Atom interactions in Fermi gases near a Feshbach resonance

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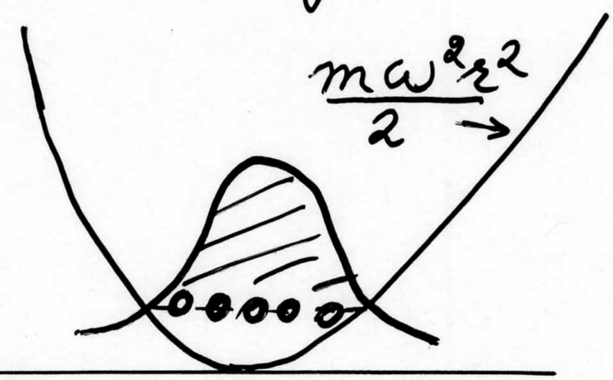
Outline

1. Introduction
2. Zero-range approximation
   Two-body problem
3. How to treat a few-body problem? Dimer-atom elastic scattering
4. Collisional relaxation
5. Role of Feshbach resonances

Trapped Bose gas

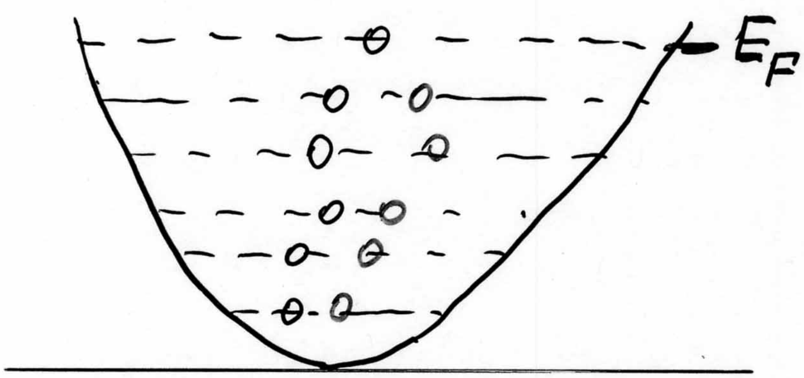
$n \Lambda_T^3 > 2,61$;

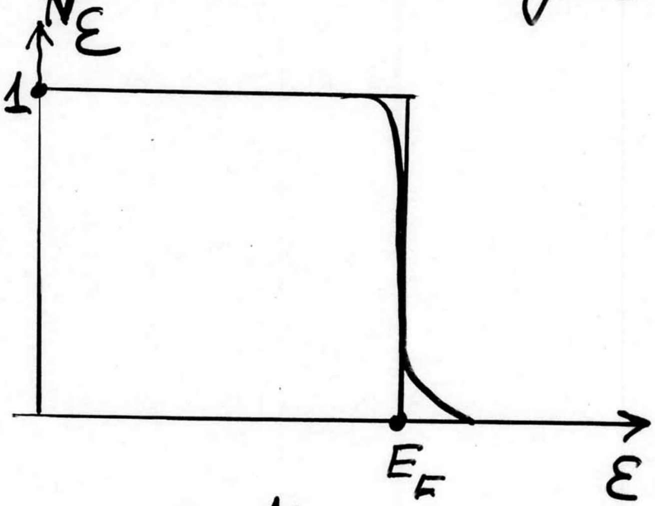
$T < T_c \simeq N^{1/3} \hbar\omega$



BEC!

Trapped two-component Fermi gas





$$E_F = \frac{\hbar^2 k_F^2}{2m}; \quad k_F = (3\pi^2 n)^{1/3}$$

Attractive intepspecies interaction

$a < 0$

Cooper pairing at sufficently low T $\rightarrow$ BCS superfluid transiti

Weakly interacting gas

$n|a|^3 \ll 1$; $k_F|a| \ll 1$

Mean-field interaction

ng ; $g = \frac{4\pi\hbar^2}{m} a$

$a < 0$

$T_c \simeq 0.49\, E_F \exp\left\{-\frac{\pi}{2k_F|a|}\right\}$

beyond experimental reach
for ordinary a ($\sim 10 \div 100$ Å)
$E_F \sim T$ (degeneracy)

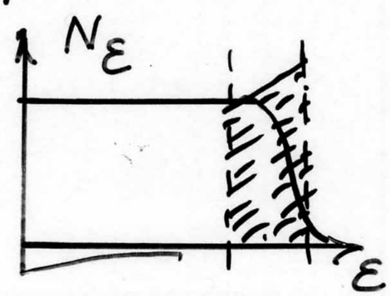
Experiments $T \sim 0.1\, E_F$

^{40}K JILA LENS

^{6}Li Rice ENS Duke MIT $\rightarrow$ 0.05

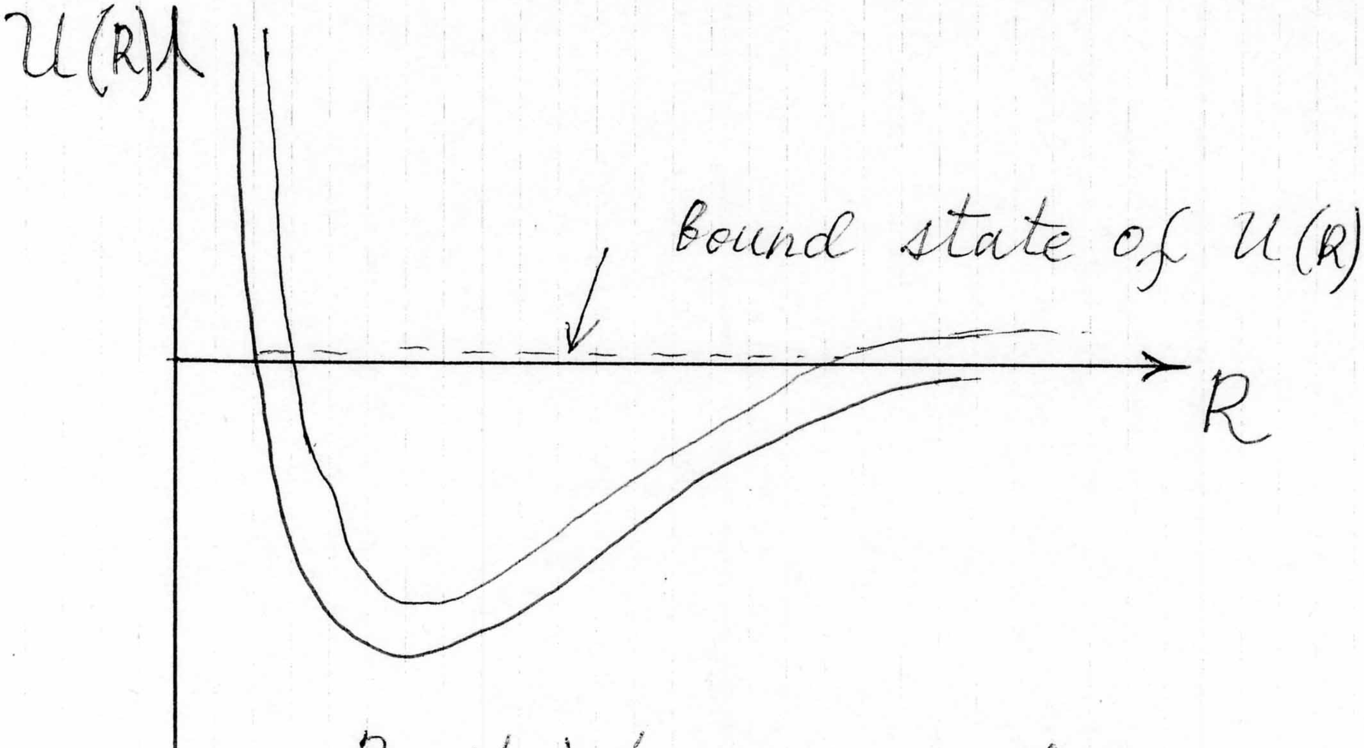
$n \sim 10^{13}\, cm^{-3}$; $E_F \sim 1\,\mu K$

Pauli blocking
Inelastic losses

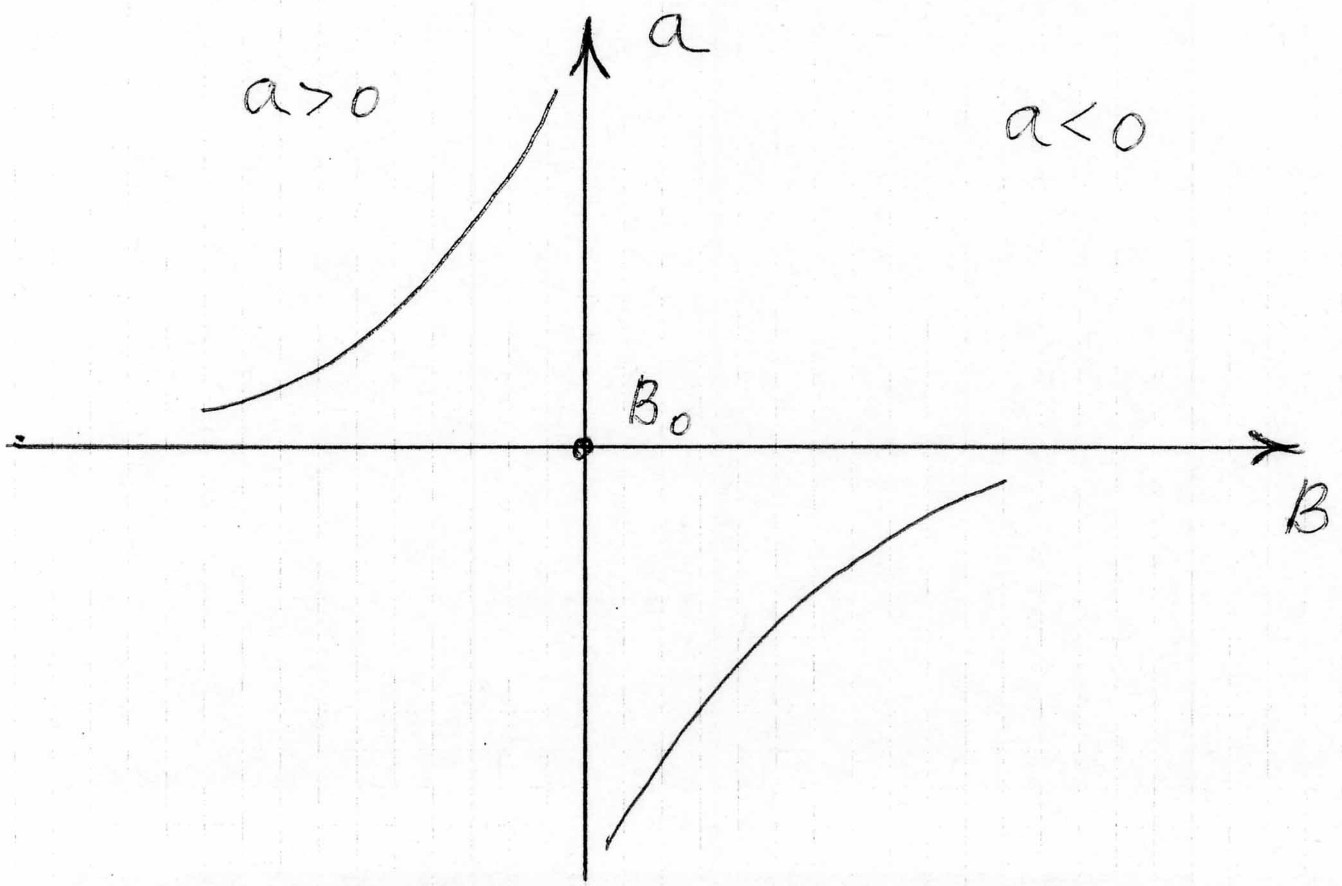


How to reach T_c!

Increase $|a| \to$ Feshbach resonance



Breit-Wigner problem

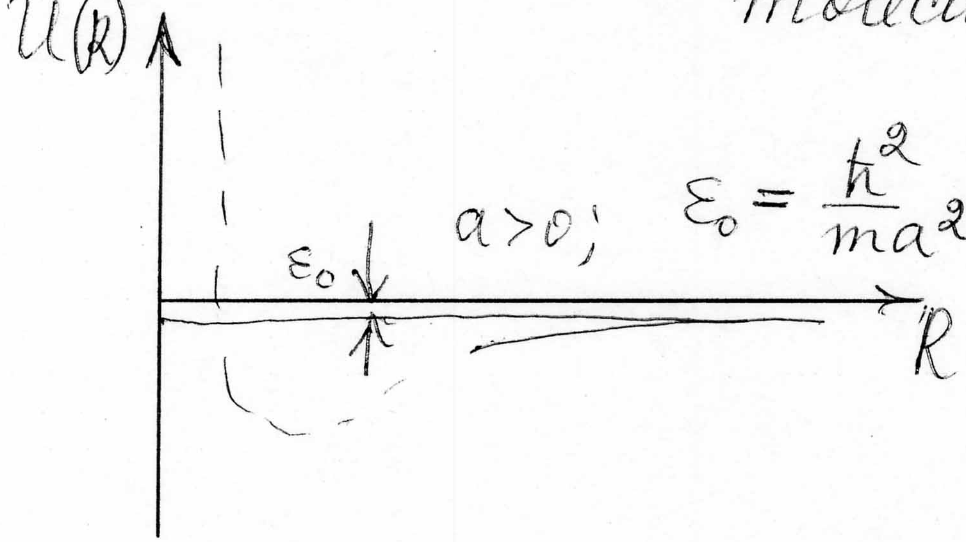


1. $a<0$;

$k_F|a| \ll 1 \rightarrow$ BCS

2. $a>0$;

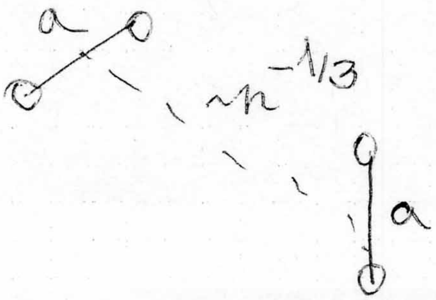
$a \gg R_e$; $\rightarrow$ weakly bound molecules (dimers)



$a>0$; gas of dimers (composite bosons)

Dimer size $\rightarrow a$

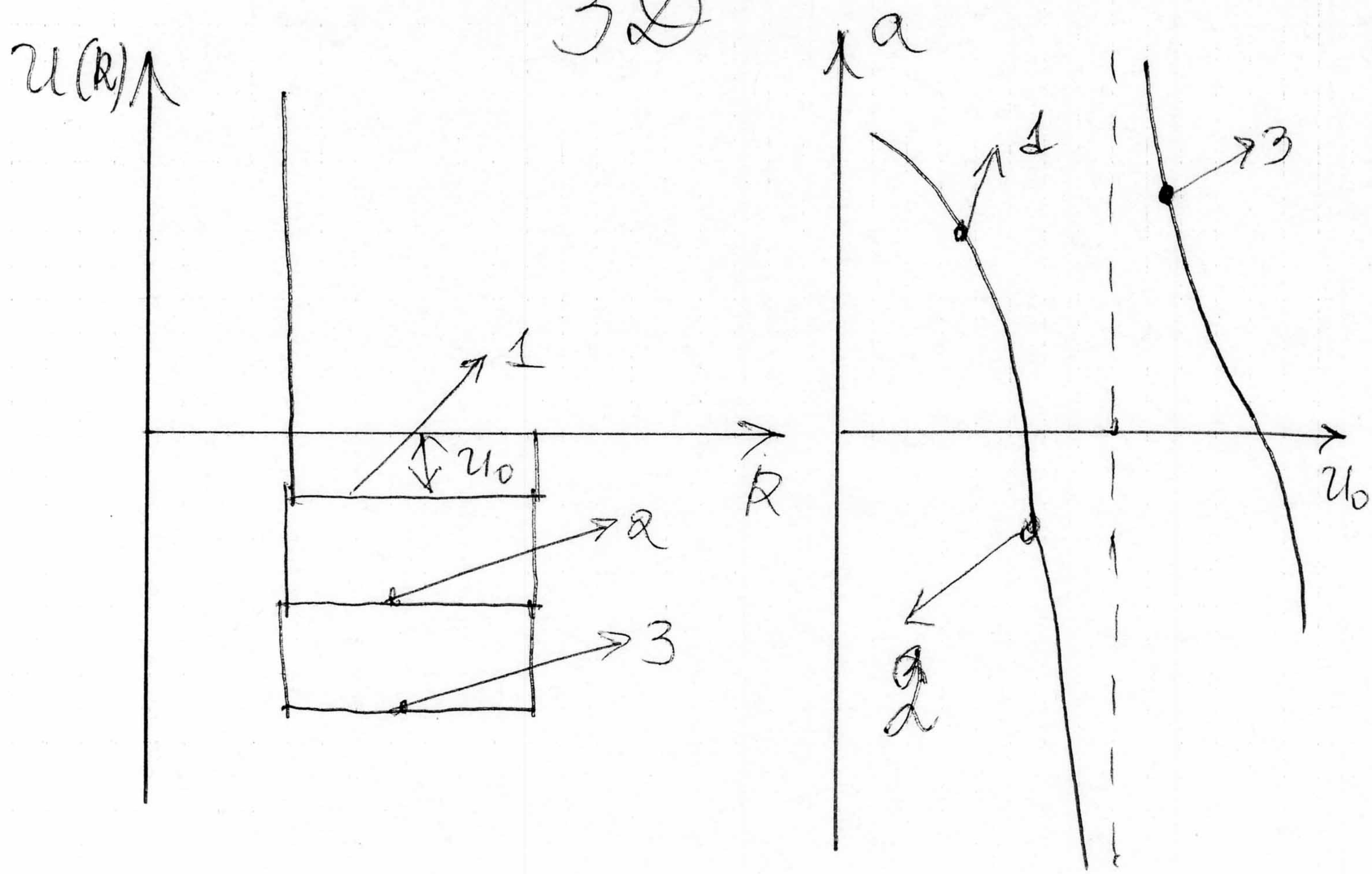
$na^3 \ll 1 \rightarrow$ weakly interacting molecular Bose gas



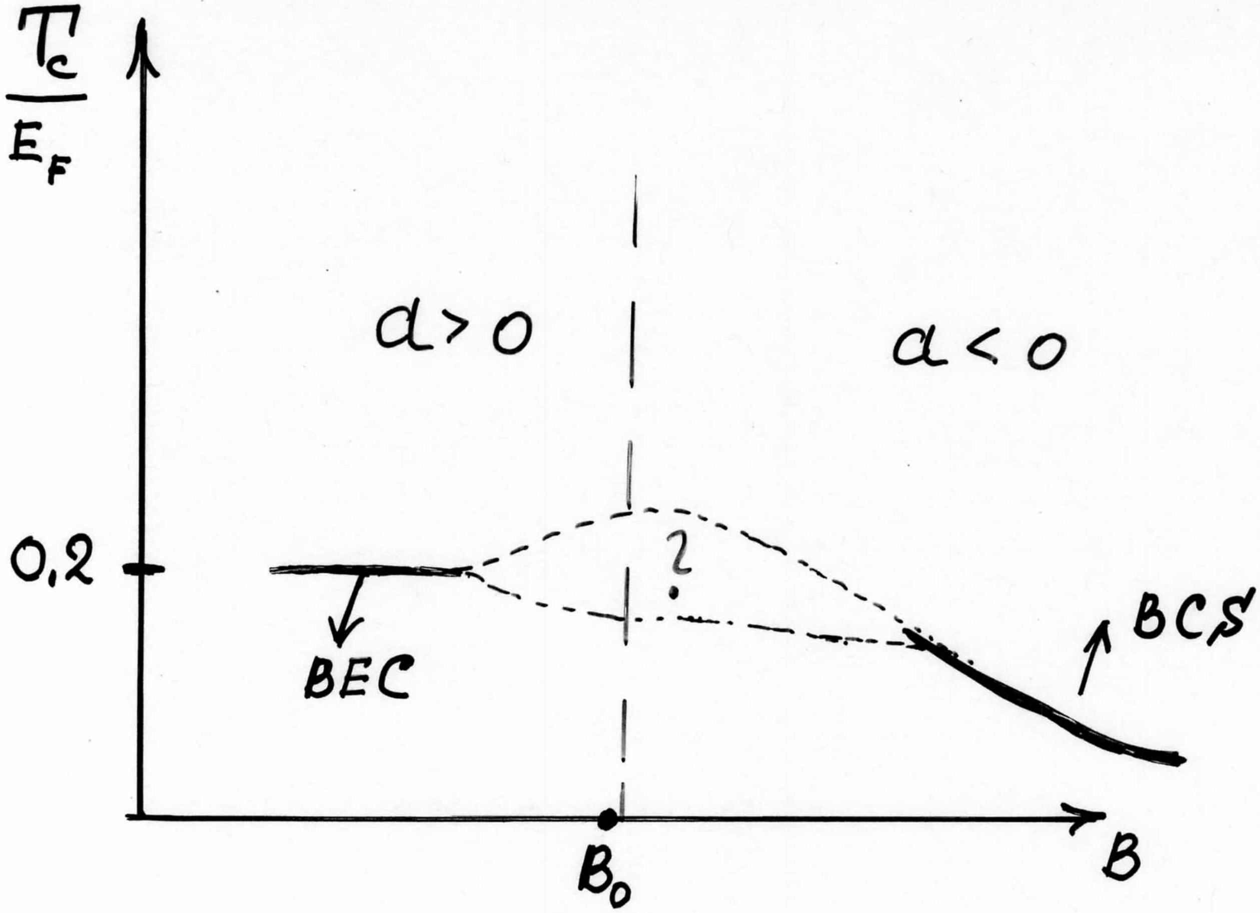
$a \ll n^{-1/3}$

Why weakly bound state for $a>0$?

3D



1. Shallow well
   No bound state; $a>0$
2. Shallow well;
   No bound state; $a<0$
3. One bound state
   $a>0$



$a<0 \Rightarrow$ BCS

$a>0 \Rightarrow$ BEC of weakly bound bosonic dimers

$$T_{BEC} \approx \frac{3{,}31\,\hbar^2}{(2m)}\left[\left(\frac{n}{2}\right)\right]^{2/3} \approx 0{,}2\,E_F$$

BCS → BEC crossover

Leggett, Nozieres / Schmidt-Rink

Widely discussed condensed matter problem

Cold gases → resonance superfluidity

3. $n|a|^3 \gtrsim 1$; $\Longleftrightarrow$ $k_F|a| \gtrsim 1$

$k_F = (3\pi^2 n)^{1/3}$

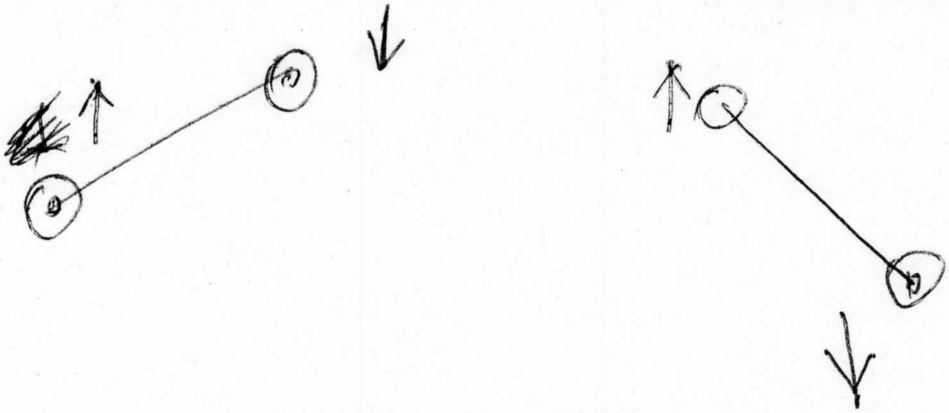
Strongly interacting regime

Problems.

1. Interactions, superfluid pairing and T_c for the strongly interacting regime

2. Interactions $\Leftrightarrow$ BEC for the gas of bosonic dimers at $a>0$

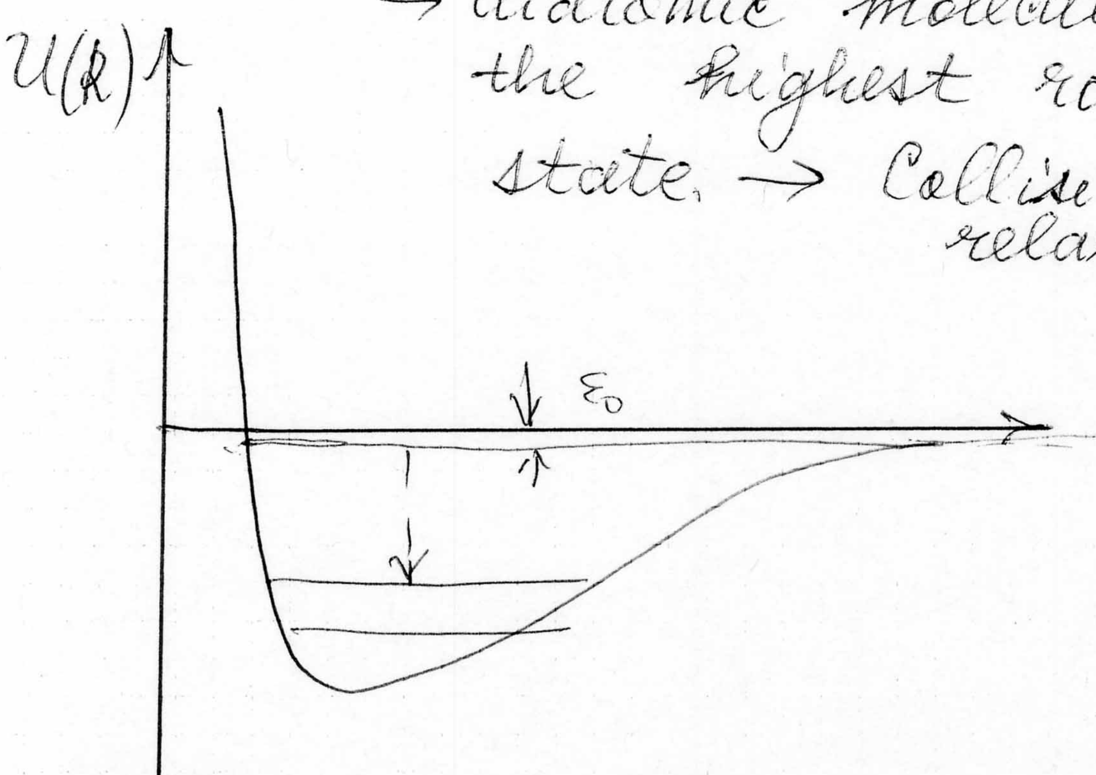
Elastic interaction



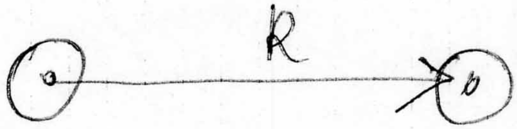
$a_{dd} = ?$ $\left(\begin{array}{l} a_{dd} > 0 \rightarrow \text{BEC} \\ a_{dd} < 0 \rightarrow \text{collapse} \end{array} \right)$

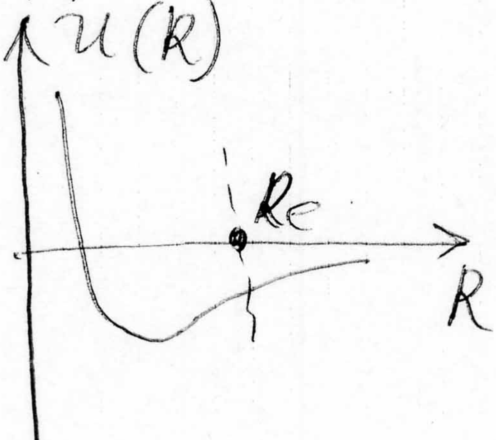
Collisional stability

Weakly bound dimers →
→ diatomic molecules in
the highest rovibrational
state. → Collisional
relaxation



Zero-range approximation





$kR_e \ll 1$

s-wave scattering

$R_e \leftarrow \quad |U(R_e)| \sim \dfrac{\hbar^2}{mR_e^2}$

$\downarrow -\dfrac{C_6}{R_e^6}$

$$R_e \approx \left(\frac{mC_6}{\hbar^2}\right)^{1/4}$$

	Li	Cs
Re(Å)	20	100

$R \gg R_e \rightarrow$ free motion

$\psi(R)$ is governed by the scattering length a

$$\delta = -\arctan(ka)$$

with Bethe-Peierls boundary condition

$$\left(\frac{R\psi}{R\psi}\right)'\Bigg|_{R\to 0} = -\frac{1}{a} \qquad (*)$$

$$\psi \sim \left(\frac{1}{R} - \frac{1}{a}\right), \quad R \to 0$$

So, we have the Schrödinger equation with the boundary condition $(*)$

$$-\left(\Delta_R + K^2\right)\psi = 0$$

$$\psi = \exp\left\{i\vec{K}\vec{R}\right\} + h\,G\left(\vec{R}, 0\right)$$

$$G\left(\vec{R}, \vec{R}'\right) = \frac{\exp\left(iK\left|\vec{R}-\vec{R}'\right|\right)}{4\pi\left|\vec{R}-\vec{R}'\right|}$$

$$\exp\left\{i\vec{K}\vec{R}\right\} \xRightarrow{\text{s-wave}} \frac{\sin KR}{KR}$$

$$h = -\frac{a}{1+ika} = \frac{\exp(2i\delta)-1}{2ik}$$

$$\psi \propto \frac{\sin(kR+\delta)}{kR}$$

$|a| \gg R_e \longrightarrow \psi$ for

R even much smaller than $|a|$ (but $R \gg R_e$)

$a > 0$; $a \gg R_e \rightarrow$ weakly bound molecular state ($E = -\varepsilon_0$)

ZRA

$$\left(-\Delta_R + \frac{m\varepsilon_0}{\hbar^2}\right)\varphi_0 = 0$$

$$\varphi_0(\vec{R}) = \psi_0(\vec{R}) + h_0 G_{\varepsilon_0}(\vec{R}, 0)$$

$$G_{\varepsilon_0}(\vec{R}, \vec{R}') = \frac{\exp\left(-\sqrt{\frac{m\varepsilon_0}{\hbar^2}}|\vec{R}-\vec{R}'|\right)}{4\pi|\vec{R}-\vec{R}'|}$$

$$\psi_0(\vec{R}) \equiv 0$$

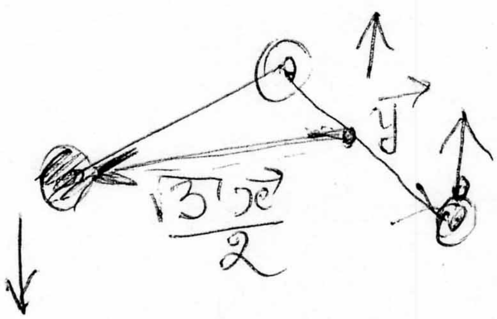
Bethe-Peierls boundary condition (*)

$$\mathcal{E}_0 = \frac{\hbar^2}{ma^2};$$

$$f_0(R) = \frac{1}{\sqrt{2\pi a}\,R} \exp\left\{-\frac{R}{a}\right\}$$

3-body problem

Atom-dimer elastic interaction



$a \gg R_e$

ZRA

$$\left[-\Delta_x - \Delta_y + U\left(\frac{\sqrt{3}\vec{x}+\vec{y}}{2}\right) + U\left(\frac{\sqrt{3}\vec{x}-\vec{y}}{2}\right)\right] \times$$

$$\times \Psi = E\Psi;$$

$\vec{r}_{\pm} = \frac{(\sqrt{3}x \pm \vec{y})}{2} \to 0$ boundary conditions

$$r = \mp \frac{2y}{\sqrt{3}} \quad \text{for} \quad r_{\pm} \to 0$$

$$E = -\underbrace{\frac{1}{a^2}}_{\varepsilon_0} + k^2 \quad (\hbar = m = 1)$$

$ka \ll 1 \to$ s-wave scattering with $E = -\frac{1}{a^2} < 0$.

$$G = \frac{1}{8\pi^2 a^2 X^2} K_2\left(\frac{X}{a}\right)$$

$$X = \sqrt{(\vec{x} - \vec{x}')^2 + (\vec{y} - \vec{y}')^2}$$

$$X \ll a \to G_+ = \frac{1}{4\pi^3 X^4};$$

$$\psi(\vec{x}, \vec{y}) = \cancel{\psi_0(\vec{x}, \vec{y})} +$$

I $$+ \int d\vec{r}' G_+\left(\sqrt{(\vec{x} - \vec{r}'/2)^2 + (\vec{y} + \frac{\sqrt{3}\,\vec{r}'}{2})^2}\right) h(\vec{r}')$$

II $$- \int d\vec{r}' G_+\left(\sqrt{\left(\vec{x} - \frac{\vec{r}'}{2}\right)^2 + \left(\vec{y} - \frac{\sqrt{3}\,\vec{r}'}{2}\right)^2}\right) h(\vec{r}')$$

Consider the limit $r_+ \to 0$

II $\quad -\int d\vec{r}' G_+\left(\sqrt{r^2 + \vec{r}' + \vec{r}\vec{r}' - \sqrt{3}\,\vec{r}\vec{r}_+ + r_+^2}\right) h(\vec{r}')$

$$\vec{r} = (\vec{x} - \sqrt{3}\,\vec{y})/2\,;$$

$$-\int d\vec{r}' G_+\left(\sqrt{r^2 + r'^2 - \vec{r}\cdot\vec{r}'}\right) h(\vec{r}')$$

I $\quad \int G_+\left(\sqrt{(\vec{r}-\vec{r}')^2 + r_+^2}\right) h(\vec{r}')\, d\vec{r}'$

Add and substract an auxiliary quantity

$$h(\vec{r}) \int G_+\left(\sqrt{(\vec{r}-\vec{r}')^2 + r_+^2}\right) d\vec{r}' =$$

$$= \frac{1}{4\pi r_+}\, h(\vec{r}) \exp\left\{-\frac{r_+}{a}\right\}$$

$$\lim_{r_+ \to 0} \int G_+\left(\sqrt{(\vec{r}-\vec{r}')^2 + r_+^2}\right)\left[h(\vec{r}') - h(\vec{r})\right] d\vec{r}'$$

$$= P\int G_+\left(|\vec{r}-\vec{r}'|\right)\left[h(\vec{r}') - h(\vec{r})\right] d\vec{r}$$

$$\psi = \frac{1}{4\pi}\left(\frac{1}{r_+} - \frac{1}{a}\right) h(\vec{r})$$

$$+ \int d\vec{r}\,' \left\{ G(|\vec{r}-\vec{r}\,'|)\left[h(\vec{r}\,') - h(\vec{r})\right] - G\left(\sqrt{r^2 + r'^2 - \vec{r}\vec{r}\,'}\right)\right\}$$

$$\psi = \frac{1}{4\pi} f(\vec{r})\left(\frac{1}{r_+} - \frac{1}{a}\right)$$

$$h(\vec{r}) \equiv f(\vec{r})$$

$$(*) \quad \int d\vec{r} \left\{ G(|\vec{r}-\vec{r}\,'|)\left[f(\vec{r}) - f(\vec{r}\,')\right] + G\left(\sqrt{r^2 + r'^2 - \vec{r}\vec{r}\,'}\right)\right\}$$

$r \gg a$

$$\psi(\vec{x}, \vec{y}) = f_0(r_+)\left(1 - \frac{2a_{ad}}{\sqrt{3}\, r}\right)$$

$a_{ad} \to$ atom-dimer scattering length

$$f_0(r_+) \xrightarrow{r_+ \to 0} \frac{1}{\sqrt{2\pi a}}\left(\frac{1}{r_+} - \frac{1}{a}\right)$$

$$\xrightarrow{r \gg a} f(r) = \sqrt{\frac{8\pi}{a}}\left(1 - \frac{2a_{ad}}{\sqrt{3}\, r}\right);$$

Integration of (*) over the angle $\to$ Integral equation of 1 variable $\to$ soft-core repulsion

Skorniakov Ter-Martirosian, (1956) $\leftarrow$ neutron-deutron scattering

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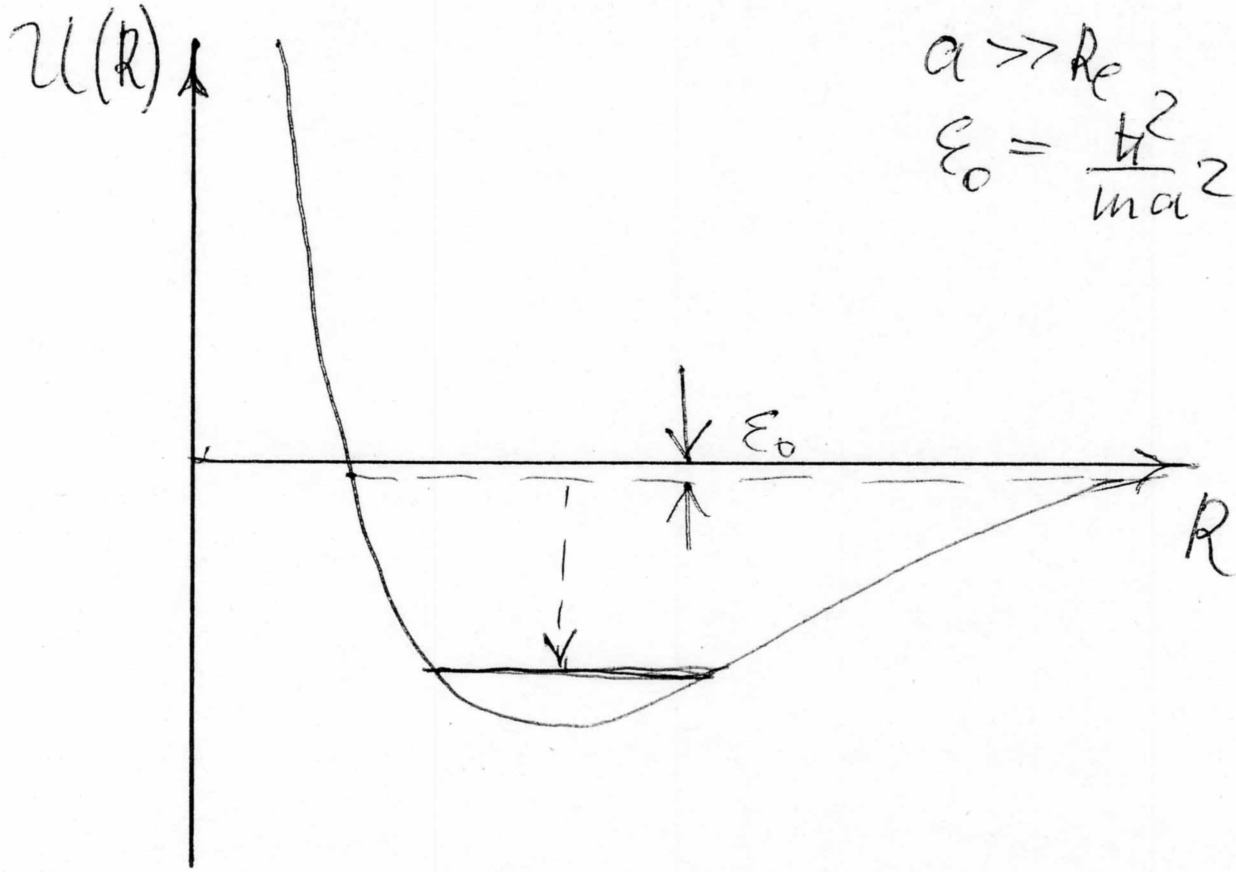
Collisional relaxation

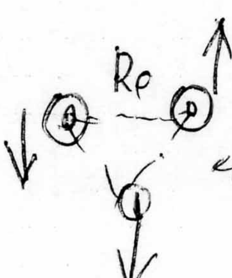


$a \gg R_e$

$\varepsilon_0 = \frac{\hbar^2}{ma^2}$

weakly bound dimer $\quad L \sim a$

deep bound state $\quad L \sim R_e \ll a$

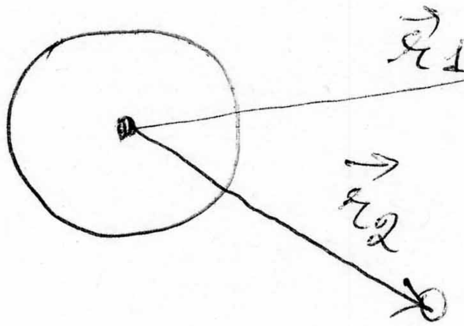


relative momentum $K \to \frac{1}{a}$

rel. amplitude $(K R_e)$

$$W_{rel} \propto \left(K R_e\right)^{2} \sim \left(\frac{R_e}{a}\right)^{2}$$

Simple toy model



$$\Gamma_{rel} = \frac{2\pi}{\hbar} \left| \int \Psi_{in} V_{int} \Psi_f \, d^3 r_i \right|^2 \delta(E_i$$

Fermi Golden rule
assuming that the
relaxation amplitude
is small

Dependence on a only in Ψ_{in}

$$\Psi_{in}(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}} \left\{ f_0(r_1)\chi(r_2) - f_0(r_2)\chi(r_1) \right\}$$

$r_1, r_2 \gg R_e$

$$f_0 = \frac{1}{\sqrt{2\pi a}} \cdot \frac{1}{r} e^{-r/a}$$

$$\chi = \left(1 - \frac{a}{r}\right)$$

For $(r_1, r_2) \ll a$ we have

$$\left[-\frac{\hbar^2}{2m}\left(\Delta_{r_1} + \Delta_{r_2}\right) + U(r_1) + U(r_2)\right]\psi_{in} = 0$$

$$\psi = A(a) f(r_1, r_2)$$

Find $A(a)$

For $(r_1, r_2) \gg R_C$

~~and (r_1, r_2)~~

ZRA

$$\psi_{in} = \frac{1}{\sqrt{2}}\left\{f_0(r_1)\chi(r_2) - f_0(r_2)\chi(r_1)\right\}$$

$$f_0 = \frac{1}{\sqrt{2\pi a}} \cdot \frac{1}{r} e^{-r/a}$$

$$\chi = \left(1 - \frac{a}{r}\right)$$

For $(r_1, r_2) \ll a$

$$\psi_{in} \Longrightarrow \frac{1}{2\sqrt{\pi a}}\left(\frac{r_2}{2r_1} - \frac{r_1}{2r_2}\right) \cdot \frac{1}{a}$$

$$A(a) \backsim \frac{1}{a^{3/2}}$$

$$\Psi_{in} = \frac{1}{a^{3/2}} f(r_1, r_2)$$

for $r_1 \sim R_e$, $r_2 \sim R_e$

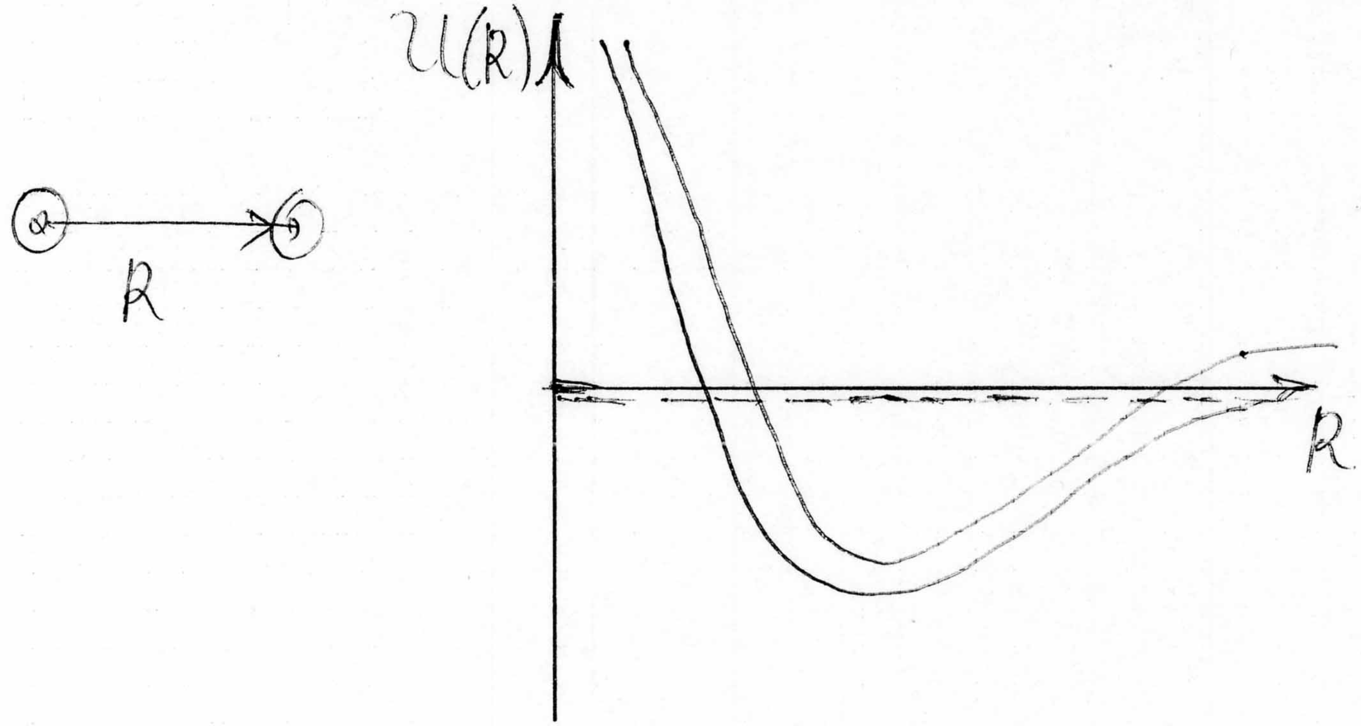
$$\alpha_{rel} \backsim \frac{1}{a^3}$$

Restoring the dimensions

$$\alpha_{rel} = C \cdot \frac{\hbar}{m} R_e \left(\frac{a}{R_e}\right)^3$$

atom-dimer	3,33
dimer-dimer	2,55

What about Feshbach resonance?



$$F(\varepsilon) = -\frac{\hbar g/\sqrt{m}}{\varepsilon + \Delta + i g\sqrt{\varepsilon}}$$

LL (volume 3)

$\frac{\hbar g}{\sqrt{m}} = W \to$ characterizes coupling between the two hyperfine domains

$$R_* = \frac{\hbar^2}{mW}$$

$\varepsilon \to 0$; $F = -a$; $a = \frac{W}{\Delta} = \frac{\hbar^2}{mR_*\Delta} > 0$

$$F = -\frac{1}{a^{-1} + R_* k^2 + ik}$$

Two-body bound state

Pole of the scattering amplitude

$$k = i\varkappa$$

$$\varepsilon_0 = \frac{\hbar\varkappa^2}{m} = \frac{\hbar^2}{mR_*^2}\left\{\frac{1}{2} + \frac{R_*}{a} - \sqrt{\frac{1}{4} + \frac{R_*}{a}}\right\}$$

$$R_* \ll a \longrightarrow \varepsilon_0 = \frac{\hbar^2}{ma^2}$$

for $ka \ll 1$

$$F = -a$$