Tensor Networks - A natural way to represent
both analytical and numerical analytic
quantum states when there is low entanglement
(namely the case): MPS, PEPS, MERA
- 1D Matrix Product States (well understood)
The evolving ground state
- higher dimension
-end of second lecture: slides showing some results
for spin liquids

References
U Schollwock, arXiv:1008.3477

Software: 1) itensor.org
           2) julia-lang.org - very nice new language
Web page
https://www.uci.edu/48480
Review: Schmidt Decays

\[ |14\rangle = \sum_{a,b} y_{ab} |a\rangle_A |b\rangle_B \]

Schmidt Form: \[ |14\rangle = \sum_{\alpha \beta} \lambda_{\alpha} |\alpha\rangle_A |\beta\rangle_B \]
- diagonal
- always possible

- Calculate: 1) SVD: \[ y_{ab} = [U D V^T]_{\alpha \beta} \]
\[ D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \]
\[ U, V \text{ unitary} \]
\[ |\alpha\rangle_A \equiv U |\alpha\rangle \]

2) Reduced density matrix
\[ \rho^A_{aa'} = \sum_b y_{ab} y^*_{a'b} = [U D V^T V^* D^{-1} U^T]_{aa'} \]
\[ = [U K^T U^T]_{aa'} \]

\[ \lambda_{\alpha} \text{ are eigenvalue of } \rho^A \text{ and } \rho^B \]
eigenvectors of \[ \rho^A \text{ from Columns of } U, \text{ etc.} \]

Von Neumann entanglement entropy
\[ S = \sum_{\alpha} \frac{1}{2} \lambda_{\alpha}^2 \ln \lambda_{\alpha}^2 \]
\[ \lambda_{\alpha} = \prod\alpha - \text{pair of } \alpha \text{ of states} \]

For ground states, \[ S \sim \text{Area-Law} \]
Tensor Network Diagrams

\[ A_{ij}, B_{jk}, C_{ik} \]

3 Tensors. Internal \( i \) and \( j \) indices are summed over. External indices define the final tensor result.

Matrix Vector \( i \) ---- Matrix \( \begin{pmatrix} A \end{pmatrix} \)

Matrix Product \( \begin{pmatrix} A \end{pmatrix} \begin{pmatrix} B \end{pmatrix} \)

\[ (AB)_{ik} = \sum_j A_{ij} B_{jk} \]

Tensor networks can get complicated.

SVD:

\[ M = UDV^T \]

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Truncating Low Probability States

Suppose we have \( x_i^2 \geq 1 \), \( \sum x_i^2 = 1 \), \( s = 1 \) charge
\[ x_i^2 \sim 10^{-10} \]
for 80 \( x_i \)s.

We want to throw away states with small \( x_i^2 \) (Schmidt states).

\[\text{Error}\ = \sum_{i=1}^{m} x_i \{ i \geq 1 \}^2 \rightarrow \sum_{i=1}^{m} x_i \{ i \geq 1 \}^2 \]

Error \( \approx 14 \) :

\[<\bar{4}|<\bar{\bar{4}}|14\rangle (14\rangle - 14\rangle)\]

\[<\bar{4}|\bar{4}\rangle = \sum_{i=1}^{m} \sum_{i' \neq i} x_i x_{i'} \]

\[\delta_{i, i'} = <\bar{i}|\bar{i}\rangle\]

\[<\bar{4}|\bar{4}\rangle - <\bar{\bar{4}}|\bar{4}\rangle = 1 - \sum_{i=1}^{m} \sum_{i' \neq i} x_i x_{i'} \]

error in \( z \) is \( \sum_{i=1}^{m} x_i^2 = \text{truncation error} \)

"discarded weight"

This is also the error in the density matrix, roughly.

This is the low TN approximation: a low entanglement approximation.
Suppose we had a wavefunction from exact diagonalization of a spin chain
\[ 4(s_1, s_2, \ldots, s_N) \times 2^N \text{ numbers} \quad (N \leq 50) \]

Diagram:
```
   s_1   s_2   s_N
```
all \( 2^N \) degrees of freedom

As an ansatz, let's propose a Matrix Product State (MPS) is a TN:
```
  s_1 s_2 s_N

A_1 \uparrow \cdots \uparrow A_N
```
\[ i = 1, \ldots, M \]

Algebraically:
\[ 4(s_1, \ldots, s_N) \xrightarrow{\text{vec}} \mathcal{A}^1 \otimes \mathcal{A}^2 \otimes \cdots \otimes \mathcal{A}^N \]
\[ 4(s_1, \ldots, s_N) = \mathcal{A}^1_{s_1} \otimes \mathcal{A}^2_{s_2} \otimes \cdots \otimes \mathcal{A}^N_{s_N} \]

Regard \( \mathcal{A}^j_{s_j} \) as two matrix:
\[ \mathcal{A}^j_{s_j} \]
\[ i_j \]
and \( \mathcal{A}^j_{s_j} \]
\[ j \]

Given \( s_1, \ldots, s_N \), pick which matrix to use on each site, take product.
This is a huge compression
\[ 2^N \rightarrow 1 \times N \exp(1) \text{ improved } \frac{M^2 \cdot 2 \cdot N}{\text{entangle}} \]
Matrix Product State (MPS)

\[ \gamma(s_1, ..., s_N) = \tilde{A}_1[s_1] \tilde{A}_2[s_2] \cdots \tilde{A}_N[s_N] \]

Very simple example: 2 spin, spin 1/2

\[ |1\downarrow > = \frac{1}{\sqrt{2}} (|1\uparrow > - |1\downarrow >) \]

\[ = \sum_{s_1, s_2} \tilde{A}_1[s_1] \cdot \tilde{A}_2[s_2] \mid 1s_1 \rangle \mid 1s_2 \rangle \]

\[ = \left( \sum_{s_1} \tilde{A}_1[s_1] \mid 1s_1 \rangle \right) \cdot \left( \sum_{s_2} \tilde{A}_2[s_2] \mid 1s_2 \rangle \right) \]

Let

\[ \tilde{A}_1 = \begin{pmatrix} |1\uparrow > \\ -\frac{1}{\sqrt{2}} |1\downarrow > \end{pmatrix}, \quad \tilde{A}_2 = \begin{pmatrix} |1\uparrow > \\ |1\downarrow > \end{pmatrix} \]

By inspection, \[ |1\downarrow > = \tilde{A}_1 \cdot \tilde{A}_2 \]

Three state: Let \[ |1\downarrow > = \frac{1}{\sqrt{3}} (|1\uparrow \uparrow > + |1\uparrow \downarrow > + |1\downarrow \downarrow >) \]

\[ \tilde{A}_1 = \begin{pmatrix} |1\uparrow > \\ |1\downarrow > \end{pmatrix}, \quad \tilde{A}_3 = \begin{pmatrix} |1\uparrow > \\ |1\downarrow > \end{pmatrix}, \quad \text{then} \quad \tilde{A}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} |1\uparrow > & |1\uparrow > \end{pmatrix} \]

(use \[ \tilde{A}_1^T \])

More states: Same structure, bigger matrices, faster entanglement compression
In 1D, low entanglement $\Rightarrow$ MPS

Test

\[ \text{from } \mathcal{E}_D = \]

Do SVD

\[ \begin{array}{c}
\alpha \quad \beta \quad \gamma \\
\end{array} \]

\[\begin{array}{c}
\mathcal{E}_D \\
\mathcal{S} \\
\mathcal{D} \\
\mathcal{L} \quad \mathcal{N}
\end{array}\]

Schmidt

for

Sites 1 - $N$ can be treated as being only $m \times m$ states

\[ \mathcal{E}_L \alpha = \mathcal{E}_R \beta \]

Repeat at our site over

\[ \begin{array}{c}
\pi \quad \pi \quad \pi \\
\end{array} \]

The site $L+2$ can be treated as being $m$ states

\[ \mathcal{S}_{L+1} \alpha = \mathcal{S} \beta \]

The $|\Psi\rangle = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} \sum_{s=1}^{L+1} \Psi_{\alpha,\beta}(\mathcal{E}_L \mathcal{S}_{L+1} \mathcal{E}_R) |\alpha \rangle |\beta \rangle |s\rangle$

We have

\[ \begin{array}{c}
\pi \quad \pi \\
\end{array} \]

\[ \begin{array}{c}
\alpha \quad \beta \\
\end{array} \]

To get the full MPS, you just need to repeat on the other sites.
Orthogonality is crucial for MPS. Drop \( V \), redefine \( U \)

In an SVD, \( M = UDV \), \( U + V \) are row/column unity

\[ U^T U = 1 \quad \text{and} \quad V V^T = 1 \]

Let \( U \) have fastest indices \( U_{\ell} \)

For left Schmidt

\[ \sum_\ell U_{\ell i} U_{\ell i} = \delta_{i i'} \quad \text{and} \quad U_{\ell i} = \delta_{i i'} \]

For right

\[ \sum_r V_{r i} V_{r i} = \delta_{i i'} \quad \text{and} \quad V_{r i} = \delta_{i i'} \]

Converting a W-fn to an MPS

\[ \begin{array}{ccccccc}
\end{array} \quad \text{Orthogonality} \]

\[ A_1 \] has \( \Omega \)

\[ A_2 \]

\[ A_1 \]

\[ A_2 \]
Keep going to the end

\[ A : I = 1 \quad \text{An this must be so} \]
\[ \text{fail} \neq N \quad \text{no orthogonality} \]

We say site \( N \) is the orthogonality center.

We could put \# on the site 1

\[ \begin{array}{c}
\text{\#} \\
\text{\#} \\
\text{\#} \\
\text{\#} \\
\end{array} \rightarrow \begin{array}{c}
\text{\#} \\
\text{\#} \\
\text{\#} \\
\text{\#} \\
\end{array} \quad J = 1
\]

\[ \begin{array}{c}
\text{\#} \\
\text{\#} \\
\text{\#} \\
\end{array} = \begin{array}{c}
\text{\#} \\
\text{\#} \\
\text{\#} \\
\end{array} + \begin{array}{c}
\text{\#} \\
\end{array} \quad D = 1
\]

e etc.

In general, it looks like

\[ \text{OC} \]

\[ \begin{array}{c}
\text{\#} \\
\text{\#} \\
\text{\#} \\
\end{array} \]

\[ \ldots \begin{array}{c}
\text{\#} \\
\text{\#} \\
\text{\#} \\
\end{array} = 1 \quad D = 1 \ldots \]

It is possible to not have an OC

--but it is useful to have one.
Normalisation of an MPS, Measuring Ops

\[ \langle 4 | 4 \rangle = \begin{array}{c|c|c|c|c|c} & & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
\end{array} \]

\[ \sum_{S_1 \ldots S_N} \langle \psi_{-} | S_1 \ldots S_N \rangle \langle S_1 \ldots S_N | \psi_{+} \rangle = \delta_{S_1 \ldots S_N} \]

To contract: left to right, or right to left.

Not top, then bottom.

"Bubbling"

Measuring Operators

\[ \langle s_1 \ldots s_n | A_0; | s_1, \ldots s_n \rangle = \langle s_1 \ldots s_n | s, \ldots (0:s) \ldots s_n \rangle \]

\(O_i\) attaches to \(A_i\).

\[ \langle 0; \rangle = \]

Also

Bubble left to right, etc.
Using an OC

\[ \text{L} \rightarrow \text{I} = \text{C} = \text{I} \]

Similarly on right

\[ \text{L} \rightarrow \text{I} \]

No need to do other contracting

Moving the OC

To move to right

1) Join two adjacent tensors, then split, moving OC
How. Using one side

\[ l = \frac{l_{u} \cdot l_{v}}{u \cdot v} \quad \text{then} \quad A \Sigma E J = 4 \]

\[ A \Sigma E J \]

\[ \text{Contract} \]

\[ A \Sigma E J + 1 \]

One side, QR

The QR is a cheaper than SVD matrix factorization

Where \( Q \) is

\[ [\text{row}] \text{ orthogonal} \quad R \text{ is} \]

upper right triangular. The "thin" version

\[ M = QR \]

\[ M = \begin{bmatrix} Q & R \end{bmatrix} \]

\[ N \times M \rightarrow N \times M \text{ MxM} \quad Q \times Q = 1 \]

Then

\[ A = Q \cdot R \]

\[ A \Sigma E J = Q \]

\[ A \Sigma E J + 1 = R \cdot A \Sigma E J + 1 \text{ old} \]
Getting the ground state in MPS Form

TEBD / O MPS

Trotter Approximation

Power method \( V' = (1 - \varepsilon H) V \) iterate
\( V \to \) ground state if \( \varepsilon \) small.

Better \( V' = e^{-\beta H} \) \( \beta \) could be big
\( e^{-\beta H} \) is hard for \( \beta \) big, but easy for \( \beta \) small, \( \beta \to 2 \gg \ll \)

Let \( H_i = \sum \hat S_i \cdot \hat S_i \), \( H = \sum H_i \).
\( e^{-2 H} = e^{-2 \sum H_i} = \prod e^{-2 H_i} \)

Errors come from \( \sum H_i, H_i H_j \neq 0 \)

\[
\begin{align*}
\varepsilon & = 1 - 2(A+B) + \frac{1}{2} 2^2 (A+B)^2 + \cdots \\
\varepsilon & = 1 - 2A + \frac{1}{2} 2^2 A^2 + \cdots \\
\varepsilon & = 1 - 2(A+B) + \frac{1}{2} 2^2 (A+B)^2 + \cdots\\
\varepsilon & = 1 - 2(A+B) + \frac{1}{2} 2^2 (A+B)^2 + \cdots
\end{align*}
\]
The error is \( \alpha \sim 2 \epsilon \). 

But to get good state, you need to do this \( O(\frac{1}{\epsilon}) \) times, so total error \( \sim \frac{1}{\epsilon} \). [Usually too lazy to do!]

If you do it twice with a reverse you get \( O(\frac{1}{\epsilon^2}) \) per sweep.

\[ e^{2i(A+B+C)} \approx e^{-2A} e^{-2B} e^{-2C} e^{-2B} e^{-2A} \]

\[ \frac{2i}{2} (A+B+C) \approx \frac{2i}{2} A \to \frac{2i}{2} B \to \frac{2i}{2} C \to \frac{2i}{2} B \to \frac{2i}{2} A \]

\[ A \text{ has } AB = BA \]
\[ B \text{ has } CA \]
\[ C \text{ has } AB \]
\[ \text{(no CAB)} \]
\[ \text{BA} \]

We apply this one at a time, moving the OC, in a sweeping algorithm.

**Diagram**

\[ H_s = S \cdot \frac{7}{5} = e^{-2H_{s}} \]

**MPS**

\[ \text{only affects } A_{2}, A_{3} \]

**Step 1**

\[ \text{Step 2} \]

\[ 2 \quad 3 \]

\[ 2 \quad 3 \]
1 sweep = \( i = 1 \text{ to } n \), then \( i = n \text{ to } -1 \text{ to } 1 \)

Moving OC in direction of sweeping

Periodically, you need to fix the norm of the state.

The splitting/moving of the OC needs the two side SVP: not QR. QR is only when we don't truncate.

\( \varepsilon \) controls error in result

But needs to converge in \# of sweeps

Need \( T = \frac{1}{\beta} \) to be \( \leq \) less than gap between ground state and 1st excited state

\( T - \frac{\Delta}{\beta} \leq \frac{\beta}{\Delta} \) \# sweeps \( \leq \frac{\beta}{\varepsilon} \)