RG for Fermions I. Notes / Shankar

RG will be applied to any problem that looks like a classical partition function and thus covers almost everything.

Eq.: Classical Stat. Mech

a. Quantum problems written as path integrals. Two examples: Boson and Fermionic Oscillators

Bosonic

\[ H = \omega_0 a^+ a \]

\[ \{ a, a^+ \} = 1 \]

Use

\[ I = \int \frac{d\phi}{2\pi i} \sqrt{1 - \phi \overline{\phi}} \]

where

\[ a|2\rangle = 2|2\rangle \quad <\overline{2}|a^+ = <\overline{2}|<21\overline{2} \]
to write
\[ Z = Tr e^{-\beta H} \quad \text{(for } \beta \to \infty, \ T = 0) \]
\[ = \int d\mathbf{\Sigma}(w) \ d\mathbf{2}(w) \ e^{-\beta \mathcal{Z}(w)} \]
\[ \langle \mathcal{Z}(\omega) \mathcal{Z}(\nu) \rangle = \frac{2\pi}{i\omega_2 - \omega_0} \delta(\omega_2 - \omega_1) \]
\[ \langle \mathcal{Z}_4 \mathcal{Z}_3 \mathcal{Z}_2 \mathcal{Z}_1 \rangle = \langle \mathcal{Z}_4 \rangle \langle \mathcal{Z}_3 \mathcal{Z}_2 \rangle + \langle \mathcal{Z}_4 \mathcal{Z}_2 \rangle \langle \mathcal{Z}_3 \rangle \]

There are many bosons
\[ H = \sum_i a_i^+ a_i + \omega_0 \]
make that many \( \mathcal{Z}_1, \mathcal{Z}_5 \) and \( \mathcal{Z}_6 \).

**Interactions**

**Hint:** \[ \langle a_1^+ a_2^+ a_3^+ a_4^+ \rangle \]
\[ \rightarrow \int \mathcal{Z}_i(\omega_1) \mathcal{Z}_j(\omega_2) \mathcal{Z}_k(\omega_3) \mathcal{Z}_l(\omega_4) \]
\[ \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \]
Fermi oscillators

\[ H = \omega_0 \hat{\Psi}^+ \hat{\Psi} \]

\[ I = \int \rho \, d\varphi \, d\psi \, \hat{\phi}^\dagger \hat{\phi} \]

\[ \langle \hat{\phi}^\dagger \hat{\phi} \rangle = \frac{1}{\sqrt{\pi}} \int \frac{d\varphi}{\sqrt{\varphi^2 + \omega_0^2}} \]

\[ \langle \hat{\phi} \rangle = \frac{2\pi}{i\varphi - \omega_0} \]

\[ Z = \int \rho \, d\varphi \, d\psi \, e^{-\varphi^2} \]

\[ \langle \hat{\phi}^3 \hat{\phi} \rangle = \langle \hat{\phi} \rangle \langle \hat{\phi}^2 \rangle - \langle \hat{\phi} \rangle^2 \]

\[ \langle \hat{\phi} \rangle = \frac{2\pi}{i\varphi - \omega_0} \]

\[ \text{Hint:} \quad \hat{\Psi}^+ \hat{\Psi} \quad \text{(all \( \hat{\Psi}^+ \hat{\Psi} \) terms)} \]

\[ \lambda \int \varphi^3 \, (\varphi^2 - \varphi^2) \Phi (\omega_2 - \omega_1) \delta (\omega_1 + \omega_2 - \omega_3 - \omega_4) \]

Note: \( \hat{\Psi} \) are Grassmann \#5.

They are not "big" or "small." There are rules for integration. Ask my RMP for details.
Renormalization:

What is it: Why do we care about it? How to do it?

$$\mathcal{Z}(a, b) = \int dx \int dy \ e^{-S(x, y, a, b)}$$

$$\langle f(x, y) \rangle = \frac{\int f(x, y) \ e^{-S} \ dx \ dy}{\mathcal{Z}}$$

If we care only about $x$, then

$$\langle f(x) \rangle = \frac{\int f(x) \left[ \int e^{-S(a, b, x, y)} \ dy \right] \ dx}{\int \left[ \int e^{-S(a, b, x, y)} \ dy \right] \ dx}$$

$$= \frac{\int f(x) \ e^{-S_{eff}(a', b', x)} \ dx}{\int e^{-S_{eff}(a', b', x)} \ dx}$$

$S_{eff}$ = effective action for $x$, that reproduces fully the contribution of $y$.

$a, b, \ldots \to a', b', \ldots$ is called Renormalization.

Eg: $S = ax^2 y^2 + b(x+y)^4$
In real life

\[ x = x_1 \ldots x_n \]
\[ y = y_1 \ldots y_m \]

ie y's are eliminated the point moves in parameter space, new coupling arise.

\[(a \ b \ c \ldots) \rightarrow (a' \ b' \ c' \ldots) \rightarrow (a'' \ b'' \ldots)\]

if \( S_{\text{eff}} = S \) ie \((a', b', c' \ldots) = (a, b, c \ldots)\)

we have a fixed point

Near fixed point - if you move away a bit and run the flow, the difference

1) Can grow relevant perturbation
2) 'Shrink Relevant
3) Stay put marginal.
In classical field theory

\[ \phi(r) = \phi(r) \phi(0) \sim e^{-r/3} \]

in general, but at "critical point"

\[ a(r) \to \frac{1}{r^{-d/2} e^{-2\pi y}} \]

\( \gamma \) is some for many different systems when they become critical. How can this be?

A: \( C_\gamma(r) \to \infty \) is controlled by \( g(k) \) for \( k \to 0 \)

\[ C(k) = \frac{\langle \phi(k) \phi(k) \rangle}{g(k)} = \delta(k-k') \]

So \( x = g(k) k < 1 \)

\( y = g(k) k > 1 \)

You can I can differ at beginning, but if all y's are eliminated over effective theories can end up being the same at critical point. So y which is
we end up here at fixed point
all systems here are critical

i.e. as far as long distance physics goes
our effective theories are same (as fixed point) though starting details are not.

Another case in a quantum problem

Let $H = H_0 + \delta H$

Does $H$, make a difference?

RG approach

let $H_0 \rightarrow \int \mathcal{L}(\phi) d^4 x$

It will turn out $S_0 \rightarrow S_0$ under RG.

Now add $\delta H$

$\int \mathcal{L}(\phi) d^4 x \rightarrow S_0 + S_5$

do to RG. Gs $\rightarrow 0$ under RG. It doesn't remain.

$\delta \rightarrow 0$ non

$\delta \rightarrow 0$ marginal case
\[ Z = \int dx \int dy \ e^{-a(x^2 + y^2) - b(x+y)^2} \]

\[ = \int dx \ e^{-a x^2 - b x^4} \text{ level} \]

\[ \times \int dy = ay^2 - b(4x^2 y + 4xy^2 + 6x^2 y^2 + y^4) \text{ consider just the} \]

\[ \int dy = e^{-ay^2} \]

\[ = \frac{1}{\sqrt{2\pi a}} \int dy e^{-ay^2} \left( \int e^{-ay^2} dy \right) \]

\[ = \langle e^{-v} \rangle \rightarrow \langle v \rangle = \text{ avg \ of} \ \int ds e^{-ay^2} \]

\[ \langle e^{-v} \rangle = e^{-\langle v \rangle + \frac{1}{2} \langle v^2 \rangle - \frac{1}{2} \langle v \rangle^2} \]

\[ \text{ consider } \langle v \rangle \]
\[ V = -b \left( 4x^3y + 4xy^3 + 6x^2y^2 \right) + y^4 \]

\[ \langle V \rangle = -b \left( 0 + 0 + 6x^2 \langle y^2 \rangle \right) + 4y^4 \]

\[ \langle y^2 \rangle = \frac{\sqrt{2a^2}}{a} \]

So:
\[ e^{-\text{Saff}(x)} = e^{ax^2 - bx^4 - 6x^2b \langle y^2 \rangle} \]

\[ a' = a \]
\[ b' = b + 6b \langle y^2 \rangle \]

\[ (x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \]

\[ \langle x \rangle = x^4 + 0 + 0 + 6 \langle x \rangle + 6 \langle x^2 \rangle \]

\[ e^{\langle y^4 \rangle} = \frac{1}{\sqrt{a}} \]

\[ \langle y^2 \rangle \]
At next order, you \( \langle V^2 \rangle - \langle V \rangle^2 \)

\[ \times \quad x \quad x \quad x \quad x + \times \]

Exactly as in \( \phi^4 \) theory but with loop variables summed only over \( y \)'s, the others being eliminated.

This in \( \phi^4 \) theory if you are eliminating \( k \) below \( \frac{\Lambda}{S} \) and \( k \) like

\[ \delta k = \times - \times = \sum \int \frac{1}{k^2 + m^2} \frac{1}{k^2 + m^2} \right) \frac{d\Phi}{N!} \]

+ other terms

The diagrams for flow are free of infrared and ultraviolet divergences since they are cut off at both ends.