Need statistical ensemble of S & M Matrices, \( d\mu(S), d\mu(M) \)

Open ballistic quantum dot

Disordered metal
Simplest possibility: completely uniform on the matrix space

- This is possible for unitary S-matrix (circular ensembles),
- Not for pseudo-unitary M-matrix (non-compact space)

Hypothesis: $P(S) = \frac{d\mu(S)}{V}$ for chaotic quantum dot

$$dS^2 = \text{Tr}\{dSdS^\dagger\} = \sum_{ij} g_{ij} dq_i dq_j \Rightarrow d\mu(S) = (\det[g])^{1/2} \prod_i dq_i$$

Dyson \quad \left\{ \begin{array}{l}
\frac{d\mu(S)}{\mu(U)} = \prod_{n<m} |\exp(i\phi_n) - \exp(i\phi_m)|^\beta \prod_i d\phi_i \\

\text{We need a parameterization in term of } \{T_n\} \\

S = \begin{pmatrix}
U & 0 \\
0 & V
\end{pmatrix}
\begin{pmatrix}
-\sqrt{1-T} & \sqrt{T} \\
\sqrt{T} & \sqrt{1-T}
\end{pmatrix}
\begin{pmatrix}
U' & 0 \\
0 & V'
\end{pmatrix}.

\mathcal{T} = \text{Diag}(T_1, T_2 \ldots T_N) \quad S_{12} = t = V \sqrt{\tau} U' \Rightarrow tt^\dagger = V \tau V^\dagger$
Coulomb Gas analogy

Circular ensemble: take $\delta V_{\beta} = 0$, $P(\{T_n\})$ only from invariant measure; what to do with jpd?

\[
\begin{align*}
\langle g \rangle &= \langle \sum_{a}^{N} T_{a} \rangle, \quad \langle g^2 \rangle = \langle \sum_{a,b}^{N} T_{a}T_{b} \rangle, \quad \langle P \rangle = \langle \sum_{a}^{N} T_{a}(1 - T_{a}) \rangle \\
< g > &= \int \prod_{c=1}^{N} dT_{c}P(\{T_{c}\}) \sum_{a}^{N} T_{a} \\
&= N \int \prod_{c=1}^{N} dT_{c}P(\{T_{c}\})T_{1} = \int_{0}^{1} \rho(T)T \\
\rho(T) &= \int \prod_{c=2}^{N} dT_{c}P(\{T_{c}\}) \\
K(T, T') &= \int \prod_{c=3}^{N} dT_{c}P(\{T_{c}\}) - \rho(T)\rho(T')
\end{align*}
\]

For Var(g) need

Need 1-pt and 2-pt correlation fcns of the jpd of $\{T_n\}$
Many methods to find these fcns and the two-pt corr. fcn is “universal” upon rescaling if only logarithmic correlations

Nice approach for $\beta=2$ is method of orthogonal polynomials

\[
\prod_{a<b}^{N} |T_a - T_b|^2 = \begin{vmatrix}
  p_1(T_1) & p_1(T_2) & \cdots & p_1(T_N) \\
  p_2(T_1) & p_2(T_2) & \cdots & p_2(T_N) \\
  \vdots & \vdots & \ddots & \vdots \\
  p_N(T_1) & p_N(T_2) & \cdots & p_N(T_N) 
\end{vmatrix}^2
\]

\[
\equiv \langle \Psi_N | \Psi_N \rangle \quad p_n = \text{orthog poly, choose Legendre, } [0,1]
\]

\[
\rho(T) = \langle \Psi_N | \sum_a \delta(T - T_a) | \Psi_N \rangle = \sum_{n=1}^{N} p_n^2(T)
\]

Use recursion relations, asymptotic form of $p_n$:

Same method gives $K(T,T')$ in terms of $p_Np_{N-1}$

(normalized to $N$ - so that $G = (e^2/h) T$)
What do we expect for this system?

Classical symmetry between reflection and transmission => \(<R> = <T> = \frac{N}{2}\)

\[
< g >= \frac{N}{\pi} \int_0^1 dT \sqrt{\frac{T}{(1-T)}} = \frac{N}{2}
\]

Need to go to next order in \(N^{-1}\) to get WL effect

\[
< P_{shot} > \propto \frac{N}{\pi} \int_0^1 dT \sqrt{T(1-T)}
\]

\[
= \frac{N}{8} \Rightarrow \frac{1}{4} P_{tunnel}
\]

\[
\rho(T) = \frac{N}{\pi \sqrt{T(1-T)}}
\]
Can get order $1/N$ effects easily for the circular ensemble - do averages over unitary group $U(2N)$

$$g = \sum_{a,b} |t_{ab}|^2 = \sum_{a,b} S_{ab}S_{ab}^*, \quad S \in U(2N)$$

$$\langle S_{ab}S_{cd}^* \rangle_{CUE} = \int d\mu(S) S_{ab}S_{cd}^* = \frac{\delta_{ac}\delta_{bd}}{2N}$$

$$\langle g \rangle_{CUE} = N^2 \cdot \frac{1}{2N} = \frac{N}{2}$$

$$\langle S_{ab}S_{ab}^* \rangle_{COE} = \int d\mu(U) [UU^T]_{ab} = \frac{1 + \delta_{ab}}{2N + 1}$$

$$\langle R \rangle_{COE} = N^2 \cdot \frac{1}{2N + 1} + N \cdot \frac{1}{2N + 1} \approx \frac{N}{2} + \frac{1}{4}$$

$$\delta G_{WL} = -(2e^2/h)(1/4)$$

**The Mystery**

Similarly $\text{Var}(g) = (1/8\beta)$
Agrees well with experiment and simulations

Actual data from ballistic junction -

M D.

CB only
Variance of $g$ for a quantum dot (Chan et al., 95),
Note factor of two reduction when $B \neq 0$ ($\beta=2$)
Disordered Wires

No $T \leftrightarrow R$ symmetry

$\langle g \rangle = N(l/L), \quad l = mfp$

$\langle R \rangle \approx N, \quad l \ll L$

Use $M$, not $S$

$T_a = \frac{1}{1 + \lambda_a}$

$M = M_1 M_2$

Parameterize $M(L)$ with polar decomposition

$M = \begin{bmatrix} u_1 & 0 \\ 0 & u_3 \end{bmatrix} \begin{bmatrix} \sqrt{(1 + \lambda)} & \sqrt{\lambda} \\ \sqrt{\lambda} & \sqrt{(1 + \lambda)} \end{bmatrix} \begin{bmatrix} u_2 & 0 \\ 0 & u_4 \end{bmatrix}$

$P_{L+dL} = \int P_L(M) P_{dL}(dM) \delta(M_{L+dL} - M_L \cdot dM_{dL})$

$M(L) \quad dM(dL) \quad \text{DMPK Equation}$

"isotropic"
1D case solved early (1959), Mello (88, 91), Beenakker (93), **RMP (97)**

Qualitative picture: Imry (86), open and closed channels

\[
\lambda_n \equiv \sinh^2 \nu_n, \quad T_n = \frac{2}{1 + \cosh 2\nu_n} \Rightarrow g = \sum_n \frac{2}{1 + \cosh 2\nu_n}
\]

**Thm:** if \( M = \prod_{i=1}^{L} M_i \), \( \lim_{L \to \infty} \frac{\nu_n}{L} = \alpha_n \) \( \{\alpha_n\} = \) Inverse localization lengths

\[
\Rightarrow \nu_n \approx \alpha_n L \quad g = \sum_n \frac{2}{1 + \cosh 2\alpha_n L}
\]
\[ g = \sum_{n=1}^{N} \frac{2}{1 + \cosh 2\alpha_n L} \]

- When \( L > 2Nl \) \( g = \exp[-L/(Nl)] \) \( \Rightarrow \) quasi-1d localization, \( \xi = Nl \).
- Fixed \( N \) (width) and increasing \( L \) always leads to localization.

\[ \text{Var}(g) = \text{Var}(N_{\text{open}}) \approx 1 \text{ (spectral rigidity, } P(\{\alpha_n\}) \propto \prod_{\text{open}} |\alpha_n - \alpha_m|^{\beta}) \]

UCF: \( \text{Var}(g) = 2/(15\beta) \), \( \delta g_{\text{WL}} = -1/3 \)
Eigenvalue density

\[ \rho(T) dT = \tilde{\rho}(\nu) d\nu \]

\[ \rho(T) = \rho_0 \frac{d\nu}{dT} \]

\[ \frac{1}{T} = \cosh^2(\nu) \]

\[ \rho(T) = \frac{Nl}{2L T \sqrt{1 - T}} \]

\[ \rho(T) = \frac{N}{\pi \sqrt{T(1 - T)}} \]

Ballistic/chaotic

Tunnel barrier

Chaotic junction

Disordered wire

Closed

Open
Semiclassical Method for Ballistic Junctions

\[ t_{ab} = -i\hbar(v_a v_b)^{1/2} \int dy' \int dy \phi_a^*(y') \phi_b(y) G(L, y'; 0, y; E). \]

\[ \int_0^T L dt \Rightarrow S_s(E) = \int r' pdq = (\hbar/2\pi)kL_s \text{ (for billiard)} \]

\[ G^{scl}(y'; y; E) = \frac{2\pi}{(2\pi i\hbar)^{3/2}} \sqrt{D_s} \exp\left(\frac{i}{\hbar} S_s(y', y, E_F) - i \frac{\pi}{2} \mu_s\right) \]

Obtained by stationary phase integration of FT of \( G^{scl}(r,r',t) \);

Now do \( \int \int \) dydy' for \( t_{ab} \) by stationary phase:

\[ \left( \frac{\partial S}{\partial y} \right)_{y'} = -p_y = -\frac{b\hbar\pi}{W}, \quad \bar{b} = \pm b \]

\[ \sin\theta_b = \pm b\pi/kW \]
\[ t_{ab} = -\frac{\sqrt{2\pi i\hbar}}{2W} \sum_{s(\bar{a},\bar{b})} \text{sgn}(\bar{a}) \text{sgn}(\bar{b}) \sqrt{\tilde{D}_s} \exp\left(\frac{i}{\hbar} \tilde{S}_s(\bar{a}, \bar{b}, E) - i \frac{\pi}{2} \tilde{\mu}_s\right), \]

\[ \tilde{S}_s/\hbar - (\pi/2)\mu_s = kL_s + \phi_s(a, b) \]

\[ |t_{ab}|^2 = \sum_{s,u} A_s A_u e^{ik(L_s - L_u) + i(\phi_s - \phi_u)} \]

\[ |t_{ab}|^2 |t_{cd}|^2 = \sum_{s,u} \sum_{v,w} A_s A_u A_v A_w e^{ik(L_s + L_v - L_u - L_w) + i \Delta \phi_{abcd}} \]

Conductance fluctuations come from random interference of paths, sensitive to B, k - \text{Var}(g) \approx 1 \text{ comes from interference of all paths, require correlated actions for different paths}

Fundamentally different from familiar speckle patterns
Can get dynamical scales just from diagonal terms

\[ t_{ab}(k) t_{ab}^*(k + \Delta k) = \sum_{s,u} A_s A_u e^{i k L_s + i \phi_s - i (k + \Delta k) L_u - i \phi_u} \]

Look at diagonal terms, \(s = u;\)

\[ \sum_s A_s^2 e^{i \Delta k L_s} \]

Sum starts to decay when \(\Delta k L_s \approx \pi \Rightarrow k_c = <\pi/L> = \gamma_c\)

For diffusive case \(L_s = v_f t_D \Rightarrow k_c = E_{th}/h v_f\)

Similar analysis give \(B_c = (h/e)/<A_{encl}>\)

Quantitative approach next time.