ except when $\vartheta_0 = 0$. Following our procedure we used in the symmetric case as closely as possible, we then make the following assumptions.

*(i)* The flow is quasi-steady. That is, for any given time of year the flow adjusts to a steady circulation on a time scale more rapid than that on which the solar zenith angle appreciably changes. Then, even though the forcing is time-dependent, we neglect local time derivatives in the momentum and thermodynamic equations.

*(ii)* The flows in the upper branches conserve angular momentum, $m$. Further assuming that $u = 0$ at $\vartheta = \vartheta_1$ so that $m = \Omega a^2 \cos^2 \vartheta_1$ we obtain

$$u(\vartheta) = \frac{\Omega a(\cos^2 \vartheta_1 - \cos^2 \vartheta)}{\cos \vartheta}. \qquad (11.52)$$

Thus, we expect to see westward (negative) winds aloft at the equator. In the lower branches the zonal flow is assumed to be approximately zero, i.e., $u(0) \approx 0$.

*(iii)* The flow satisfies cyclostrophic and hydrostatic balance. Cyclostrophic balance in the meridional momentum equation is

$$fu + \frac{u^2 \tan \vartheta}{a} = -\frac{1}{a}\frac{\partial \phi}{\partial \vartheta}, \qquad (11.53)$$

and because the flow crosses the equator we cannot neglect the second term on the left-hand side. Combining this with hydrostatic balance ($\partial\phi/\partial z = g\delta\theta/\theta_0$) leads to a generalized thermal wind balance, which may be written as

$$m\frac{\partial m}{\partial z} = -\frac{ga^2\cos^2\vartheta}{2\theta_0\tan\vartheta}\frac{\partial\theta}{\partial\vartheta}, \tag{11.54}$$

where here and henceforth we write $\theta$ in place of $\delta theta$. If the undifferentiated $m$ is approximated by $\Omega a^2\cos^2\vartheta$, this reduces to conventional thermal wind balance, (11.10). The form of (11.54) is useful because we are assuming that $m$ is conserved, and so from it we can immediately infer the temperature distribution.

*(iv)* Potential temperature in each cell is conserved when integrated over the extent of the cell. Thus,

$$\int_{\vartheta_1}^{\vartheta_s}(\theta-\theta_E)\cos\vartheta\,\mathrm{d}\vartheta = 0, \qquad \int_{\vartheta_1}^{\vartheta_w}(\theta-\theta_E)\cos\vartheta\,\mathrm{d}\vartheta = 0, \tag{11.55}$$

for the summer and winter cells, respectively, where $\theta$ is the vertically averaged potential temperature.

*(v)* Potential temperature is continuous at the edge of each cell, so that

$$\theta(\vartheta_s) = \theta_E(\vartheta_s), \qquad \theta(\vartheta_w) = \theta_E(\vartheta_w), \tag{11.56}$$

and is also continuous at $\vartheta_1$. This last condition must be explicitly imposed in the asymmetric model, whereas in the symmetric model it holds by symmetry. Now, recall from the symmetric model that the value of the temperature at the equator was determined by the integral constraint (11.18) and the continuity constraint (11.19). We have analogues of these in each hemisphere [(11.55) and (11.56)] and thus, if $\vartheta_1$ is set equal to $\vartheta_0$ we cannot expect that they each would give the same temperature at $\vartheta_0$. Thus. $\vartheta_1$ must be a free parameter to be determined.

Given these assumptions, the solution may be calculated. Using thermal wind balance, (11.54), with $m(H) = \Omega a^2\cos^2\vartheta_1$ and $m(0) = \Omega a^2\cos^2\vartheta$ we find

$$-\frac{1}{\theta_0}\frac{\partial\theta}{\partial\vartheta} = \frac{\Omega^2a^2}{gH}\left(\frac{\sin\vartheta}{\cos^3\vartheta}\cos^4\vartheta_1 - \sin\vartheta\cos\vartheta\right), \tag{11.57}$$

which integrates to

$$\theta(\vartheta) - \theta(\vartheta_1) = -\frac{\theta_0\Omega^2a^2}{2gH}\frac{(\sin^2\vartheta - \sin^2\vartheta_1)^2}{\cos^2\vartheta}. \tag{11.58}$$

The value of $\vartheta_1$, and the value of $\theta(\vartheta_1)$, are determined by the constraints (11.55) and (11.56). It is not in general possible to obtain a solution analytically, but one may be found numerically by an iterative procedure and one such is illustrated in Fig. 11.10. The zonal wind of the solution is always symmetric around the equator, because it is determined solely by angular momentum conservation. The temperature is therefore also symmetric, as (11.58) explicitly shows. However, the width of the solution in each hemisphere will, in general, be different. Furthermore, because the strength of the circulation increases with difference between the temperature of the solution and the radiative equilibrium temperature, the

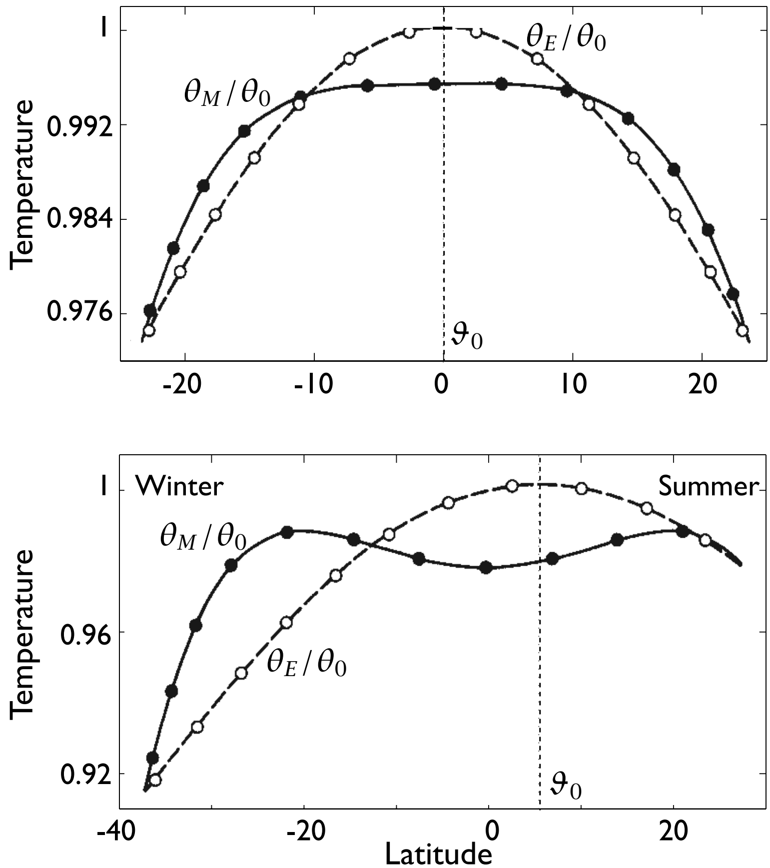


**Fig. 11.10** Solutions of the Hadley Cell model with heating centred at the equator ($\vartheta_0 = 0°$, top) and off the equator ($\vartheta_0 = +6°$N, bottom), with $\Delta_H = 1/6$. The dashed line is the radiative equilibrium temperature and the solid line is the angular-momentum-conserving solution. In the lower panel, $\vartheta_1 \approx +18°$, and the circulation is dominated by the cell extending from $+18°$ to $-36°$.[9]

circulation in the winter hemisphere will also be much stronger than that in the summer, a prediction that is qualitatively consistent with the observations (see Fig. 11.3). More detailed calculations show that, because the strength of the model Hadley Cell increases nonlinearly with $\vartheta_0$, the time-average strength of the Hadley Cell with seasonal forcing is stronger that that produced by annually averaged forcing. However, this does not appear to be a feature of either the observations or more complete numerical simulations, suggesting that an angular-momentum-conserving model has, at the least, quantitative deficiencies, as follows.[10]

*(i)* The lack of consideration of zonal asymmetries, such as monsoonal circulations.

*(ii)* The quasi-steady assumption, given the presence of a temporally progressing seasonal cycle (even in presence of zonally symmetric boundary conditions). Because the latitude of the upward branch of the Hadley Cell varies with season, the value of the angular

momentum entering the system also varies, and so a homogenized value of angular momentum is hard to achieve. For quasi-steadiness to hold we require

$$\tau_s \gg \tau_e, \tag{11.59}$$

where $\tau_e$ is some dynamical equilibration time scale, similar to the dynamical timescale $\tau_d = H/w$ (section 11.2.6), and $\tau_s$ is the seasonal time scale

*(iii)* The lack of angular momentum conservation in reality (a criticism that would also apply to a steady model with zonally symmetric boundary conditions). Such non-conservation will arise if either diffusion of momentum caused by small-scale turbulence, or the angular momentum transport by baroclinic eddies, are significant.

Nonetheless, the overall picture that the model paints, and its qualitative explanation of the strengthened and extended winter Hadley Cell, are invaluable aids to our understanding of the circulation.

## 11.5 EDDIES, VISCOSITY AND THE HADLEY CELL

So far, we have ignored the effects of baroclinic eddies on the Hadley circulation — 'ignored' rather than 'neglected', because we have no a priori or observational reason to believe that their effects will be negligible. If their effects are strong, then none of the models we discussed above will be quantitatively valid. With this in mind, in this section we look at the Hadley circulation from a quite different perspective, by supposing that the zonal momentum equation is linear, except for the effects of eddy fluxes on the right-hand side. Our approach is illustrative, not quantitative, and we again stay within the Boussinesq approximation.

We might expect eddy fluxes to be important because the angular momentum conserving solution will develop a large vertical shear and if this extends sufficiently far polewards it will become baroclinically unstable (compare Fig. 11.7 with the minimum shear needed for baroclinic instability sketched in Fig. 6.16). It is a quantitative issue as to whether the Hadley flow becomes strongly unstable before it reaches its poleward extent, and if it does not the angular-momentum-conserving solution might be expected to be a good one. But here let us assume that the flow is strongly unstable and that the ensuing instability transfers both heat and angular momentum polewards (the mechanisms of this are discussed in the next chapter). This transfer will lead to the nonconservation of angular momentum and, potentially, render invalid the models of the previous sections.

### 11.5.1 Qualitative considerations

The zonally averaged zonal momentum equation, (11.4), may be written as an equation for angular momentum, $\overline{m}$. Referring back to section 2.2 if needs be, the equation may be written as

$$\begin{aligned}
\frac{\partial \overline{m}}{\partial t} + \frac{1}{\cos\vartheta}\frac{\partial}{\partial y}(\overline{v}\,\overline{m}\cos\vartheta) + \frac{\partial}{\partial z}(\overline{w}\,\overline{m}) \\
&= -\frac{1}{\cos\vartheta}\frac{\partial}{\partial y}(\overline{m'v'}\cos\vartheta) - \frac{\partial}{\partial z}(\overline{m'w'}) \\
&= -\frac{1}{\cos\vartheta}\frac{\partial}{\partial y}(\overline{u'v'}\,a\cos^2\vartheta) - \frac{\partial}{\partial z}(\overline{u'w'}\,a\cos\vartheta),
\end{aligned} \tag{11.60}$$

where $\overline{m} = (\overline{u} + \Omega a \cos\vartheta) a \cos\vartheta$, $m' = u' a \cos\vartheta$, $y = a\vartheta$, and the vertical and meridional velocities are related by the mass continuity relation

$$\frac{1}{\cos\vartheta}\frac{\partial}{\partial y}(\overline{v}\cos\vartheta) + \frac{\partial \overline{w}}{\partial z} = 0. \tag{11.61}$$

In the angular-momentum-conserving model the eddy fluxes were neglected and (11.60) was approximated by the simple expression $\partial\overline{m}/\partial y = 0$, and by construction the Rossby number is $\mathcal{O}(1)$, because $\zeta = -f$.

The observed eddy heat and momentum fluxes are shown in Fig. 11.11. The eddy momentum flux is generally polewards, converging in the region of the surface westerlies. Its magnitude, and more particularly its meridional gradient, is as large or larger than the momentum flux associated with the mean flow. Neglecting vertical advection and vertical eddy fluxes, and using (11.61), (11.60) may be written as

$$\frac{\partial \overline{m}}{\partial t} + \overline{v}\frac{\partial \overline{m}}{\partial y} = -\frac{1}{\cos\vartheta}\frac{\partial}{\partial y}(\overline{u'v'} a \cos^2\vartheta). \tag{11.62}$$

Thus, if $\overline{v} > 0$ (as in the upper branch of the Northern Hemisphere Hadley Cell) and the flow is steady, the observed eddy fluxes are such as to cause the angular momentum of the zonal flow to decrease as it moves polewards, and the zonal velocity is lower than it would be in the absence of eddies. (In the Southern Hemisphere the signs of $v$ and the eddy momentum flux are reversed, but the dynamics are equivalent.) Note that we cannot a priori determine whether eddies are likely to be important by comparing the magnitudes of the eddy terms with the terms on the left-hand side of (11.62) in the angular-momentum-conserving solution, because in that solution these terms are individually zero. Rather, we should compare the eddy fluxes to $\overline{v}\partial m_e/\partial y$, where $m_e = \Omega a^2 \cos^2\vartheta$ is the angular momentum of the solid Earth.

The eddy flux of heat will also affect the Hadley Cell, although in a different fashion. We see from Fig. 11.11 that the eddy flux of temperature is predominantly polewards, and therefore that eddies export heat from the subtropics to higher latitudes. Now, the zonally averaged thermodynamic equation may be written

$$\begin{aligned}\frac{\partial \overline{b}}{\partial t} + \frac{1}{\cos\vartheta}\frac{\partial}{\partial y}(\overline{v}\overline{b}\cos\vartheta) + \frac{\partial}{\partial z}(\overline{w}\overline{b}) \\ = -\frac{1}{\cos\vartheta}\frac{\partial}{\partial y}(\overline{v'b'}\cos\vartheta) - \frac{\partial}{\partial z}(\overline{w'b'}) + Q[b].\end{aligned} \tag{11.63}$$

where $Q[b]$ represents the heating. After vertical averaging, the vertical advection terms vanish and the resulting equation is the thermodynamic equation implicitly used in the angular-momentum-conserving model, with the addition of the meridional eddy flux on the right-hand side. A diverging eddy heat flux in the subtropics (as in Fig. 11.11) is plainly equivalent to increasing the meridional gradient of the radiative equilibrium temperature, and therefore will increase the intensity of the overturning circulation.

### 11.5.2 An idealized eddy-driven model

Consider now the extreme case of an 'eddy-driven' Hadley Cell. (The driving for the Hadley Cell, and the atmospheric circulation in general, ultimately comes from the differential

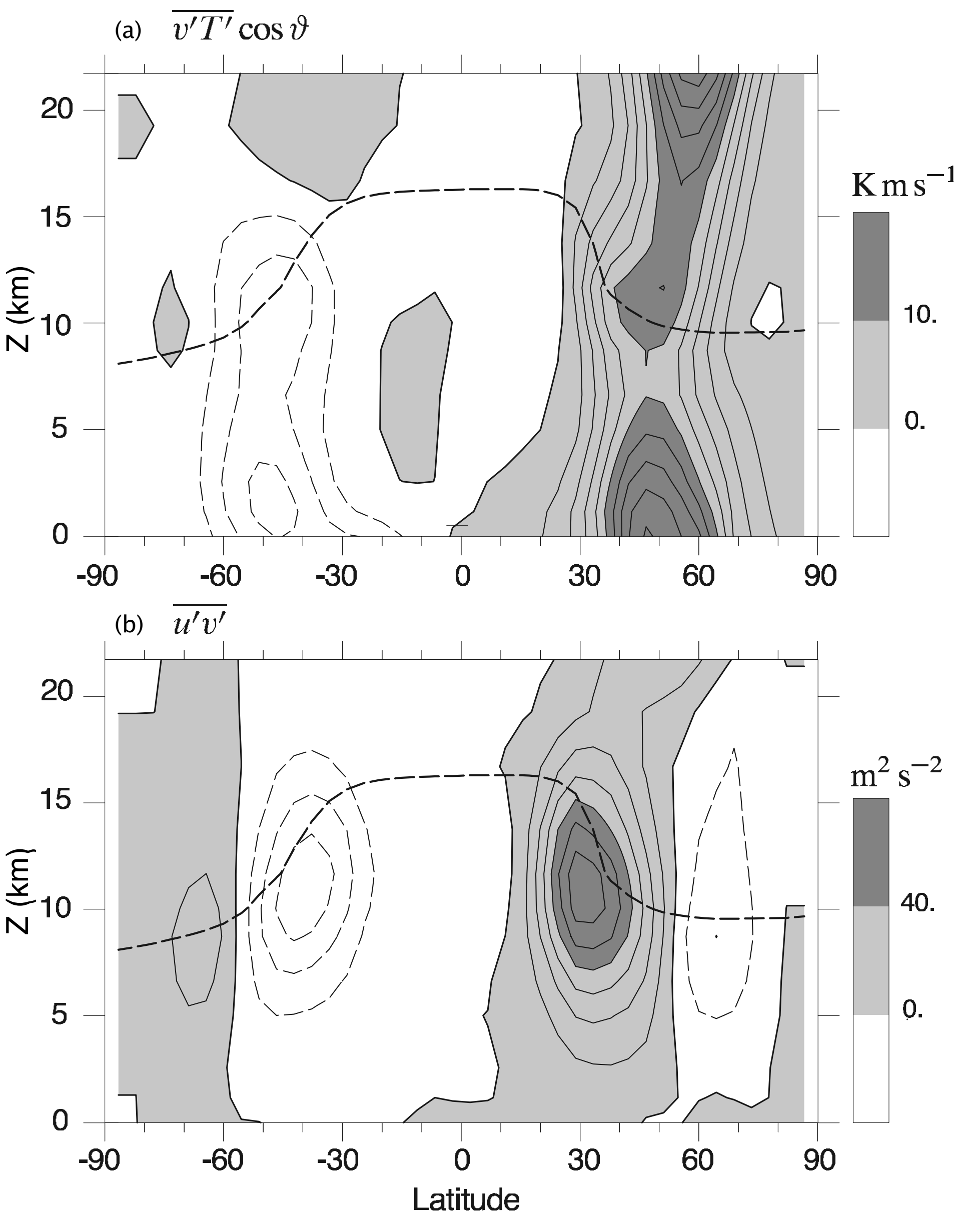


**Fig. 11.11** (a) The average meridional eddy heat flux and (b) the eddy momentum flux in the northern hemisphere winter (DJF). The ordinate is log-pressure, with scale height $H = 7.5$ km. Positive (northward) fluxes are shaded in both cases, and the dashed line marks the thermal tropopause. The eddy heat flux (contour interval $2\,\mathrm{K\,m\,s^{-1}}$) is largely polewards, and down the temperature gradient, in both hemispheres. The eddy momentum flux (contour interval $10\,\mathrm{m^2\,s^{-2}}$) *converges* in mid-latitudes in the region of the mean jet, and must be upgradient there.[11]

heating between equator and pole. Recognizing this, 'eddy driving' is a convenient way to refer to the mediating role of eddies in producing a zonally averaged circulation.) The model is over-simple, but revealing. The zonally averaged zonal momentum equation (11.4) may be written as

$$\frac{\partial \overline{u}}{\partial t} - (f + \overline{\zeta})\overline{v} = -\frac{1}{\cos^2\vartheta}\frac{\partial}{\partial \vartheta}(\cos^2\vartheta\,\overline{u'v'}). \tag{11.64}$$

If the Rossby number is sufficiently low this becomes simply

$$\frac{\partial \overline{u}}{\partial t} - f\overline{v} = M, \tag{11.65}$$

where $M$ represents the eddy terms. This approximation is not quantitatively accurate but it will highlight the role of the eddies. (Note the contrast between this model and the angular-momentum-conserving model. In the latter we assumed $f + \overline{\zeta} \approx 0$, and $Ro = \mathcal{O}(1)$; now we are neglecting $\overline{\zeta}$ and assuming the Rossby number is small.) At a similar level of approximation let us write the thermodynamic equation, (11.63), as

$$\frac{\partial \overline{b}}{\partial t} + N^2 w = J, \tag{11.66}$$

where $J = Q[b] - (\cos\vartheta)^{-1}\partial_y(\overline{v'b'}\cos\vartheta)$ represents the diabatic terms and eddy forcing. We are assuming, as in quasi-geostrophic theory, that the mean stratification, $N^2$ is fixed, and now $\overline{b}$ represents only the (zonally averaged) deviations from this. If we simplify further by using Cartesian geometry then the mass conservation is

$$\frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0, \tag{11.67}$$

and we may define a meridional streamfunction $\Psi$ such that

$$\overline{w} = \frac{\partial \Psi}{\partial y}, \qquad \overline{v} = -\frac{\partial \Psi}{\partial z}. \tag{11.68}$$

We may then use the thermal wind relation,

$$f\frac{\partial \overline{u}}{\partial z} = -\frac{\partial \overline{b}}{\partial y}, \tag{11.69}$$

to eliminate time derivatives in (11.65) and (11.66), giving

$$f^2\frac{\partial^2 \Psi}{\partial z^2} + N^2\frac{\partial^2 \Psi}{\partial y^2} = f\frac{\partial M}{\partial z} + \frac{\partial J}{\partial y}. \tag{11.70}$$

This is a linear equation for the overturning streamfunction, one that holds even if the flow is not in a steady state, and we see that the overturning circulation is forced by eddy fluxes of heat and momentum, as well as heating and other terms that might appear on the right-hand sides of (11.65) and (11.66). If we rescale the vertical coordinate by the Prandtl ratio (i.e., let $z = z'f/N$) then (11.70) is a Poisson equation for the streamfunction. A few other germane points are as follows.

*(i)* The horizontal gradient of the thermodynamic forcing partially drives the circulation. At low latitudes, both the heating term and the horizontal eddy flux divergence act in the same sense. An overturning circulation that is forced by diabatic terms, and so with warm fluid rising and cold fluid sinking, is called a 'direct cell'.

*(ii)* The vertical gradient of the horizontal eddy momentum divergence also partially drives the circulation, and from Fig. 11.11 it is clear these fluxes will intensify the circulation. That is, the same terms that cause angular momentum non-conservation act to strengthen the overturning circulation. This balance is reflected in the momentum equation — the Coriolis term $fv$ is balanced by the eddy momentum flux convergence.

*(iii)* If $M$ contains frictional terms, such as $\nu\partial^2 u/\partial z^2$, then these may also act to strengthen the meridional circulation, and weaken angular momentum conservation.

*(iv)* If $N$ is small, then the circulation will become stronger if the other terms remain the same. That is, a dry atmosphere with a lapse rate close to that of a dry adiabat (i.e., $N = 0$) may have a stronger overturning circulation than otherwise, because the air can circulate without transporting any heat.

*(v)* In winter, the increased strength of eddy momentum and buoyancy fluxes will drive a stronger Hadley Cell. This constitutes a different mechanism from that given in section 11.4 for the increased strength of the winter cell.

** A slight generalization*

We can generalize (11.70) somewhat by replacing (11.65) by (11.62), namely

$$\frac{\partial \overline{m}}{\partial t} + \overline{v}\frac{\partial \overline{m}}{\partial y} = M. \tag{11.71}$$

Then, using thermal wind equation in the form

$$\frac{f}{a\cos\vartheta}\frac{\partial \overline{m}}{\partial z} = -\frac{\partial \overline{b}}{\partial y}, \tag{11.72}$$

an equation very similar to (11.70) may be derived. However, the coefficients on the left-hand side are functions of the solution, and $f^2\partial_{zz}\Psi$ in (11.70) is replaced by a term like $(f\partial_y\overline{m})(\partial_{zz}\Psi)$. Then, to the extent that $\partial\overline{m}/\partial y < f$ and the other terms are the same, the overturning will be stronger than that given using (11.70).[12]

## 11.6 THE HADLEY CELL: SUMMARY AND NUMERICAL SOLUTIONS

We have presented two models for the Hadley Cell: (i) an angular momentum conserving model and (ii) a largely eddy-driven model. The two models are opposite extremes, both being severe approximations to a more complete representation of the Hadley Cell that might comprise the zonal momentum equation with eddies (11.60), the thermodynamic equation (11.63) (with the effects of moisture included) and the meridional momentum equation, perhaps approximated as cyclostrophic wind balance. In reality, both the conservative effects of angular momentum advection and the effects of eddy fluxes likely play a role, and delineating the importance of their respective effects is a task that must be guided by observations and numerical simulations.

Illustrative results from two idealized GCM experiments are shown in Figs. 11.12 and 11.13. The GCM has no explicit representation of moisture, except that the lapse rate is adjusted to a value close to the moist adiabatic lapse rate if it exceeds that value. In one experiment the model is constrained to produce an axisymmetric solution (top panels of

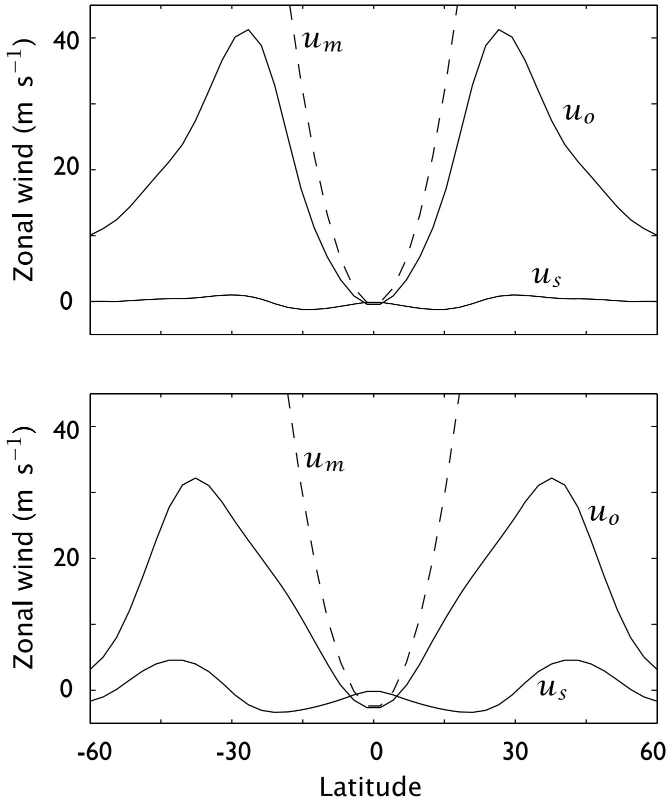


**Fig. 11.12** The zonal wind in two numerical simulations. The lower panel is from an idealized dry, three-dimensional atmospheric GCM, and the upper panel is an axisymmetric version of the same model. Plotted are the zonal wind at the level of the Hadley Cell outflow, $u_o$; the surface wind, $u_s$; and the angular-momentum-conserving value, $u_m$.[13]

the figures), and the zonal wind produced by the model in the Hadley Cell outflow is fairly close to being angular-momentum-conserving. (The lack of perfect angular momentum conservation is due to the presence of a small but finite vertical viscosity that is necessary in order to reach a steady state; without it, the steady state becomes symmetrically unstable.) In a three-dimensional version of the model, in which baroclinic eddies are allowed to form, the zonal wind is significantly reduced from its angular-momentum-conserving value, and correspondingly the overturning circulation is much stronger. Indeed, the strength of the Hadley Cell increases roughly linearly with the strength of the eddies in a sequence of numerical integrations similar to those shown, as suggested by (11.70). Qualitatively similar results are found in a model with no convective parameterization. In this case, the lapse rate is closer to neutral, $N^2$ is small, and the overturning circulation is generally stronger, as also expected from (11.70).

Is the real Hadley circulation 'eddy-driven', as in section 11.5.2, or is it a largely zonally symmetric structure constrained by angular momentum conservation, as in section 11.2 and

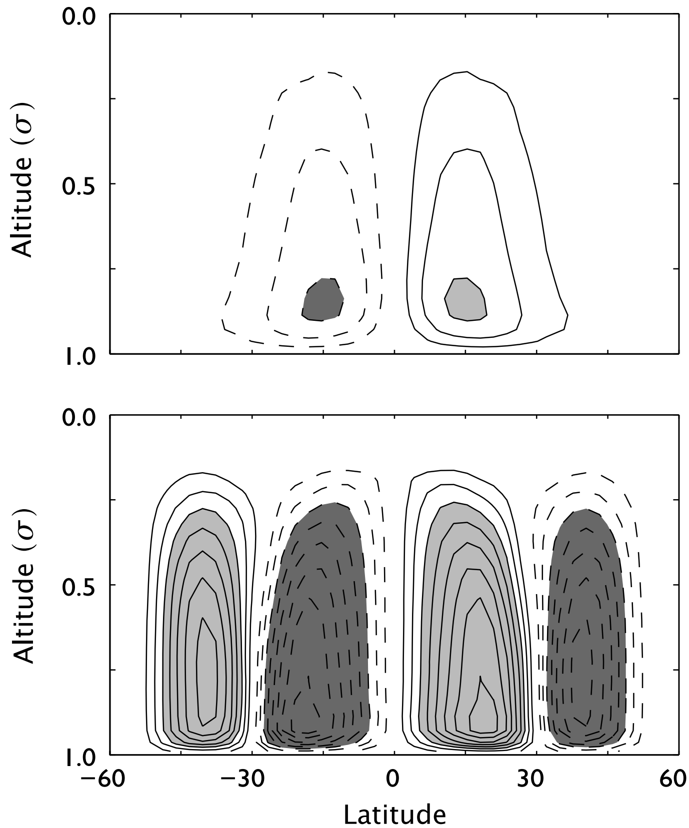


**Fig. 11.13** As for Fig. 11.12, but now showing the streamfunction of the overturning circulation. 'Altitude' is $\sigma = p/p_s$, where $p_s$ is surface pressure, and contour interval is 5 Sv (i.e., $5 \times 10^9$ kg s$^{-1}$).

its hemispherically asymmetric extensions? Observations of the overturning flow in summer and winter provide a guide. Figure 11.14 shows the thickness weighted transport overturning circulation in isentropic coordinates, and (as discussed more in chapter 7) this circulation includes both the Eulerian mean transport and the transport due to eddies. In winter there is considerable recirculation within the Hadley Cell, most of it coming from the zonally symmetric flow, and it is a quite distinct structure from the mid-latitude circulation. This suggests that the poleward extent of the winter Hadley Cell is influenced by axisymmetric dynamics, for if it were solely a response to eddy heat fluxes one might expect it to join more smoothly with the mid-latitude Ferrel Cell. It may be that the effects of condensation and the concentration of the thermodynamic source act to give the axisymmetric circulation a significant role. However, a winter Hadley Cell strongly influenced by eddy momentum fluxes might still strongly recirculate, and eddy effects do almost certainly play an important role. In summer, there is virtually no recirculation within the Hadley Cell and it does not appear as a self-contained structure, and this is suggestive of eddy effects and/or a strong mid-latitude influence. However, the above remarks are very speculative.

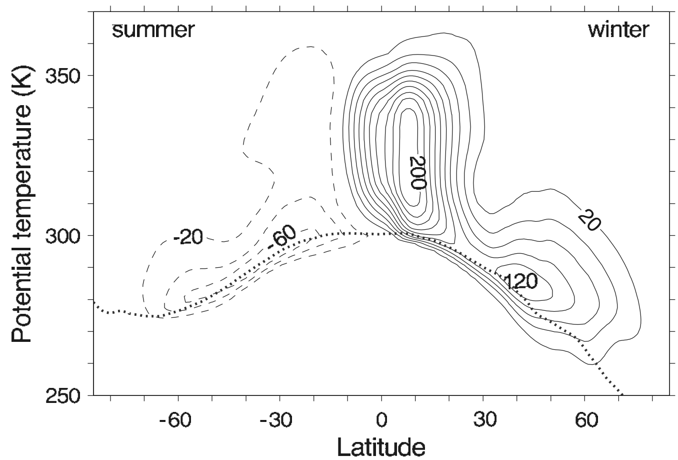


**Fig. 11.14** The observed mass transport streamfunction in isentropic coordinates in northern hemisphere winter (DJF). The dotted line is the median surface temperature. The return flow is nearly all in a layer near the surface, much of it at a lower temperature than the median surface temperature. Note the more vigorous circulation in the winter hemisphere.[14]

## 11.7 THE FERREL CELL

In this section we give a brief introduction to the Ferrel Cell, taking the eddy fluxes of heat and momentum to be given and viewing the circulation from a zonally averaged and Eulerian perspective. We investigate the associated dynamics in the next chapter.

The Ferrel Cell is an indirect meridional overturning circulation in mid-latitudes (see Fig. 11.3) that is apparent in the zonally averaged $v$ and $w$ fields, or the meridional overturning circulation defined by (11.3) or (11.68). It is 'indirect' because cool air apparently rises in high latitudes, moves equatorwards and sinks in the subtropics. Why should such a circulation exist? The answer, in short, is that it is there to balance the eddy momentum convergence of the mid-latitude eddies and it is effectively driven by those eddies. To see this, consider the zonally averaged zonal momentum equation in mid-latitudes; at low Rossby number, and for steady flow this is just

$$-f\overline{v} = -\frac{1}{\cos^2\vartheta}\frac{\partial}{\partial\vartheta}(\cos^2\vartheta\,\overline{u'v'}) + \frac{1}{\rho}\frac{\partial\tau}{\partial z}. \tag{11.73}$$

This is a steady version of (11.65) with the addition of a frictional term $\partial\tau/\partial z$ on the right-hand side. At the surface we might approximate the stress by a drag, $\tau = r\overline{u}_s$, where $r$ is a constant, with the stress falling away with height so that it is important only in the lowest kilometre or so of the atmosphere, in the atmospheric Ekman layer. Above this layer, the eddy momentum flux convergence is balanced by the Coriolis force on the meridional flow. In mid-latitudes (from about 30° to 70°) the eddy momentum flux divergence is negative in both hemispheres (Fig. 11.11) and therefore, from (11.73), the averaged meridional flow must be equatorwards, as illustrated schematically in Fig. 11.15.

The flow cannot be equatorwards everywhere, simply by mass continuity, and the return flow occurs largely in the Ekman layer, of depth $d$ say. Here the eddy balance is between the Coriolis term and the frictional term, and integrating over this layer gives

$$-fV \approx -r\overline{u}_s, \tag{11.74}$$

where $V = \int_0^d \rho\overline{v}\,dz$ is the meridional transport in the boundary layer, above which the stress vanishes. The return flow is polewards (i.e., $V > 0$ in the Northern Hemisphere) producing

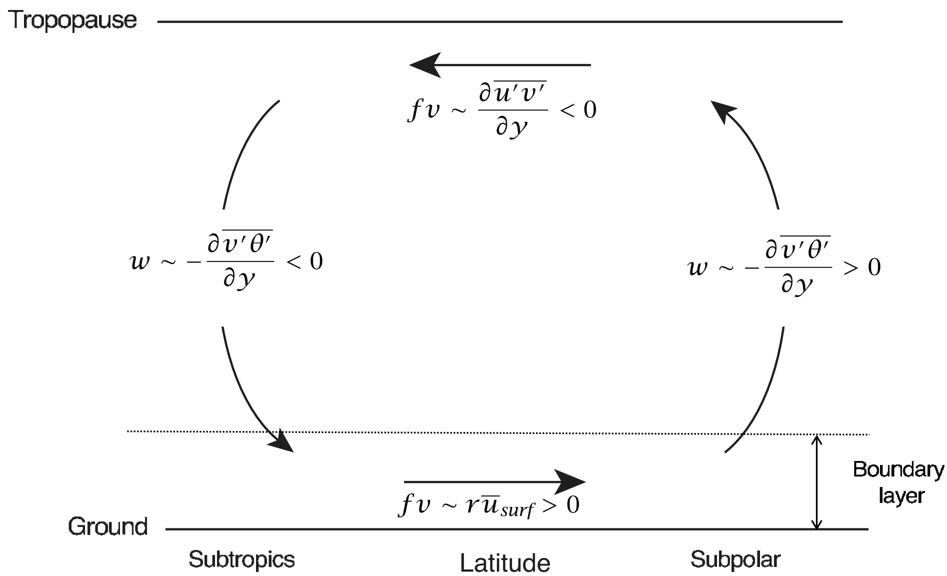


**Fig. 11.15** The eddy-driven Ferrel Cell, from an Eulerian point of view. Above the planetary boundary layer the mean flow is largely in balance with the eddy heat and momentum fluxes, as shown. The lower branch of the Ferrel Cell is largely confined to the boundary layer, where it is in a frictional–geostrophic balance.

an eastward Coriolis force. This can be balanced by a westward frictional force provided that the surface flow has an eastward component. In this picture, then, the mid-latitude eastward zonal flow at the surface is a consequence of the polewards flowing surface branch of the Ferrel Cell, this poleward flow being required by mass continuity given the equatorward flow in the upper branch of the cell. In this way, the Ferrel Cell is responsible for bringing the mid-latitude eddy momentum flux convergence to the surface where it may be balanced by friction (refer again to Fig. 11.15).

A more direct way to see that the surface flow must be eastwards, given the eddy momentum flux convergence, is to vertically integrate (11.73) from the surface to the top of the atmosphere. By mass conservation, the Coriolis term vanishes (i.e., $\int_0^\infty f\rho\overline{v}\,\mathrm{d}z = 0$) and we obtain

$$\int_0^\infty \frac{1}{\cos^2\vartheta}\frac{\partial}{\partial\vartheta}(\cos^2\vartheta\,\overline{u'v'})\rho\,\mathrm{d}z = [\tau]_0^\infty = -r\overline{u}_s. \tag{11.75}$$

That is, the surface wind is proportional to the vertically integrated eddy momentum flux convergence. Because there *is* a momentum flux convergence, the left-hand side is negative and the surface winds are eastwards.

The eddy heat flux also plays a role in the Ferrel Cell, for in a steady state we have, from (11.66)

$$w = \frac{1}{N^2}\left[Q[b] - \frac{1}{\cos\vartheta}\frac{\partial(\overline{v'b'}\cos\vartheta)}{\partial y}\right] \tag{11.76}$$

and inspection of Fig. 11.11 shows that the observed eddy heat flux produces an overturning circulation in the same sense as the observed Ferrel Cell (again see Fig. 11.15).

Is the circulation produced by the heat fluxes *necessarily* the same as that produced by the momentum fluxes? In a non-steady state the effects of both heat and momentum fluxes on the Ferrel Cell are determined by (11.70) (an equation which in fact applies more accurately at mid-latitudes than at low ones because of the low-Rossby number assumption), and there is no particular need for the heat and momentum fluxes to act in the same way. But in a steady state they must act to produce a consistent circulation. To show this, for simplicity let us take $f$ and $N^2$ to be constant, let us suppose the fluid is incompressible and work in Cartesian coordinates. Take the $y$-derivative of (11.73) and the $z$-derivative of (11.76) and use the mass continuity equation. Noting that $\overline{v'\zeta'} = -\partial\overline{u'v'}/\partial y$ we obtain

$$\frac{\partial}{\partial y}\left(\overline{v'\zeta'} + \frac{f_0}{N^2}\frac{\partial\overline{v'b'}}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}Q[b]\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial\tau}{\partial z}\right). \tag{11.77}$$

The expression on the left-hand side is the divergence of the eddy flux of quasi-geostrophic potential vorticity! That the heat and momentum fluxes act to produce a consistent overturning circulation is thus equivalent to requiring that the terms in the quasi-geostrophic potential vorticity equation are in a steady-state balance.

## Notes

1 Many of the observations presented here are so-called 'reanalyses', prepared by the National Centers for Environmental Prediction (NCEP) and the European Centre for Medium-Range Weather Forecasts (ECMWF) (e.g., Kalnay 1996). Unless stated, we use the NCEP reanalysis with data from 1958–2003. Reanalysis products are syntheses of observations and model results and so are not wholly accurate representations of the atmosphere. However, especially in data-sparse regions of the globe and for poorly measured fields, they are likely to be more accurate representations of the atmosphere than could be achieved using only the raw data. Of course, this in turn means they contain biases introduced by the models.

2 For example, Trenberth & Caron (2001).

3 George Hadley (1685–1768) was a British meteorologist who formulated the first dynamical theory for the trade winds, presented in a paper (Hadley 1735) entitled 'Concerning the cause of the general trade winds.' (At that time, trade winds referred to any large-scale prevailing wind, and not just tropical winds. Some etymologists have associated the name with the commercial (i.e., trade) exploitation of the wind by mariners on long ocean journeys, but such an origin has been disputed: trade also means customary, and the winds customarily blow in one direction. Relatedly, in Middle English the word trade means path or track — hence the phrase 'the wind blows trade', meaning the wind is on track.) Hadley realized that in order to account for the zonal winds, the Earth's rotation makes it necessary for there also to be a meridional circulation. His vision was of air heated at low latitudes, cooled at high latitudes, giving rise to a single meridional cell between the equator and each pole. Although he thought of the cell as essentially filling the hemisphere, and he did not account for the instability of such a flow, it was nevertheless a foundational contribution to meteorology. The thermally direct cell in low latitudes is now named after him.

A three-celled circulation was proposed by William Ferrel (1817–1891), an American school teacher and meteorologist, and the middle of these cells is now named for him. His explanation of the cell (Ferrel 1856a) was not correct, but this is hardly surprising because the eddy motion producing the angular momentum convergence that drives the Ferrel Cell was not understood for another 100 years or so. Ferrel in fact abandoned his three-celled picture in favour of a something more akin to a two-celled picture (Ferrel 1859), similar to that proposed by J. Thompson in 1857. The history of these ideas, and those of Hadley, is discussed by Thomson (1892). Ferrel did however give the first essentially correct description of the role of the Coriolis force and the geostrophic wind in the general circulation (Ferrel 1858, a paper with a quite modern style), a key development in the history of geophysical fluid dynamics. Ferrel also contributed to tidal theory [in Ferrel (1856b) he noted, for example, that the tidal force due to the moon would slow the Earth's rotation] and to ocean dynamics. (See http://www.history.noaa.gov/giants/ferrel2.html).

Although Hadley's single-celled viewpoint was in part superseded by the three-celled and two-celled structures, the modern view of the overturning circulation is, ironically, that of a single cell of 'residual circulation', which, although having distinct tropical and extratropical components, in some ways qualitatively resembles Hadley's original picture. See chapter 12.

4 Following Held & Hou (1980). Schneider (1977) and Schneider & Lindzen (1977) were particularly influential precursors.

5 Nevertheless, Fang & Tung (1996) do find some analytic solutions in the presence of moisture and convection.

6 After Hide (1969).

7 Gierasch (1975).

8 Largely following Lindzen & Hou (1988).

9 Solutions from Lindzen & Hou (1988).

10 See Dima & Wallace (2003) for some relevant observations. They noted that the asymmetry of the Hadley Cell is affected by monsoonal circulations (which are, of course, not accounted for in the model presented here). Fang & Tung (1999) investigated the effects of time dependence, and essentially noted that (11.59) is not well satisfied, although this alone was unable to limit the nonlinear amplification effect. Walker & Schneider (2005) showed that the effects of vertical momentum diffusion and of momentum transport by baroclinic eddies are both significant in a GCM, and these limit the nonlinear amplification.

11 Figure courtesy of M. Juckes, using an ECMWF reanalysis.

12 A still more general, usually elliptic equation for the overturning circulation may be derived from the zonally averaged primitive equations, assuming only that the zonally averaged zonal wind is in cyclostrophic balance with the pressure field (Vallis 1982).

13 Simulations kindly performed by C. Walker. See also Walker and Schneider (2005).

14 Figure courtesy of T. Schneider, using an ECMWF reanalysis.

## *Further reading*

Lorenz, E. N., 1967. *The Nature and Theory of the General Circulation of the Atmosphere.*
A classic monograph on the atmospheric general circulation.

Peixoto, J. P. & Oort, A. H., 1992. *Physics of Climate.*
A descriptive but physically based discussion of the climate and the general circulation, with an emphasis on observations.

## *Problems*

11.1 Explicitly derive equations (11.20) and (11.21).

11.2 Suppose that, in the vertically integrated Hadley Cell model considered in section 11.2 the radiative equilibrium temperature falls linearly from the equator to the pole. For example, suppose that $\theta_E = \theta_{E0} - \Delta\Theta(|y|/a)$, rather the quadratic fall-off in (11.16). Obtain and discuss the solutions to the Held–Hou problem. Include an expression for the latitudinal extent of the Hadley Cell, and comment on any discontinuities at the edge of the Hadley Cell and at the equator.

11.3 By considering the form of the viscous term in spherical coordinates, show explicitly that an atmosphere in solid body rotation experiences no viscous stresses.

11.4 *(a)* Suppose the zonal wind is in thermal wind balance with the radiative equilibrium temperature, $\theta_E$. Obtain a condition for the meridional variation of the radiative equilibrium temperature so that the wind does not violate Hide's theorem (that is, it does not produce an interior maximum of angular momentum). For example, show the radiative equilibrium temperature must not vary more rapidly with latitude faster than the latitude raised to some power. You may make the small-angle approximation. Alternatively, you could use thermal wind balance in the form (2.206) or (2.208). You may assume the surface wind is zero if needs be.

*(b)* Suppose the radiative equilibrium temperature falls off with latitude as $P_4(\sin\vartheta)$, where $P_4$ is the fourth Legendre polynomial. Show that the zonal velocity that is in thermal wind balance with this does not violate Hide's theorem. Comment on the relevance of this to the issue of whether the radiative equilibrium solution is physically realizable.

11.5 In the angular-momentum-conserving model of the Hadley Cell, air that starts at rest at the equator develops a large zonal velocity, and hence a large kinetic energy, as it moves polewards. Explain carefully where this energy comes from. (Note that the Coriolis force itself does no work on a fluid parcel.)

11.6 A spinning ice skater with arms outstretched lowers his arms. Show that if the skater's angular momentum is conserved his kinetic energy increases. Where has this energy come from? Is it different if the skater raises his arms?

11.7 *(a)* Derive and plot the layer thickness as a function of latitude in the shallow water Hadley Cell model, and the corresponding zonal wind.

*(b)* Suppose that the equilibrium thickness, $h^*$, falls quadratically with latitude, rather than linearly as we assume in (11.47). Obtain and plot expressions for the extent of the Hadley Cell, the thickness and the zonal wind.

11.8 ♦ The oceanic thermohaline circulation seems similar to Hadley's vision of the atmospheric circulation, with a large thermally driven cell between pole and equator. Discuss. Is conservation of angular momentum an important factor in the thermohaline circulation? If so, what are its manifestations? If not, why not?