What is a mesoscopic system?

A system where certain size-dependent energy scales are greater than thermal energy scales, which are still much larger than 1-particle level spacings.

- Not an absolute size, T-dependent size scale (1 µm typical)
- Not related to discreteness of energy levels (micro-system)
  Quantum dot can be exception
- Second meaning: engineered micro and nanostructures
  New regimes available
Thermal energy/time scales?
\[ k_B T, \ h/\tau_\phi \propto (k_B T)^\alpha \]

**Transport energy/time scales:**

**Ballistic limit:**
- \[ \tau_{\text{erg}} = \frac{L}{v_f} \quad (E_{\text{erg}} = \hbar v_f/L = 1 \text{d level spacing} \gg \Delta \varepsilon) \]
- \[ \tau_{\text{esc}} = \frac{L}{(v_f p_1)} \quad (p_1 = \text{escape prob/bounce}) \]
  open system: \[ p_1 > \frac{1}{(k_f L)^{d-1}} \approx N_c \Rightarrow E_{\text{esc}} \geq \Delta \varepsilon \]

**Diffusive limit:** \( l < L \)
- \[ \tau_{\text{erg}} = \frac{L^2}{D} \Rightarrow E_{\text{Th}} = \frac{\hbar D}{L^2} \quad (\text{Thouless energy}) \]
- \[ \tau_{\text{esc}} = \frac{L^2}{D p_1} \Rightarrow E_{\text{esc}} = p_1 E_{\text{Th}} \]
  (often assume \( p_1 \approx 1, E_{\text{esc}} \approx E_{\text{th}} \))

\[ g = \frac{E_{\text{erg}}}{\Delta \varepsilon} = \text{dimensionless “conductance”} - \text{only the true conductance when } E_{\text{esc}} \approx E_{\text{erg}} \]
Interaction energy/time scale

- Charging energy, $E_c = \frac{e^2}{C}$, $\tau_c = R_QC$
  
  $C \propto \frac{1}{L^2}$

When $E_{\text{erg}}, E_c > kT, \frac{h}{\tau_\phi} \Rightarrow$ characteristic mesoscopic phenomena:

Will focus on coherence and fluctuations
The Mesoscopic Fermi Gas

If all $\mu_i$ are not equal current flows between $N$ reservoirs

Landauer-Buttiker “Octopus”

$S$-matrix

Mesoscopic system connected by perfect leads to phase-randomizing, thermal equilibrium non-interacting fermion reservoirs at $\mu_1, T_1, \mu_2, T_2$...
Landauer counting argument (1d): (two reservoirs)

\[ I = I_1 - I_2 = e\Delta N_1 v_f T \]
\[ = e\left(\frac{dN_1^+}{d\epsilon} eV\right) v_f T \]
\[ = e\left(\frac{1}{2\pi} \frac{dk}{d\epsilon} eV\right) \frac{1}{\hbar} \frac{d\epsilon}{dk} T \]
\[ = \left(\frac{e^2}{\hbar}\right) V \]

Two-probe Landauer Formula

Generalizations: \( G = \left(\frac{e^2}{h}\right) T \) per incident degree of freedom, i.e. transverse channels, spin … cancellation of velocity and DOS relies only on trans. invar. in leads

\[ G = \frac{e^2}{h} T \text{Tr}\{t(\epsilon_f)t^{\dagger}(\epsilon_f)\} \]

(Two-probe, \( T=0 \))
Two-probe, Temperature $\Theta \neq 0$

$$S = \begin{bmatrix} r_{11} & t_{12} \\ t_{21} & r_{22} \end{bmatrix}$$

$$\mu_2, \Theta_2 \quad |t_{21}|^2 = T_1, \quad |t_{12}|^2 = T_2$$

Assume $\Theta_1 = \Theta_2 = \Theta$

$$\Delta N_1 v_f T(\epsilon_f) \rightarrow \int d\epsilon [n_1(\epsilon)v_1(\epsilon)T_1(\epsilon)f_1(\epsilon-\mu_1/\Theta_1)-n_2(\epsilon)v_2(\epsilon)T_2(\epsilon)f_2(\epsilon-\mu_2/\Theta_2)]$$

$$n_i v_i = \frac{1}{h}, \quad T_1 = T_2, \quad f_1(\mu_1 - \epsilon) - f(\mu_2 - \epsilon) \approx -\frac{df}{d\epsilon}|_{\epsilon_f} eV$$

$$\Rightarrow G = -\frac{e^2}{h} \int d\epsilon (df/d\epsilon) T(\epsilon)$$

Many-channel case:

$$r_{11} \rightarrow r_{11,ab}, \quad t_{21} \rightarrow t_{21,ab}$$

$$a,b = 1,2...N$$

If $\Theta_1 \neq \Theta_2$ can calculate thermoelectric coefficients in terms of $S$-matrix
Final Generalization: $N_L$ leads

$$I_m = \frac{e^2}{\hbar} \sum_{n=1}^{N_L} (T_{mn} - N\delta_{mn})V_n \equiv \sum_{n=1}^{N_L} G_{mn}V_n$$

- $G_{mn}$ are conductance coefficients, necessary to describe 4-probe measurements, Hall resistance measurements
- Unitarity of S-matrix implies Kirchoff’s Laws in general
- $G_{mn} = G_{nm}$ only if B=0 or if only two probes, general TR symmetry of S-matrix implies $G_{mn}(B) = G_{nm}(-B)$ only.
- leads to van der Pauw reciprocity relations

- Properties of mesoscopic conductance: violates macrosymmetries, depends on measurement geometry, non-local (see Les Houches)
- If $T_{mn}$ are integers then resistance is quantized to $\hbar/qe^2$, $q=$integer
• Universal conductance fluctuations, sample-specific reproducible “noise” as fcn of B

• Looks like a longitudinal resistance measurement but \( G(B) \neq G(-B) \)
Making a quantum model for the LB counting argument

Non-interacting fermions at T=0, state is a Slater det or Fock state of single-particle fermion states - what are the sp states?

\[ \psi_{m,a}(x, y) = e^{-ik_0x_m}\phi_a(y) + \sum_b r_{mm,ba}e^{ik_bx_m}\phi_b(y) \quad (x \in x_m) \]
\[ + \sum_b t_{nm,ba}e^{ik_bx_n}\phi_b(y) \quad (x \in x_n) \]

\[ \phi_a(y) = \text{channel wavefunction, normalized to unit flux} \Rightarrow \text{unitary S-matrix, } \psi = \text{orthonormal basis} \]

- Linear response: fill up these states to common \( \varepsilon_f \) and calculate current response to linear order in potential \( \{V_n\} \) imposed on leads - see Les Houches Notes

- Mesoscopic fermi gas: fill each scattering state to appropriate \( \mu_n \) and calculate the currents \( I_m \) which flow in this states - see Buttiker, PRB, 46, 12485 (1992)

\[ \Psi_{meso} = \prod_{\alpha} c_\alpha^\dagger |0> \]
\[ J(x, y) = \sum_{\alpha, \beta} J_{\alpha\beta}(x, y)c_\alpha^\dagger c_\beta \]
\[ I_m = \int dy_m \hat{x}_m \cdot \hat{J}(x_m, y_m) \]
\[ 1D : \hat{I}_m = \hat{J}(x_m) \]

Drop channel indices, treat as 1D
Expectation Value at $T=0$ and $T \neq 0$

$\langle c_{m,k_m}^\dagger c_{n,k_n} \rangle = \left\{ \begin{array}{lr}
\delta_{m,n} \delta_{k_m,k_n} \theta(\epsilon_{k_n} - \mu_n) \\
\delta_{m,n} \delta_{k_m,k_n} f(\epsilon_{k_n} - \mu_n, T)
\end{array} \right.$

Expectation values $\langle c^\dagger c c^\dagger c \rangle$, $\langle c^\dagger c c^\dagger c c^\dagger c \rangle$, given by Wick’s Thm, can calculate correlations and fluctuations in term of S-matrix; note presence of $\mu_n$, this is not an equilibrium state.

$I_m = \langle \hat{I}_m \rangle = \langle \Psi | \hat{J}_m(x_m) | \Psi \rangle = \sum_{n} \sum_{k_n}^{occ} \langle n, k_n | J(x_m) | n, k_n \rangle$

$I_{mn} = \frac{e \hbar}{2\pi M} \int_{k_0}^{k_f^{(n)}} dk_n k_n | t_{mn} |^2 = \frac{e}{\hbar} \int_{\mu_0}^{\mu_n} d\epsilon | t_{mn} |^2(\epsilon) = \frac{e^2}{\hbar} | t_{mn} |^2 V_n$

$I_m = \sum_{n}^{NL} I_{mn} = \frac{e^2}{\hbar} \sum_{n}^{NL} | t_{mn} |^2 V_n$

LB equations!
Current noise in mesoscopic fermi gas

\[ C_{mn}(t - t') \equiv \langle \Delta \hat{I}_m(t) \Delta \hat{I}_n(t') + \Delta \hat{I}_n(t') \Delta \hat{I}_m(t) \rangle \]

\[ \Delta \hat{I}(t) = \hat{I}(t) - \langle \hat{I}(t) \rangle \]

\[ = \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \langle \Delta \hat{I}_m(\omega) \Delta \hat{I}_n(\omega') + \Delta \hat{I}_n(\omega') \Delta \hat{I}_m(\omega) \rangle \exp[-i(\omega t + \omega' t')] \]

\[ < \Delta \hat{I}_m \Delta \hat{I}_n + \Delta \hat{I}_n \Delta \hat{I}_m > \equiv (S_{mn}(\omega)) \delta(\omega + \omega') \]

\[ \hat{I}_m(t) = \sum \sum < n, k_n | J_m | p, k_p > c_n^\dagger(k_n) c_p(k_p) e^{i(\epsilon_n - \epsilon_p)t} \]

Need OD current matrix element

\[ < n, k_n | J_m | p, k_p > = \frac{\hbar k_m e}{M} [S_{mn}^*(k_n) S_{mp}(k_p) - \delta_{nm} \delta_{mp}] \]

\[ \equiv \frac{\hbar k_m e}{M} A_{np}^m(\epsilon_n, \epsilon_p) \]

\[ \hat{I}_m(\omega) = (e/\hbar) \sum \sum_{n} \int d\epsilon A_{np}^m(\epsilon, \epsilon + \omega) c_n^\dagger(\epsilon) c_p(\epsilon + \omega) \]
Simplify: 2-probe
\[ \langle \Delta I_m^2(\omega) \rangle \equiv S(\omega) \]
\[ \sum_{npqr} \langle c_n^\dagger(\epsilon)c_p(\epsilon + \omega)c_q^\dagger(\epsilon')c_r(\epsilon' + \omega') \rangle \]

“direct” contraction cancels with \( \langle I_m \rangle \), leaving “exchange”
\[ \propto \delta_{nr}\delta_{pq}\delta(\epsilon - \epsilon' - \omega')\delta(\epsilon' - \epsilon - \omega) f_n(\epsilon)(1 - f_p(\epsilon')) \]
\[ P_m(\omega \rightarrow 0) = \langle S_m(\omega) \rangle_{\Delta \nu} = \frac{2e^2}{h} \Delta \nu \sum_{np} \int d\epsilon A^m_{np}(\epsilon) A^m_{pn}(\epsilon) f_n(\epsilon)(1 - f_p(\epsilon)) \]
\[ A^m_{np} = S^*_{mn}(\epsilon_n) S_{mp}(\epsilon_p) \]

\( T=0, \mu_1 = \mu_2 + eV, A_{12} A_{21} \) only
\[ P_1(\omega \rightarrow 0) = (2e^2/h) T(1-T)eV \Delta \nu \]

\( r^* t t^* r = RT = T(1-T) \)

\{ \text{Mesoscopic shot noise} \}
Many-channel: \[ A_{12}^1 A_{21}^1 = Tr\{r_{11}^{\dagger} t_{21} t_{21}^{\dagger} r_{11}\} = \sum_{a=1}^{N} T_a (1 - T_a) \]

Weak transmission:
\[
\frac{e^2}{\hbar} \sum_{a=1}^{N} T_a (1 - T_a) \approx \frac{e^2}{\hbar} \sum_{a=1}^{N} T_a = G
\]

\[ P(\omega \to 0) = 2eGV \Delta \nu = 2eI \Delta \nu \quad \text{Tunneling shot noise} \]

Temp \( \Theta \neq 0, V=0 \Rightarrow f_n(1-f_n) \neq 0 \),
\[
[2A_{12}^1 A_{21}^1 + A_{11}^1 A_{11}^1 + A_{22}^1 A_{22}^1]f(1 - f) = 2T - k_B \Theta \partial f / \partial \epsilon
\]

\[
4k_B \Theta \Delta \nu \int d\epsilon \frac{e^2}{\hbar} T(-\partial f / \partial \epsilon) = 4k_B \Theta \Delta \nu G(T) \quad \text{Johnson Noise}
\]

\[ P(\omega \to 0, \Theta, V) = 2\Delta \nu \frac{e^2}{\hbar} \sum_{a=1}^{N} 2k_B \Theta T_a^2 + T_a (1 - T_a)eV \coth[eV/2k_B \Theta]
\]

Cross-over function
\[
\approx 2eI \Delta \nu \coth[eV/2k_B \Theta]
\]