d-wave quasiparticles in the vortex state

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Contents

1 Basic lengthscales

Estimate of the coherence length:

$$\xi \approx \frac{\hbar v_F}{\Delta} \approx \frac{(1.5eV \times \text{Å})}{40meV} \sim 35\text{Å} \tag{1}$$

(Vishik et.al. PRL 104, 207002 (2010); PNAS vol. 109 no. 45 18332 (2012)) Recall, this is the lengthscale over which the (single component) Ginzburg-Landau order parameter recovers near a defect. It sets the scale of the core size.

Penetration depth:

$$\lambda \approx 1500\text{Å}. \tag{2}$$

(see e.g. Blatter et.al. RMP 1994, p1162).

Abrikosov parameter:

$$\kappa = \frac{\lambda}{\xi} \approx 45. \tag{3}$$

This places HTS in the extreme type-II limit.

Magnetic length (typical separation between vortices)

$$\ell_v = \frac{1}{\sqrt{2}}\ell = \sqrt{\frac{hc}{2eB}} = 454\text{Å}\frac{1}{\sqrt{B[T]}} \tag{4}$$

(material independent, involves only fundamental constants)

We are interested in the limit

$$\xi \ll \ell \ll \lambda \tag{5}$$

(Equivalent to $H_{c1} \ll H \ll H_{c2}$.)

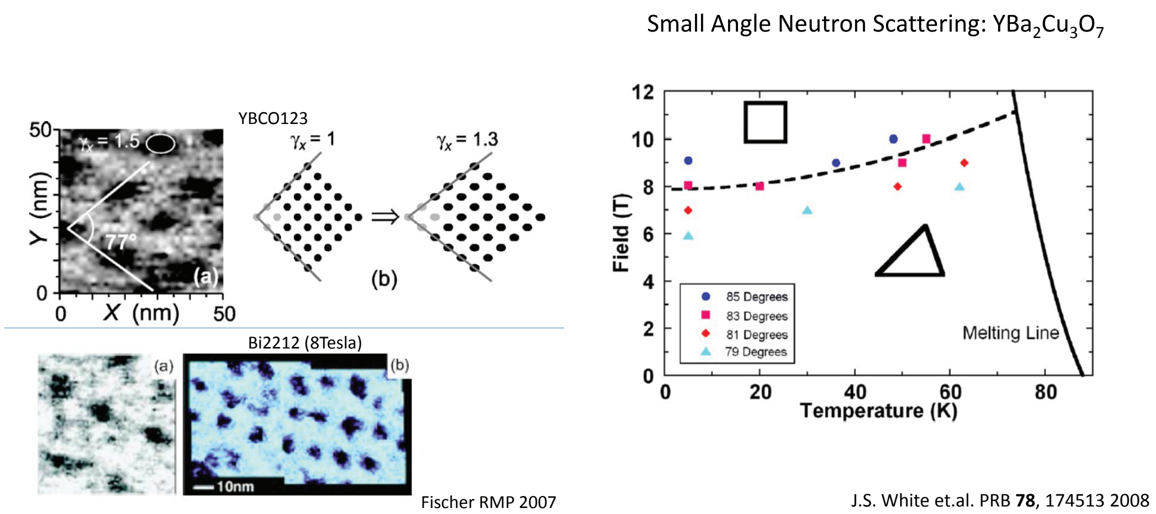

Figure 1: (Left) STM images of vortex lattices in HTS. (Right) Vortex lattice geometries from the small angle neutron scattering in YBCO

2 Volovik's semiclassical effect and the spectrum of a d-wave superconductor in the presence of a uniform supercurrent

2.1 qp spectrum for an s-wave superconductor with a uniform supercurrent

As a warmup, consider the model of an s-wave superconducting pairing term at finite momentum (coresponding a finite supercurrent) $\Delta(\mathbf{r})c^\dagger_{\mathbf{r}\uparrow}c^\dagger_{\mathbf{r}\downarrow} + h.c. = e^{2i\mathbf{q}\cdot\mathbf{r}}\Delta c^\dagger_{\mathbf{r}\uparrow}c^\dagger_{\mathbf{r}\downarrow} + e^{-2i\mathbf{q}\cdot\mathbf{r}}\Delta c_{\mathbf{r}\downarrow}c_{\mathbf{r}\uparrow}$. We will assume a normal state dispersion (with the chemical potential shift included) $\epsilon_{\mathbf{k}} = \xi_{\mathbf{k}} - \mu$. (The distinction between being in real and reciprocal space is in using the subscript $\mathbf{r}$ rather than $\mathbf{k}$, respectively).

You can find the solution to this in Tinkham's book (Chapter 10, special topics).

The Heisenberg equations of motion $i\hbar\frac{\partial c}{\partial t} = [c, \mathcal{H}]$ give

$$i\hbar\frac{\partial}{\partial t}c_{\mathbf{r}\uparrow} = \sum_{\mathbf{r}'}\epsilon_{\mathbf{r}-\mathbf{r}'}c_{\mathbf{r}'\uparrow} + e^{2i\mathbf{q}\cdot\mathbf{r}}\Delta c^\dagger_{\mathbf{r}\downarrow} \tag{6}$$

$$i\hbar\frac{\partial}{\partial t}c^\dagger_{\mathbf{r}\downarrow} = -\sum_{\mathbf{r}'}\epsilon_{\mathbf{r}-\mathbf{r}'}c^\dagger_{\mathbf{r}'\downarrow} + e^{-2i\mathbf{q}\cdot\mathbf{r}}\Delta c_{\mathbf{r}\uparrow}. \tag{7}$$

Let,

$$\begin{aligned} c_{\mathbf{r}\uparrow} &= \frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}}\psi_{\mathbf{k}\uparrow} \\ c^\dagger_{\mathbf{r}\downarrow} &= \frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{i(-\mathbf{q}+\mathbf{k})\cdot\mathbf{r}}\psi_{\mathbf{k}\downarrow} \end{aligned} \tag{8}$$

$$\frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}} i\hbar\frac{\partial}{\partial t}\psi_{\mathbf{k}\uparrow} = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}}\left(\sum_{\mathbf{r}'}\epsilon_{\mathbf{r}-\mathbf{r}'}e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}'}\psi_{\mathbf{k}\uparrow} + e^{2i\mathbf{q}\cdot\mathbf{r}}\Delta e^{i(-\mathbf{q}+\mathbf{k})\cdot\mathbf{r}}\psi_{\mathbf{k}\downarrow}\right) \tag{9}$$

$$\frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{i(-\mathbf{q}+\mathbf{k})\cdot\mathbf{r}} i\hbar\frac{\partial}{\partial t}\psi_{\mathbf{k}\downarrow} = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}}\left(-\sum_{\mathbf{r}'}\epsilon_{\mathbf{r}-\mathbf{r}'}e^{i(-\mathbf{q}+\mathbf{k})\cdot\mathbf{r}'}\psi_{\mathbf{k}\downarrow} + e^{-2i\mathbf{q}\cdot\mathbf{r}}\Delta e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}}\psi_{\mathbf{k}\uparrow}\right) \tag{10}$$

Defining the Fourier transform $\epsilon_{\mathbf{k}} = \sum_{\mathbf{r}}\epsilon_{\mathbf{r}}e^{-i\mathbf{k}\cdot\mathbf{r}}$, we have $\sum_{\mathbf{r}'}\epsilon_{\mathbf{r}-\mathbf{r}'}e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}'} = \sum_{\mathbf{r}'}\epsilon_{-\mathbf{r}'}e^{i(\mathbf{k}+\mathbf{q})\cdot(\mathbf{r}'+\mathbf{r})} = e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}}\epsilon_{\mathbf{k}+\mathbf{q}}$. Also, $\sum_{\mathbf{r}'}\epsilon_{\mathbf{r}-\mathbf{r}'}e^{i(-\mathbf{q}+\mathbf{k})\cdot\mathbf{r}'} = \sum_{\mathbf{r}'}\epsilon_{-\mathbf{r}'}e^{i(-\mathbf{q}+\mathbf{k})\cdot(\mathbf{r}'+\mathbf{r})} = \epsilon_{\mathbf{q}-\mathbf{k}}e^{i(-\mathbf{q}+\mathbf{k})\cdot\mathbf{r}}$. Therefore,

$$\frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}} i\hbar\frac{\partial}{\partial t}\psi_{\mathbf{k}\uparrow} = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}}\left(\epsilon_{\mathbf{k}+\mathbf{q}}e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}}\psi_{\mathbf{k}\uparrow} + \Delta e^{i(\mathbf{q}+\mathbf{k})\cdot\mathbf{r}}\psi_{\mathbf{k}\downarrow}\right) \tag{11}$$

$$\frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{i(-\mathbf{q}+\mathbf{k})\cdot\mathbf{r}} i\hbar\frac{\partial}{\partial t}\psi_{\mathbf{k}\downarrow} = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}}\left(-\epsilon_{\mathbf{q}-\mathbf{k}}e^{i(-\mathbf{q}+\mathbf{k})\cdot\mathbf{r}}\psi_{\mathbf{k}\downarrow} + \Delta e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}}\psi_{\mathbf{k}\uparrow}\right) \tag{12}$$

Multiply the first equation from the left by $N^{-1/2}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{r}}$ and the second by $N^{-1/2}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}}$. Then sum over $\mathbf{r}$ which give $N\delta_{\mathbf{kp}}$. We therefore find:

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_{\mathbf{k}\uparrow}\\ \psi_{\mathbf{k}\downarrow}\end{pmatrix} = \begin{pmatrix}\epsilon_{\mathbf{k}+\mathbf{q}} & \Delta\\ \Delta & -\epsilon_{\mathbf{q}-\mathbf{k}}\end{pmatrix}\begin{pmatrix}\psi_{\mathbf{k}\uparrow}\\ \psi_{\mathbf{k}\downarrow}\end{pmatrix} \tag{13}$$

This is solved by diagonalizing the above 2×2 Bogolyubov-de Gennes (BdG) matrix Hamiltonian which we write as (using TRS $\epsilon_{\mathbf{k}} = \epsilon_{-\mathbf{k}}$)

$$\begin{pmatrix}\epsilon_{\mathbf{k}+\mathbf{q}} & \Delta\\ \Delta & -\epsilon_{\mathbf{q}-\mathbf{k}}\end{pmatrix} = \frac{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}-\mathbf{q}}}{2}1_2 + \frac{\epsilon_{\mathbf{k}+\mathbf{q}} + \epsilon_{\mathbf{k}-\mathbf{q}}}{2}\sigma_3 + \Delta\sigma_1 \tag{14}$$

Now, we can readily read off the qp energies

$$E(\mathbf{k}) \quad = \quad \frac{\epsilon_{\mathbf{k+q}} - \epsilon_{\mathbf{k-q}}}{2} \pm \sqrt{\left(\frac{\epsilon_{\mathbf{k+q}} + \epsilon_{\mathbf{k-q}}}{2}\right)^2 + \Delta^2} \tag{15}$$

At small $\mathbf{q}$

$$E(\mathbf{k}) \quad \approx \quad \mathbf{q} \cdot \nabla \epsilon_{\mathbf{k}} \pm \sqrt{\epsilon_{\mathbf{k}}^2 + \Delta^2} \approx \mathbf{q} \cdot \mathbf{v}_F(\hat{\mathbf{k}}) \pm \sqrt{\epsilon_{\mathbf{k}}^2 + \Delta^2}. \tag{16}$$

The first term shifts the overall energy of qp's by a $\mathbf{k}$-dependent amount; if $\mathbf{q}$ is parallel with $\mathbf{v}_F$, then the energy is shifted up, if it is anti-parallel, then the energy is shifted down. If it is perpendicular, then it is unaffected to leading order in $\mathbf{q}$.

Obviously we are dealing with a Doppler shift.

2.2 qp spectrum for a nodal d-wave superconductor with a uniform supercurrent

Consider now the model of a nodal d-wave superconducting pairing term at finite momentum. If we think of $\Delta(\mathbf{r}, \mathbf{r}')$ as being a function of the center of mass position $\frac{\mathbf{r}+\mathbf{r}'}{2}$ and a relative position $\mathbf{r} - \mathbf{r}'$, then having an overall supercurrent means that the center of the mass component of the Cooper pair carries $2\mathbf{q}$. So, the pairing term is

$$\sum_{\mathbf{r}} \left(e^{i\mathbf{q}\cdot(2\mathbf{r}+\hat{x})} \Delta \left(c^\dagger_{\mathbf{r}\uparrow} c^\dagger_{\mathbf{r}+\hat{x}\downarrow} - c^\dagger_{\mathbf{r}\downarrow} c^\dagger_{\mathbf{r}+\hat{x}\uparrow} \right) - e^{i\mathbf{q}\cdot(2\mathbf{r}+\hat{y})} \Delta \left(c^\dagger_{\mathbf{r}\uparrow} c^\dagger_{\mathbf{r}+\hat{y}\downarrow} - c^\dagger_{\mathbf{r}\downarrow} c^\dagger_{\mathbf{r}+\hat{y}\uparrow} \right) + h.c. \right) \tag{17}$$

The Heisenberg equations of motion

$$i\hbar \frac{\partial}{\partial t} c_{\mathbf{r}\uparrow} \quad = \quad \sum_{\mathbf{r}'} \epsilon_{\mathbf{r}-\mathbf{r}'} c_{\mathbf{r}'\uparrow} + \sum_{\delta=\hat{x},\hat{y}} \left(e^{i\mathbf{q}\cdot(2\mathbf{r}+\delta)} \Delta_\delta c^\dagger_{\mathbf{r}+\delta\downarrow} + e^{i\mathbf{q}\cdot(2\mathbf{r}-\delta)} \Delta_\delta c^\dagger_{\mathbf{r}-\delta\downarrow} \right) \tag{18}$$

$$i\hbar \frac{\partial}{\partial t} c^\dagger_{\mathbf{r}\downarrow} \quad = \quad -\sum_{\mathbf{r}'} \epsilon_{\mathbf{r}-\mathbf{r}'} c^\dagger_{\mathbf{r}'\downarrow} + \sum_{\delta=\hat{x},\hat{y}} \left(e^{-i\mathbf{q}\cdot(2\mathbf{r}+\delta)} \Delta_\delta c_{\mathbf{r}+\delta\uparrow} + e^{-i\mathbf{q}\cdot(2\mathbf{r}-\delta)} \Delta_\delta c_{\mathbf{r}-\delta\uparrow} \right) \tag{19}$$

Using the substitution in Eq.(8) we find

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_{\mathbf{k}\uparrow} \\ \psi_{\mathbf{k}\downarrow} \end{pmatrix} &= \begin{pmatrix} \epsilon_{\mathbf{k+q}} & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}} & -\epsilon_{\mathbf{q-k}} \end{pmatrix} \begin{pmatrix} \psi_{\mathbf{k}\uparrow} \\ \psi_{\mathbf{k}\downarrow} \end{pmatrix} \\ \Rightarrow E(\mathbf{k}) \begin{pmatrix} \psi_{\mathbf{k}\uparrow} \\ \psi_{\mathbf{k}\downarrow} \end{pmatrix} &= \begin{pmatrix} \epsilon_{\mathbf{k+q}} & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}} & -\epsilon_{\mathbf{q-k}} \end{pmatrix} \begin{pmatrix} \psi_{\mathbf{k}\uparrow} \\ \psi_{\mathbf{k}\downarrow} \end{pmatrix} \end{aligned} \tag{20}$$

where $\Delta_{\mathbf{k}} = 2\Delta\left(\cos k_x - \cos k_y\right)$.

Just as before:

$$E(\mathbf{k}) \quad = \quad \frac{\epsilon_{\mathbf{k+q}} - \epsilon_{\mathbf{k-q}}}{2} \pm \sqrt{\left(\frac{\epsilon_{\mathbf{k+q}} + \epsilon_{\mathbf{k-q}}}{2}\right)^2 + \Delta_{\mathbf{k}}^2} \tag{21}$$

and at small $\mathbf{q}$

$$E(\mathbf{k}) \quad \approx \quad \mathbf{q} \cdot \nabla \epsilon_{\mathbf{k}} \pm \sqrt{\epsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2} \approx \mathbf{q} \cdot \mathbf{v}_F\big|_{node} \pm \sqrt{v_F^2 k_\perp^2 + v_\Delta^2 k_\parallel^2}. \tag{22}$$

2.3 Volovik's semiclassical approximation

Volovik's semiclassical approximation amounts to replacing $\mathbf{q}/m$ with a position dependent superfluid velocity $\mathbf{v}(\mathbf{r})$. Then,

$$E(\mathbf{k}) \quad \rightarrow \quad E(\mathbf{k},\mathbf{r}) \approx \mathbf{v}(\mathbf{r}) \cdot \mathbf{p}_F\big|_{node} \pm \sqrt{v_F^2 k_\perp^2 + v_\Delta^2 k_\parallel^2}. \tag{23}$$

The thermodynamic density of states per area in this approximation is

$$N(\omega) = \frac{1}{Area} \sum_{\mathbf{r}} \sum_{\mathbf{k}} \delta\left(\omega - E(\mathbf{k},\mathbf{r})\right). \tag{24}$$

In the absence of the superflow (i.e. for $\mathbf{v}(\mathbf{r}) = 0$), the low-energy qp density of states is

$$N(\omega)\big|_{\mathbf{v}(\mathbf{r})=0} = 4 \int \frac{d^2\mathbf{k}}{(2\pi)^2} \delta\left(|\omega| - \sqrt{v_F^2 k_\perp^2 + v_\Delta^2 k_\parallel^2}\right) = \frac{2}{\pi} \frac{|\omega|}{v_F v_\Delta}. \tag{25}$$

Since in the absence of the superflow the density of states at $\omega = 0$ is at its minimum, the presence of superflow, which in some regions shifts the qp energy by a positive or negative amount, leading to finite density of states.

$$N(\omega) = \sum_{j=1}^{4} \frac{1}{2\pi} \frac{\langle |\omega - \mathbf{v}(\mathbf{r}) \cdot \mathbf{p}_F\big|_{node_j}| \rangle}{v_F v_\Delta}. \tag{26}$$

where the $\langle \ldots \rangle$ represent the spatial average. Since the energy without the superflow is linear in momentum, and since the only lengthscale (appart from the core size $\xi \rightarrow 0$) associated with the phase twist is the inverse magnetic length, $1/\ell$, the zero energy density of states in this approximation at low energy is

$$N(\omega = 0) \sim \frac{1}{v_\Delta \ell} \sim \frac{\sqrt{H}}{v_\Delta}. \tag{27}$$

This is the famous Volovik's result (G.E. Volovik JETP Lett. 58, 469-473 (1993)). (E.g., see also Ioffe and Millis, Journal of Physics and Chemistry of Solids 63, 2259 (2002)).

3 Quantum mechanics of d-wave qps in the vortex state

Is there any physics beyond the semiclassical approximation? What can we learn from the full quantum solution that is absent in the semiclassical approximation?

The semiclassical approximation relies on our ability to construct wavepackets, since it is only if we have well localized wavepackets that we can think of a local superflow Doppler shift. On the other hand, the massless Dirac fermion in the vicinity of the nodes are described by a scale-free equation, therefore the size of the wavepacket is set by the thermal length times a function of the Dirac cone anisotropy $\alpha_D = v_F/v_\Delta$. For isotropic system, the size is basically the thermal length $\hbar v_F/(k_B T)$. This lengthscale diverges as $T \rightarrow 0$, which means that there is a temerature/energy scale below which the size of the wavepacket is longer than the intervortex separation. The superflow is therefore not uniform and we expect quantum interference to play a role at the lowest energies.

When we deal with an ideal, perfectly periodic, vortex lattice, as we will see, we can chose to work in the basis when vortex crystalline momentum $\mathbf{k}$ is a good quantum number. The eigenvalues $E_n(\mathbf{k})$ can be labeled by a discrete (magnetic) band index and the continuous $\mathbf{k}$ which lies in the (magnetic) Brillouin zone. Ignoring the Zeeman effect, for a moment, which corresponds to an overall shift in each qp energy, we can show that for a space-inversion symmetric vortex lattice, given an eigenstate at $\mathbf{k}$, with energy $E_{\mathbf{k}}$, there is an orthogonal state at energy $-E_{\mathbf{k}}$ and the same $\mathbf{k}$. This means that the qp spectrum must have a mirror symmetry about $E=0$. Therefore, if there are states at zero energy, as would be required if the semiclassical result was correct, they have to appear as degenerate doublets corresponding to band crossings. However, we are dealing with a 2D problem which break time reversal symmetry. Any accidental band crossing requires tuning of at least 3-parameters according to the Wigner-von Neumann argument. Holding everything else fixed, we only have two parameters to vary k_x and k_y. Therefore, the bands should avoid, and there should be a gap at zero energy. Gapless points can appear only if an additional parameter, such as the Fermi energy, is fine-tuned.

3.1 Choice of the pairing term in the mixed state

$$\sum_{\mathbf{r}}\left(e^{i\theta_{\mathbf{r},\mathbf{r}+\hat{x}}}\Delta\left(c^\dagger_{\mathbf{r}\uparrow}c^\dagger_{\mathbf{r}+\hat{x}\downarrow}-c^\dagger_{\mathbf{r}\downarrow}c^\dagger_{\mathbf{r}+\hat{x}\uparrow}\right)-e^{i\theta_{\mathbf{r},\mathbf{r}+\hat{y}}}\Delta\left(c^\dagger_{\mathbf{r}\uparrow}c^\dagger_{\mathbf{r}+\hat{y}\downarrow}-c^\dagger_{\mathbf{r}\downarrow}c^\dagger_{\mathbf{r}+\hat{y}\uparrow}\right)+h.c.\right) \tag{28}$$

We would like to have vortex lattice in the center-of-the-mass coordinate of the phase field. So, our ansatz for the pairing term is

$$e^{i\theta_{\mathbf{r},\mathbf{r}+\delta}}\approx e^{i\theta_{\mathbf{r}+\frac{1}{2}\delta}}\approx e^{i\theta_{\mathbf{r}}}e^{\frac{i}{2}\int_{\mathbf{r}}^{\mathbf{r}+\delta}d\mathbf{l}\cdot\nabla\theta}$$

where the line integral in the last expression is to be performed along the straight line joining the sites $\mathbf{r}$ and $\mathbf{r}+\delta$. Schematically, $e^{i\theta_{\mathbf{r},\mathbf{r}+\delta}}\to e^{i(\theta_{\mathbf{r}}+\theta_{\mathbf{r}+\delta})/2}\approx\frac{e^{i\theta_{\mathbf{r}}}+e^{i\theta_{\mathbf{r}+\delta}}}{|e^{i\theta_{\mathbf{r}}}+e^{i\theta_{\mathbf{r}+\delta}}|}$ i.e. the above choice for the bond phase variable is to average the neighboring site phase variables.

In order to describe the magnetic field ($\nabla\times\mathbf{A}$) induced vortex lattice, we choose the phase field to be the solution of the London's equations for the gauge invariant superfluid velocity:

$$\nabla\cdot\left(\frac{1}{2}\nabla\theta-\frac{e}{\hbar c}\mathbf{A}\right) = 0, \tag{29}$$

$$\nabla\times\left(\frac{1}{2}\nabla\theta-\frac{e}{\hbar c}\mathbf{A}\right) = \hat{z}\pi\sum_j\delta(\mathbf{r}-\mathbf{r}_j)-\hat{z}\frac{e}{\hbar c}B, \tag{30}$$

subject to an additional constraint that $\frac{1}{2}\nabla\theta-\frac{e}{\hbar c}\mathbf{A}$ is periodic and vanishes on average. Vortex positions are denoted by $\mathbf{r}_j$. The explicit solution to these equations is presented in the Appendix.

3.2 Tight binding formulation: BCS-Hoffstadter hamiltonian

$$\begin{aligned}\mathcal{H} &= \sum_{\mathbf{r}}\left(\left(\sum_{\delta=\hat{x},\hat{y}}t_{\mathbf{r},\mathbf{r}+\delta}c^\dagger_{\mathbf{r},\sigma}c_{\mathbf{r}+\delta,\sigma}+H.c.\right)-(\mu+h_Z)c^\dagger_{\mathbf{r},\uparrow}c_{\mathbf{r},\uparrow}-(\mu-h_Z)c^\dagger_{\mathbf{r},\downarrow}c_{\mathbf{r},\downarrow}\right)\\ &+ \sum_{\mathbf{r}}\Delta\left(e^{i\theta_{\mathbf{r}}}e^{\frac{i}{2}\int_{\mathbf{r}}^{\mathbf{r}+\hat{x}}d\mathbf{l}\cdot\nabla\theta}\left(c^\dagger_{\mathbf{r}\uparrow}c^\dagger_{\mathbf{r}+\hat{x}\downarrow}-c^\dagger_{\mathbf{r}\downarrow}c^\dagger_{\mathbf{r}+\hat{x}\uparrow}\right)-e^{i\theta_{\mathbf{r}}}e^{\frac{i}{2}\int_{\mathbf{r}}^{\mathbf{r}+\hat{y}}d\mathbf{l}\cdot\nabla\theta}\left(c^\dagger_{\mathbf{r},\uparrow}c^\dagger_{\mathbf{r}+\hat{y},\downarrow}-c^\dagger_{\mathbf{r},\downarrow}c^\dagger_{\mathbf{r}+\hat{y},\uparrow}\right)+H.c.\right).\end{aligned} \tag{31}$$

This model has been discussed extensively in OV et.al. PRB 63, 134509 (2001), OV et.al. PRB 64, 224508 (2001), OV and A. Melikyan PRL 96, 167005 (2006), A. Melikyan and Z. Tesanovic, PRB 74, 144501 (2006). The hopping term is the standard Hoffstadter hamiltonian for an electron hopping on a square tight-binding lattice

$$t_{\mathbf{r},\mathbf{r}+\delta} = -te^{-iA_{\mathbf{r},\mathbf{r}+\delta}} = -te^{-i\frac{e}{\hbar c}\int_{\mathbf{r}}^{\mathbf{r}+\delta} d\mathbf{l}\cdot\mathbf{A}},$$

where again, the line integral in the last expression is to be performed along the straight line joining the sites $\mathbf{r}$ and $\mathbf{r}+\delta$. In the symmetric gauge $\mathbf{A} = \frac{B}{2}(-y,x)$ and so the magnetic flux Φ through an elementary plaquette enters the Peierls factors as

$$A_{\mathbf{r},\mathbf{r}+\hat{\mathbf{x}}} = -\pi y\Phi/\phi_0 \tag{32}$$

$$A_{\mathbf{r},\mathbf{r}+\hat{\mathbf{y}}} = \pi x\Phi/\phi_0. \tag{33}$$

where $\phi_0 = hc/e$.

The chemical potential is μ and the Zeeman coupling enters via h_Z.

The Heisenberg equations of motion are

$$i\hbar\frac{\partial}{\partial t}c_{\mathbf{r}\uparrow} = \left(\sum_{\delta=\pm\hat{x},\pm\hat{y}} t_{\mathbf{r},\mathbf{r}+\delta}c_{\mathbf{r}+\delta\uparrow}\right) - (\mu+h_Z)c_{\mathbf{r}\uparrow} + \left(\sum_{\delta=\pm\hat{x},\pm\hat{y}} \Delta_\delta e^{i\theta_{\mathbf{r}}} e^{\frac{i}{2}\int_{\mathbf{r}}^{\mathbf{r}+\delta} d\mathbf{l}\cdot\nabla\theta} c^\dagger_{\mathbf{r}+\delta\downarrow}\right) \tag{34}$$

$$i\hbar\frac{\partial}{\partial t}c^\dagger_{\mathbf{r}\downarrow} = -\left(\sum_{\delta=\pm\hat{x},\pm\hat{y}} t^*_{\mathbf{r},\mathbf{r}+\delta}c_{\mathbf{r}+\delta\uparrow}\right) + (\mu-h_Z)c^\dagger_{\mathbf{r}\downarrow} + \left(\sum_{\delta=\pm\hat{x},\pm\hat{y}} \Delta_\delta e^{-i\theta_{\mathbf{r}}} e^{-\frac{i}{2}\int_{\mathbf{r}}^{\mathbf{r}+\delta} d\mathbf{l}\cdot\nabla\theta} c_{\mathbf{r}+\delta\uparrow}\right) \tag{35}$$

where $\Delta_{\hat{x}} = -\Delta_{\hat{y}} = \Delta$.

3.3 Translational symmetry and Franz-Tesanovic singular gauge transformation

Even though the vortices form a periodic array, the above BdG Hamiltonian is not invariant under discrete translations. Instead the pure translations must be followed by a gauge transformation. Such transformations are called magnetic translations.

In order to better display the periodicity of the Hamiltonian, and in order to remove the phase variable from the off-diagonal pairing term, Franz and Tesanovic (M. Franz and Z. Tesanovic, PRL 84, 554 (2000)) devised a singular gauge transformation. Their original formulation was in continuum. Here we work on the lattice, and perform the symmetric version of the transformation, which instead of ending up with a $U(1)$ vector-fields, has Z_2 fields.

To this end, define site variables

$$e^{\frac{i}{2}\theta_{\mathbf{r}}} \equiv \sqrt{e^{i\theta_{\mathbf{r}}}} \tag{36}$$

to be either one of the two solutions to $\left(e^{\frac{i}{2}\theta_{\mathbf{r}}}\right)^2 = e^{i\theta_{\mathbf{r}}}$. We will fix which one of the two it is shortly.

Now, let

$$c_{\mathbf{r}\uparrow} = e^{\frac{i}{2}\theta_{\mathbf{r}}}\psi_{\mathbf{r}\uparrow} \tag{37}$$

$$c^\dagger_{\mathbf{r}\downarrow} = e^{-\frac{i}{2}\theta_{\mathbf{r}}}\psi_{\mathbf{r}\downarrow}. \tag{38}$$

The resulting Heisenberg equations of motions for ψ's are

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\psi_{\mathbf{r}\uparrow} &= \sum_{\delta=\pm\hat{x},\pm\hat{y}} e^{\frac{i}{2}\theta_{\mathbf{r}+\delta}}e^{-\frac{i}{2}\theta_{\mathbf{r}}}t_{\mathbf{r},\mathbf{r}+\delta}\psi_{\mathbf{r}+\delta\uparrow} - (\mu+h_Z)\psi_{\mathbf{r}\uparrow} + \sum_{\delta=\pm\hat{x},\pm\hat{y}} \Delta_\delta e^{-\frac{i}{2}\theta_{\mathbf{r}+\delta}}e^{\frac{i}{2}\theta_{\mathbf{r}}}e^{\frac{i}{2}\int_{\mathbf{r}}^{\mathbf{r}+\delta} d\mathbf{l}\cdot\nabla\theta}\psi_{\mathbf{r}+\delta\downarrow} \\
i\hbar\frac{\partial}{\partial t}\psi_{\mathbf{r}\downarrow} &= -\sum_{\delta=\pm\hat{x},\pm\hat{y}} e^{-\frac{i}{2}\theta_{\mathbf{r}+\delta}}e^{\frac{i}{2}\theta_{\mathbf{r}}}t^*_{\mathbf{r},\mathbf{r}+\delta}\psi_{\mathbf{r}+\delta\downarrow} + (\mu-h_Z)\psi_{\mathbf{r}\downarrow} + \sum_{\delta=\pm\hat{x},\pm\hat{y}} \Delta_\delta e^{\frac{i}{2}\theta_{\mathbf{r}+\delta}}e^{-\frac{i}{2}\theta_{\mathbf{r}}}e^{-\frac{i}{2}\int_{\mathbf{r}}^{\mathbf{r}+\delta} d\mathbf{l}\cdot\nabla\theta}\psi_{\mathbf{r}+\delta\uparrow}
\end{aligned} \tag{39}
$$

Consider first the off-diagonal term $e^{-\frac{i}{2}\theta_{\mathbf{r}+\delta}}e^{\frac{i}{2}\theta_{\mathbf{r}}}e^{\frac{i}{2}\int_{\mathbf{r}}^{\mathbf{r}+\delta} d\mathbf{l}\cdot\nabla\theta}$. Its lattice curl is

$$
e^{\frac{i}{2}\oint d\mathbf{l}\cdot\nabla\theta} = (-1)^{n_v} \tag{40}
$$

where n_v is the number of vortices encircled. Note, that the site variables do not contribute to the lattice curl. Moreover, on each bond, the product of the site variables

$$
e^{\frac{i}{2}\theta_{\mathbf{r}+\delta}}e^{-\frac{i}{2}\theta_{\mathbf{r}}}
$$

equals

$$
e^{\frac{i}{2}\int_{\mathbf{r}}^{\mathbf{r}+\delta} d\mathbf{l}\cdot\nabla\theta}
$$

up to an overall sign. Therefore, we can choose the minus signs in the square roots such that

$$
e^{-\frac{i}{2}\theta_{\mathbf{r}+\delta}}e^{\frac{i}{2}\theta_{\mathbf{r}}}e^{\frac{i}{2}\int_{\mathbf{r}}^{\mathbf{r}+\delta} d\mathbf{l}\cdot\nabla\theta} = z^{(2)}_{\mathbf{r},\mathbf{r}+\delta} \tag{41}
$$

where $z^{(2)}_{\mathbf{r},\mathbf{r}+\delta} = -1$ on all bonds cut by the branch-cuts and $+1$ otherwise. We can choose the branch-cuts to connect the vortices pairwise.

Therefore:

$$
\begin{aligned}
E\psi_{\mathbf{r}\uparrow} &= -t\sum_{\delta=\pm\hat{x},\pm\hat{y}} z^{(2)}_{\mathbf{r},\mathbf{r}+\delta}e^{i\int_{\mathbf{r}}^{\mathbf{r}+\delta} d\mathbf{l}\cdot\left(\frac{1}{2}\nabla\theta-\frac{e}{\hbar c}\mathbf{A}\right)}\psi_{\mathbf{r}+\delta\uparrow} - (\mu+h_Z)\psi_{\mathbf{r}\uparrow} + \sum_{\delta=\pm\hat{x},\pm\hat{y}} \Delta_\delta z^{(2)}_{\mathbf{r},\mathbf{r}+\delta}\psi_{\mathbf{r}+\delta\downarrow} \\
E\psi_{\mathbf{r}\downarrow} &= +t\sum_{\delta=\pm\hat{x},\pm\hat{y}} z^{(2)}_{\mathbf{r},\mathbf{r}+\delta}e^{-i\int_{\mathbf{r}}^{\mathbf{r}+\delta} d\mathbf{l}\cdot\left(\frac{1}{2}\nabla\theta-\frac{e}{\hbar c}\mathbf{A}\right)}\psi_{\mathbf{r}+\delta\downarrow} + (\mu-h_Z)\psi_{\mathbf{r}\downarrow} + \sum_{\delta=\pm\hat{x},\pm\hat{y}} \Delta_\delta z^{(2)}_{\mathbf{r},\mathbf{r}+\delta}\psi_{\mathbf{r}+\delta\uparrow} \\
\Rightarrow E\begin{pmatrix}\psi_{\mathbf{r}\uparrow}\\ \psi_{\mathbf{r}\downarrow}\end{pmatrix} &= \hat{H}_{BdG}\begin{pmatrix}\psi_{\mathbf{r}\uparrow}\\ \psi_{\mathbf{r}\downarrow}\end{pmatrix}
\end{aligned} \tag{42}
$$

To make it more transparent, consider the continuum limit of this Hamiltonian, where the branch-cuts are accounted for via boundary conditions on the wavefunctions. (We will not use this Hamiltonian in explicit calculations)

$$
\hat{H}^{cont.}_{BdG} = \begin{pmatrix} \frac{1}{2m}\left(\mathbf{p}+\frac{\hbar}{2}\nabla\theta-\frac{e}{c}\mathbf{A}\right)^2-\mu-h_z & \Delta(\mathbf{p}) \\ \Delta(\mathbf{p}) & -\frac{1}{2m}\left(\mathbf{p}-\frac{\hbar}{2}\nabla\theta+\frac{e}{c}\mathbf{A}\right)^2+\mu-h_z \end{pmatrix} \tag{43}
$$

$$
= \sigma_3\frac{1}{2m}\left(\mathbf{p}+\sigma_3 m\mathbf{v}(\mathbf{r})\right)^2 - \sigma_3\mu - 1_2 h_z + \sigma_1\Delta(\mathbf{p}) \tag{44}
$$

where $\Delta(\mathbf{p}) = \Delta(p_x^2-p_y^2)/p_F^2$ the superfluid velocity is

$$
m\mathbf{v}(\mathbf{r}) = \frac{\hbar}{2}\nabla\theta - \frac{e}{c}\mathbf{A} \tag{45}
$$

Note that the Zeeman term, h_Z, amounts to a simple overall constant shift of all qp energies.

It is clear from this formulation, that there is no minimal coupling of ψ's to the external electromagnetic vector potential. Therefore, the qps do not experience the Lorentz force. Nevertheless, the time reversal symmetry is broken. Expanding near each of the nodes gives

$$\hat{H}_{BdG}^{cont.}\big|_{node} \;=\; 1_2\left(\mathbf{v}(\mathbf{r})\cdot\mathbf{p}_F\big|_{node} - h_z\right) + \sigma_3 v_F p_\perp + \sigma_1 v_\Delta p_\parallel \tag{46}$$

With a periodic choice of the branch-cuts, the Hamiltonian $\hat{H}_{BdG}$ in Eq.(42), is periodic. However, the unit cell contains at least two vortices, i.e. is pierced by a multiple of the electronic flux quantum hc/e. We can therefore use the Bloch theorem, and write the wavefunctions in terms of the plane-wave part and the periodic part. The resulting Hamiltonian acting on periodic functions is

$$\hat{H}_{BdG}(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{H}_{BdG}e^{i\mathbf{k}\cdot\mathbf{r}} \tag{47}$$

where $\mathbf{k}$ resides in the magnetic Brillouin zone.

Now, without loss of generality, set $h_Z = 0$. Then,

$$\sigma_2\hat{H}^*_{BdG}(\mathbf{k})\sigma_2 = -\hat{H}_{BdG}(-\mathbf{k}). \tag{48}$$

Therefore, for each eigenstate with energy E at $\mathbf{k}$, there is an orthogonal eigenstate at $-\mathbf{k}$ with energy $-E$. If the vortex lattice has an inversion symmetry, then

$$\mathcal{I}\hat{H}_{BdG}(\mathbf{k})\mathcal{I} = \tilde{H}_{BdG}(-\mathbf{k}). \tag{49}$$

Now, $\tilde{H}_{BdG}(-\mathbf{k}) \neq \hat{H}_{BdG}(-\mathbf{k})$, because the branch-cuts may not necessarily be chosen in inversion symmetric fashion. Nevertheless, there is always a suitable Z_2 gauge transformation, $\gamma_\mathbf{r}$, which will bring the branch-cut back to its original place:

$$\gamma_\mathbf{r}\mathcal{I}\hat{H}_{BdG}(\mathbf{k})\mathcal{I}\gamma_\mathbf{r} = \hat{H}_{BdG}(-\mathbf{k}). \tag{50}$$

Therefore, for each eigenstate with energy E at $\mathbf{k}$, there is an eigenstate at $-\mathbf{k}$ with energy E. Combining the two transformations gives

$$\gamma_\mathbf{r}\mathcal{I}\sigma_2\hat{H}^*_{BdG}(\mathbf{k})\sigma_2\mathcal{I}\gamma_\mathbf{r} = -\hat{H}_{BdG}(\mathbf{k}). \tag{51}$$

This guarantees that for each eigenstate with energy E at $\mathbf{k}$, there is an orthogonal eigenstate at $\mathbf{k}$ with energy $-E$. According to the previous argument, which I repeat for convenience, the spectrum is gapped: This means that the qp spectrum must have a mirror symmetry about $E = 0$. Therefore, if there are states at zero energy, as would be required if the semiclassicla result was correct, they have to appear as degenerate doublets corresponding to band crossings. However, we are dealing with a 2D problem which break time reversal symmetry. Any accidental band crossing requires tuning of at least 3-parameters according to the Wigner-von Neumann argument. Holding everything else fixed, we only have two parameters to vary k_x and k_y. Therefore, the bands should avoid, and there should be a gap at zero energy. Gapless points can appear only if an additional parameter, such as the Fermi energy, is fine-tuned.

From the numerical studies, the typical size of the minigaps is $E_1 \approx 3.5\hbar v_F/(\ell\alpha_D^2)$. The width of the lowest magnetic subband is $E_2 \approx 1.5\hbar v_F/(\ell\alpha_D)$. At temperatures $T \gtrsim E_2$, semiclassical description applies.

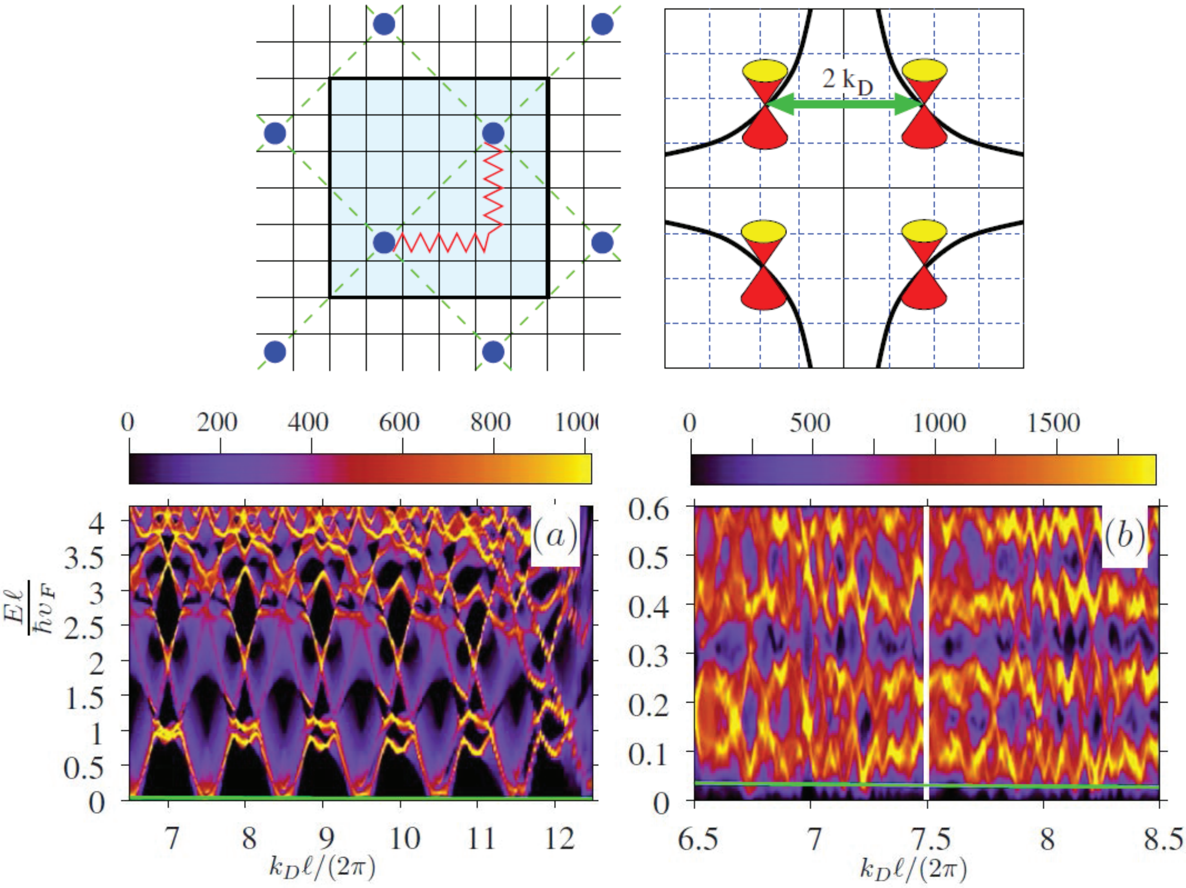

Figure 2: Top left: Schematic of the magnetic unit cell with the branch-cuts shown. Top right: schematic of the Fermi surface and the nodal points. Bottom: Scaled density of states for (a) $v_F/v_\Delta = 1$ and (b) $v_F/v_\Delta = 7$. From: A. Melikyan and O.Vafek PRB 78, 020502(R) 2008.

3.4 Effects of the branch-cuts via the index theorem

Please see O. Vafek and A. Melikyan Phys. Rev. Lett. 96, 167005 (2006), where we show that the number of Dirac nodes can double from 4 to 8 as a consequence of the branch-cuts. At $\mu = h_Z = 0$,

$$(-1)^{x+y}\gamma_{\mathbf{r}}\mathcal{I}\hat{H}_{BdG}(\mathbf{k})\mathcal{I}\gamma_{\mathbf{r}}(-1)^{x+y} = -\tilde{H}_{BdG}(-\mathbf{k}). \tag{52}$$

Therefore, at $\mathbf{k} = 0$ and other points on the Brillouin zone edge, the Hamiltonian anti-commutes with the combine inversion and particle-hole transformation

$$\left\{(-1)^{x+y}\gamma_{\mathbf{r}}\mathcal{I}, \hat{H}_{BdG}(\mathbf{k})\right\} = 0. \tag{53}$$

Moreover, $\left((-1)^{x+y}\gamma_{\mathbf{r}}\mathcal{I}\right)^2 = 1$ and therefore

$$\mathrm{Tr}\left[(-1)^{x+y}\gamma_{\mathbf{r}}\mathcal{I}\right] = n_+ - n_-,$$

where $n_\pm$ is the number of zero energy states with $(-1)^{x+y}\gamma_{\mathbf{r}}\mathcal{I}$ eigenvalue equal to ± 1.

3.5 Scaling functions from the tight-binding solutions

Based on the analysis of the linearized equations, Simon and Lee (S.H. Simon and P.A. Lee PRL 78, 1548 (1997)) argued that the qp eigenenergies follow the scaling form

$$E_n(\mathbf{k}) = \frac{\hbar v_F}{\ell}\mathcal{F}_n\left(\mathbf{k}\ell, \frac{v_F}{v_\Delta}\right). \tag{54}$$

This implies that the density of states per area should follow

$$\begin{aligned} N(\omega) &= \frac{1}{Area}\sum_n\sum_{\mathbf{k}}\delta\left(\omega - E_n(\mathbf{k})\right) &(55)\\ N_{SL}(\omega) &= \sum_n\int_{mBZ}\frac{d^2\mathbf{k}}{(2\pi)^2}\delta\left(\omega-\frac{\hbar v_F}{\ell}\mathcal{F}_n\left(\mathbf{k}\ell,\frac{v_F}{v_\Delta}\right)\right) = \frac{1}{\hbar v_F\ell}\Phi\left(\frac{\omega\ell}{\hbar v_F},\frac{v_F}{v_\Delta}\right) &(56)\\ &= \frac{1}{\hbar v_\Delta \ell}\tilde{\Phi}\left(\frac{\omega\ell}{\hbar v_F},\frac{v_F}{v_\Delta}\right). &(57)\end{aligned}$$

The integrated density of states, which is smoother than the density of states, shows, that apart from small deviations, it indeed follows the proposed scaling form for $\alpha_D = v_F/v_\Delta \gtrsim 3$. The resulting scaling function is thus determined numerically. For the density of states per layer, per spin, per area, we have

$$N(0) = b\frac{1}{\hbar v_\Delta}\sqrt{\frac{eB}{hc}}; \quad b \approx 0.92. \tag{58}$$

The qp contribution to specific heat is

$$\begin{aligned} \lim_{T\to 0} C &= 2k_B n_{layers}\int_0^\infty dE N(0)\frac{E^2}{4T^2\cosh^2\frac{E}{2T}} &(59)\\ &= k_B T n_{layers}\frac{\pi^2}{3}N(0). &(60)\end{aligned}$$

For YBCO, $n_{layers} = 2$ so

$$\gamma(H) = \lim_{T\to 0}\frac{C}{T} \approx 0.1\frac{\sqrt{H[T]}}{\hbar v_\Delta[eV\text{Å}]}\frac{mJ}{molK^2}. \tag{61}$$

We can use this formula to extract the value of v_Δ from the prefactor of $\sqrt{H}$ in $\gamma(H)$.

4 Appendix: solution to the London's equations

Vortex positions $\mathbf{r}_j$ enter through $\theta_\mathbf{r}$ which is chosen to be the solution of the (continuum) London's equations

$$\nabla\times\nabla\theta(\mathbf{r}) = 2\pi\hat{z}\sum_j\delta(\mathbf{r}-\mathbf{r}_j),$$

$$\nabla\cdot\nabla\theta(\mathbf{r}) = 0.$$

Vortices are arranged within a unit cell as shown in Fig. 1b. We study a variety of vortex lattices (VL): when $L_x = L_y$ the vortices form a square lattice, for $L_x/L_y \approx \sqrt{3}$ the lattice is triangular, the intermediate ratio $L_x/L_y \approx 1.4$ yields oblique VL.

In order to solve the London equations for the phase we start by noting that $\frac{1}{2}\nabla\theta - \frac{e}{\hbar c}\mathbf{A}$ is proportional to the physical superfluid velocity, which is periodic and vanishes upon spatial averaging.

Let,

$$\theta(\mathbf{r}) \quad = \quad \theta_0(\mathbf{r}) + \sum_j \tan^{-1}\frac{y-y_j}{x-x_j} \tag{62}$$

then

$$\begin{aligned}
\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial x}\right)\theta(\mathbf{r}) &= \left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial x}\right)\theta_0(\mathbf{r}) + \sum_{j=1}^{n_V}\sum_{\tau}\frac{1}{z-Z_j-\tau} \\
&= \left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial x}\right)\theta_0(\mathbf{r}) + C_0 + C_1 z + \sum_{j=1}^{n_V}\left(\frac{1}{z-Z_j} + \sum_{\tau\neq 0}\left(\frac{1}{z-Z_j-\tau} + \frac{1}{\tau} + \frac{z-Z_j}{\tau^2}\right)\right)
\end{aligned}$$

The Weierstrass zeta function is defined as

$$\zeta(z;\omega_1,\omega_2) = \frac{1}{z} + \sum_{\tau\neq 0}\left(\frac{1}{z-\tau} + \frac{1}{\tau} + \frac{z}{\tau^2}\right)$$

where τ forms a lattice with periods ω_1 and ω_2. Therefore:

$$\begin{aligned}
\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial x}\right)\theta(\mathbf{r}) &= \left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial x}\right)\theta_0(\mathbf{r}) + \sum_{j=1}^{n_V}\sum_{\tau}\frac{1}{z-Z_j-\tau} \\
&= \left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial x}\right)\theta_0(\mathbf{r}) + C_0 + C_1 z + \sum_{j=1}^{n_V}\zeta(z-Z_j;\omega_1,\omega_2)
\end{aligned}$$

In the symmetric gauge $A_y + iA_x = \frac{1}{2}B(x-iy) = \frac{1}{2}B\bar{z}$. Therefore,

$$\begin{aligned}
\frac{1}{2}\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial x}\right)\theta(\mathbf{r}) + \frac{e}{\hbar c}\left(A_y + iA_x\right) &= \frac{1}{2}\left(\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial x}\right)\theta_0(\mathbf{r}) + C_0 + C_1 z + \sum_{j=1}^{n_V}\zeta(z-Z_j;\omega_1,\omega_2)\right) \\
&- \frac{eB}{2\hbar c}\bar{z}
\end{aligned} \tag{63}$$

We demand that

$$\nabla\cdot\left(\frac{1}{2}\nabla\theta - \frac{e}{\hbar c}\mathbf{A}\right) = 0$$

and that

$$\nabla\times\left(\frac{1}{2}\nabla\theta - \frac{e}{\hbar c}\mathbf{A}\right) = \hat{z}\pi\sum_j\delta(\mathbf{r}-\mathbf{r}_j) - \hat{z}\frac{e}{\hbar c}B.$$

Note that the first can be written as

$$\nabla\cdot\mathbf{v} = Re\left(\frac{2}{i}\frac{\partial}{\partial\bar{z}}(v_y + iv_x)\right)$$

and the second as

$$\hat{z}\left(\frac{\partial}{\partial x}v_y - \frac{\partial}{\partial y}v_x\right) = \nabla\times\mathbf{v} = Re\left(2\frac{\partial}{\partial\bar{z}}(v_y + iv_x)\right).$$

We know that the Weierstrass $\zeta-$function is odd $\zeta(-z)=-\zeta(z)$. Therefore, we can guarantee that the above expression vanishes upon spatial averaging if we choose the unknown coefficients as:

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial x}\right)\theta(\mathbf{r})+\frac{e}{\hbar c}\left(A_y+iA_x\right) &= \frac{1}{2}\sum_{j=1}^{n_V}\zeta(z-Z_j;\omega_1,\omega_2)-\frac{eB}{2\hbar c}\frac{1}{n_V}\sum_{j=1}^{n_V}\left(\bar{z}-\bar{Z}_j\right)\\ &+ \frac{C}{n_V}\sum_{j=1}^{n_V}(z-Z_j)+f(z)\end{aligned}$$

where $f(z)$ is everywhere analytic (i.e. no poles) and vanishes upon spatial average.

Next, we demand this to be periodic:

$$\begin{aligned}&\frac{1}{2}\sum_{j=1}^{n_V}\zeta(z-Z_j;\omega_1,\omega_2)-\frac{eB}{2\hbar c}\frac{1}{n_V}\sum_{j=1}^{n_V}\left(\bar{z}-\bar{Z}_j\right)+\frac{C}{n_V}\sum_{j=1}^{n_V}(z-Z_j)+f(z)\\ =&\frac{1}{2}\sum_{j=1}^{n_V}\zeta(z+\omega_1-Z_j;\omega_1,\omega_2)-\frac{eB}{2\hbar c}\frac{1}{n_V}\sum_{j=1}^{n_V}\left(\bar{z}+\bar{\omega}_1-\bar{Z}_j\right)+\frac{C}{n_V}\sum_{j=1}^{n_V}(z+\omega_1-Z_j)+f(z+\omega_1)\\ =&\frac{1}{2}\sum_{j=1}^{n_V}\zeta(z+\omega_2-Z_j;\omega_1,\omega_2)-\frac{eB}{2\hbar c}\frac{1}{n_V}\sum_{j=1}^{n_V}\left(\bar{z}+\bar{\omega}_2-\bar{Z}_j\right)+\frac{C}{n_V}\sum_{j=1}^{n_V}(z+\omega_2-Z_j)+f(z+\omega_2)\end{aligned}$$

Now,

$$\zeta(z+\omega_1;\omega_1,\omega_2) = \zeta(z;\omega_1,\omega_2)+2\eta_1 \tag{64}$$

$$\zeta(z+\omega_2;\omega_1,\omega_2) = \zeta(z;\omega_1,\omega_2)+2\eta_2 \tag{65}$$

where

$$\eta_{1,2}=\zeta\left(\frac{\omega_{1,2}}{2};\omega_1,\omega_2\right).$$

So, we need

$$n_V\eta_1-\frac{eB}{2\hbar c}\bar{\omega}_1+C\omega_1+f(z+\omega_1)-f(z)=0 \tag{66}$$

$$n_V\eta_2-\frac{eB}{2\hbar c}\bar{\omega}_2+C\omega_2+f(z+\omega_2)-f(z)=0. \tag{67}$$

Since this must hold for every z, $f(z)$ must be periodic. But since it is everywhere finite and analytic, it must be a constant. Moreover, since it vanishes on average $f(z)=0$.

Therefore

$$n_V\eta_1-\frac{eB}{2\hbar c}\bar{\omega}_1+C\omega_1=0 \tag{68}$$

$$n_V\eta_2-\frac{eB}{2\hbar c}\bar{\omega}_2+C\omega_2=0. \tag{69}$$

Multiplying both sides of the first equation by ω_2 and of the second equation by ω_1, subtracting, and using the Legendre theorem

$$\eta_1\omega_2-\eta_2\omega_1=i\pi, \tag{70}$$

we see that both equations are satisfied if we solve either one of them for C. So,

$$C=-n_V\frac{\eta_1}{\omega_1}+\pi\frac{eB}{hc}\frac{\bar{\omega}_1}{\omega_1}, \tag{71}$$

and

$$\frac{1}{2}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial x}\right)\theta(\mathbf{r})+\frac{e}{\hbar c}\left(A_y+iA_x\right) = \frac{1}{2}\sum_{j=1}^{n_V}\zeta(z-Z_j;\omega_1,\omega_2)-\frac{eB}{2\hbar c}\frac{1}{n_V}\sum_{j=1}^{n_V}\left(\bar{z}-\bar{Z}_j\right)$$
$$+\left(-\frac{\eta_1}{\omega_1}+\frac{\pi}{n_V}\frac{eB}{hc}\frac{\bar{\omega}_1}{\omega_1}\right)\sum_{j=1}^{n_V}(z-Z_j)$$

Finally,

$$\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial x}\right)\theta(\mathbf{r}) = \sum_{j=1}^{n_V}\zeta(z-Z_j;\omega_1,\omega_2)+\frac{eB}{\hbar c}\frac{1}{n_V}\sum_{j=1}^{n_V}\left(\bar{Z}_j\right)$$
$$+\left(-2\frac{\eta_1}{\omega_1}+\frac{2\pi}{n_V}\frac{eB}{hc}\frac{\bar{\omega}_1}{\omega_1}\right)\sum_{j=1}^{n_V}(z-Z_j)$$

We can obtain the phase field by integrating $\nabla\theta$:

$$Im\int_{z_0}^{z}dz'\left(\frac{\partial}{\partial y'}+i\frac{\partial}{\partial x'}\right)\theta(\mathbf{r}') = \sum_{j=1}^{n_V}Im\int_{z_0}^{z}dz'\zeta(z'-Z_j;\omega_1,\omega_2)+\frac{eB}{\hbar c}\frac{1}{n_V}\sum_{j=1}^{n_V}Im\int_{z_0}^{z}dz'\left(\bar{Z}_j\right)$$
$$+\ Im\left[\left(-2\frac{\eta_1}{\omega_1}+\frac{2\pi}{n_V}\frac{eB}{hc}\frac{\bar{\omega}_1}{\omega_1}\right)\int_{z_0}^{z}dz'\sum_{j=1}^{n_V}\left(z'-Z_j\right)\right].$$

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