Exactly solvable models for 2D topological phases

A topological phase is a quantum phase with:

1. Energy gap
2. Quasiparticle excitations have non-trivial braiding statistic

Examples:

1. FQH states
2. Spin liquids

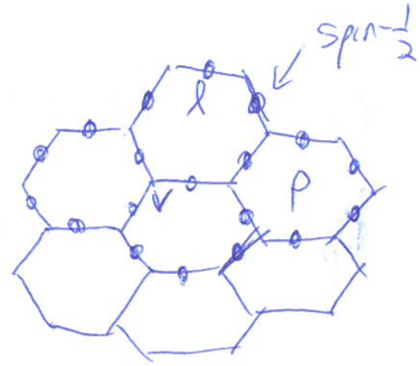
Goal of next 2 lectures: construct a large class of exactly solvable lattice spin models for top. phases

Applications:

1. Build intuition
2. Derive universal properties: boundaries, topological entanglement entropy, ...
3. Numerics: tensor networks
4. Experimental realizations (?)

Simple example: "Toric code" model

(Kitaev, 1997) "$\mathbb{Z}_2$ topological phase"



$$H = -\sum_v \underbrace{\prod_{\curlywedge} \sigma_\ell^x}_{Q_v} - \sum_p \underbrace{\prod_{\hexagon} \sigma_\ell^z}_{B_p}$$

Note: $[Q_v, Q_{v'}] = [B_p, B_{p'}] = [Q_v, B_p] = 0$

$\Rightarrow$ Can simultaneously diagonalize B_p, Q_v: $|\{b_p\}, \{q_v\}\rangle$

Eigenvalues: $b_p, q_v = \pm 1$ (since $B_p^2 = Q_v^2 = 1$)

Energy: $E = -\sum_v q_v - \sum_p b_p$

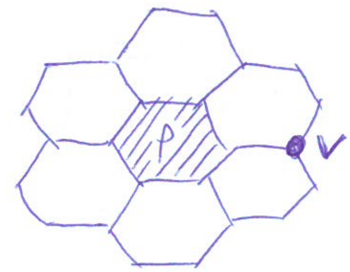
Gd. state: $|\{b_p = q_v = 1\}\rangle$

2 types of excitations:

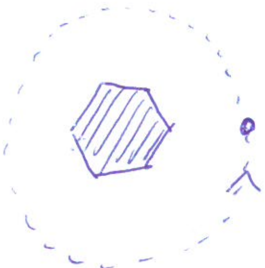
$q_v = -1$ for some v $\Longrightarrow$ "charge" $E = 2$

$b_p = -1$ for some p $\Longrightarrow$ "flux" $E = 2$

$\Rightarrow$ Gapped

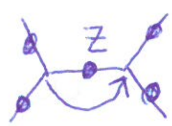


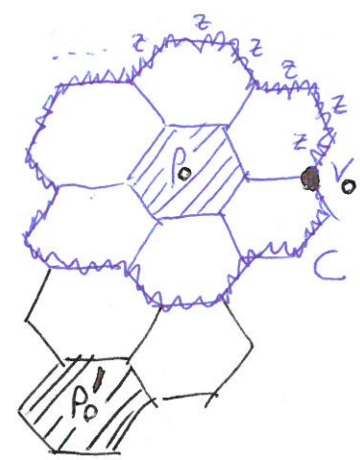
Excitations have non-triv. mutual statistics:



$e^{i\theta} = -1$

To see this:

σ^z moves charge: 



$$\prod_{l \in C} \sigma^z_l |p_0, v_0\rangle = \prod_{p \in \mathrm{Int}(C)} B_p |p_0, v_0\rangle = -|p_0, v_0\rangle$$

Without flux in center:

$$\prod_{l \in C} \sigma^z_l |p_0', v_0\rangle = \prod_{p \in \mathrm{Int}(C)} B_p |p_0', v_0\rangle = |p_0', v_0\rangle$$

String picture

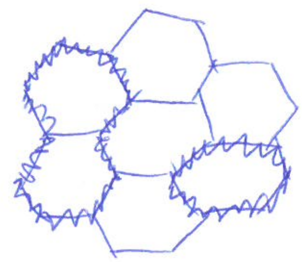
$\sigma_\ell^x = -1 \iff$ "string on link ℓ"

$\sigma_\ell^x = +1 \iff$ "no string on link ℓ"

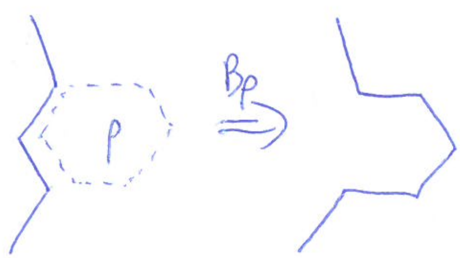
spin states $\iff$ string states

Q_v term: Favors configurations where $\prod_{\curlywedge} \sigma^x = 1$

$\Rightarrow$ strings form closed loops



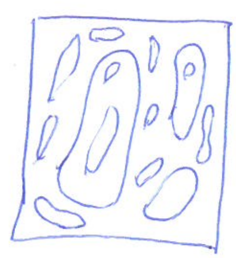
B_p term: string hopping



Gd state: $|\Phi\rangle = \sum_{X \text{ closed}} |X\rangle$

$\iff$ $\Phi(X) = 1$ for all closed string config. X

Looks like a "string condensate"



In fact Φ is an "ideal" condensate in the sense that its correlation length $\xi = 0$

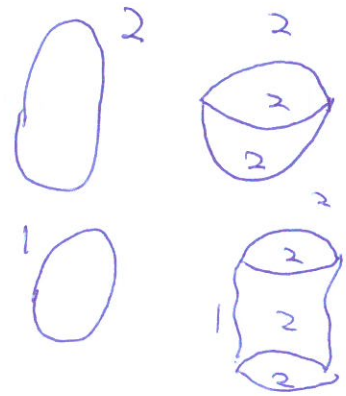
"Fixed point" wave function

Generalizing toric code

(ML + X.-G. Wen, 2005)

"String-nets": networks of strings
- strings can come in different types
- strings can branch (3-fold branching only)

$\Longrightarrow$ labeled trivalent graphs



"string-net models": QM models that describe dynamics of string-nets

To specify a string-net model, need to provide data.

[Note: will focus on special class of string-net models]

Data:

1. Number of string types: N $\qquad$ $\overset{i}{\text{———}}$ $\qquad$ $i = 1, \dots, N$

2. Branching rules: Triplets $\{i, j, k\}$ allowed to meet at a point

String-net Hilbert space:

Orthonormal basis = { string-net configs. (on honeycomb lattice) obeying branching rules }

Example:

Number of string types: $N = 2$

Branching rules: $\{1, 2, 2\}$, $\{2, 2, 2\}$

Strategy for building string-net models:

1. Construct ground state wave functions Φ

2. Construct corresponding Hamiltonians H

Constructing wave functions

Want "fixed pt" wave function:

$\Phi($  $) = \ldots$

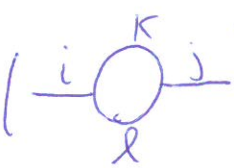
How can we write down Φ?

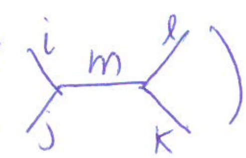
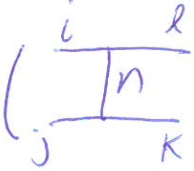
Define implicitly using local constraint eqs:

1. Φ only depends on topology of string-net:

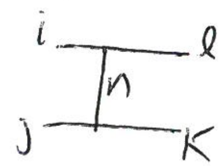
$\Phi($  $) = \Phi($  $)$

2(a) $\Phi($  $) = d_i \, \Phi(\circ)$

(b) $\Phi($  $) = 0$ if $i \neq j$

(c) $\Phi($  $) = \sum_n F^{ijm}_{kln} \, \Phi($  $)$

Here (d_i, F^{ijm}_{kln}) are complex constants that define Φ.

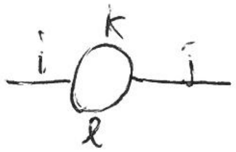
[Note: We define $F^{ijm}_{kln} = 0$ if branching rules are not obeyed by either  or ]

New notation

Index i runs from $0, 1, \ldots, N$

$\uparrow$ "vacuum string"

shorthand for summarizing eqs:

$\Phi($  $) = 0$ if $i \neq j$ $\Longleftrightarrow$

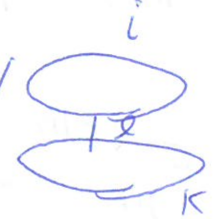
$\Phi($  $) = 0$ if $i \neq 0$,

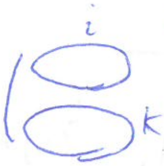
$\Phi($  $) = 0$ if $j \neq 0$,

$\Phi($  $) = 0$ if $i \neq j \neq 0$

Local rules completely specify Φ:

Example

$\Phi\Big($  $\Big) = \sum_{\ell} F^{ikj}_{ki\ell}\, \Phi\Big($  $\Big)$

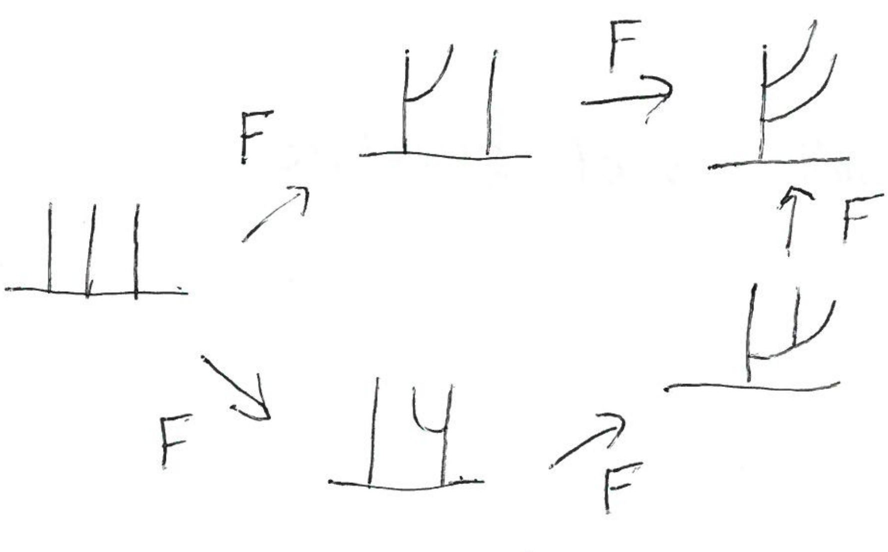
$= F^{ikj}_{ki0} \cdot \Phi\Big($  $\Big)$

$$= F^{ikj}_{ki0} \cdot d_i \cdot d_k \cdot \underbrace{\Phi(\text{vacuum})}_{=1}$$

$$= F^{ikj}_{ki0} \cdot d_i \cdot d_k$$

Trick for writing down complicated w.f.'s.

But rules are not usually self-consistent:



Self-consistency conditions

(i) $\sum_n F^{mlq}_{kpn} F^{jip}_{mns} F^{jsn}_{lkr} = F^{jip}_{qkr} F^{riq}_{mls}$

(ii) $F^{ijm}_{kln} = F^{lkm}_{jin} = F^{jim}_{lkn} = F^{imj}_{knl} \sqrt{\frac{d_m d_n}{d_j d_l}}$

(iii) $F^{ijk}_{ji0} = \sqrt{\frac{d_k}{d_i d_j}} \, \delta_{ijk}$

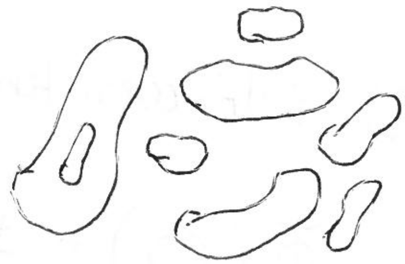
where $\delta_{ijk} = \begin{cases} 1 & \text{if } \{i,j,k\} \text{ allowed} \\ 0 & \text{otherwise} \end{cases}$

Each solution $(F^{ijm}_{kln}, d_i, \delta_{ijk})$ defines a string-net wave function Φ.

Later, will construct an exactly solvable Hamiltonian H.

Solutions are very sparse with lots of structure.

Example #1

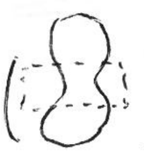


1. String types: $N=1$
2. Branching rules: No branching

Two nontrivial local rules:

$$\Phi(\bigcirc) = d_1 \cdot \Phi(\cdot)$$

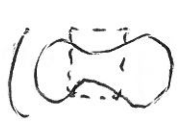
$$\Phi(\,)\,(\,) = F^{110}_{110} \cdot \Phi(\asymp)$$

Notice: $\Phi\Big($  $\Big) = F^{110}_{110}\ \Phi\Big($  $\Big)$

$$\| \qquad\qquad\qquad \|$$

$$d_1 \qquad\qquad F^{110}_{110} \cdot d_1^2$$

$$\Rightarrow \quad F^{110}_{110} = d_1^{-1}$$

Also $\Phi\Big($  $\Big) = F^{110}_{110}\ \Phi\Big($  $\Big)$

$$\| \qquad\qquad\qquad \|$$

$$d_1^2 \qquad\qquad F^{110}_{110} \cdot d_1$$

$$\Rightarrow \quad F^{110}_{110} = d_1$$

$\therefore\ d_1 = F^{110}_{110} = \pm 1$

2 self-consistent sets of rules:

$$\Phi(\bigcirc) = \pm\,\Phi(\cdot)$$

$$\Phi(\,)(\,) = \pm\,\Phi(\asymp)$$

2 string-net wave functions:

$\Phi_+(X) = 1$ for all string config. X $\Longleftrightarrow$ Gd-state of toric code

$\Phi_-(X) = (-1)^{N_{loops}(X)}$ " " " " $\Longleftrightarrow$ Gd-state of "doubled semion model"

Example #2

1. String types: $N = 1$
2. Branching rules: $\{1,1,1\}$

In this case, find only two solutions to (i)-(iii).

Corresponding rules:

$$\Phi(\bigcirc) = \tau \cdot \Phi(\cdot)$$

$$\Phi(-\!\bigcirc) = 0$$

$$\Phi(\,)(\,) = \tau^{-1}\Phi(=) + \tau^{-1/2}\Phi(\mathrm{I})$$

$$\Phi(\succ\!\!-\!\!\prec) = \tau^{-1/2}\Phi(=) - \tau^{-1}\Phi(\mathrm{I})$$

where $\tau = \dfrac{1 \pm \sqrt{5}}{2}$

Easy to check rigidity with respect to τ:

$$\Phi(\bigcirc\bigcirc) = \tau^{-1}\Phi(\bigcirc) + \tau^{-1/2}\Phi(\ominus)$$

$$\overset{\shortparallel}{\tau^2} \qquad = 1 + \tau^{-1/2}\Phi(\ominus)$$

$$= 1 + \tau^{-1/2}\left[\tau^{-1/2}\Phi(\bigcirc\bigcirc) - \tau^{-1}\Phi(\bigcirc\!-\!\bigcirc)\right.$$

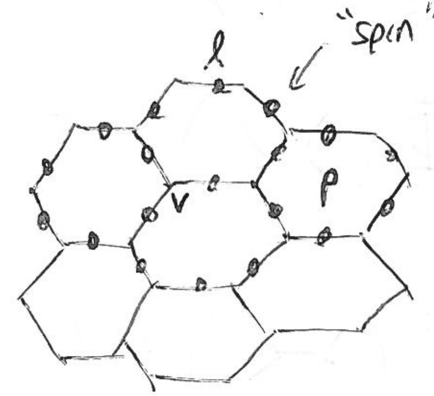
$$= 1 + \tau$$

$$\Rightarrow \tau = \frac{1 \pm \sqrt{5}}{2}$$

Formula for Φ?

Specified implicitly, but no closed form expression.

Hamiltonians



Each "spin" can be in

$N+1$ states: $|0\rangle, |1\rangle, \dots, |N\rangle$

String language:

$\Rightarrow$ $\overset{i}{\overline{\qquad\qquad}}$

$\Rightarrow$ No string

$$H = -\sum_v Q_v - \sum_p B_p$$

Q_v

Defined by $Q_v \Big|$  $\Big\rangle = \delta_{ijk} \Big|$  $\Big\rangle$

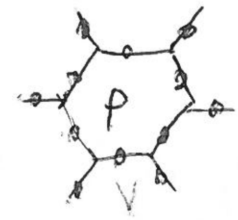
Favors string-nets that obey branching rules

B_p

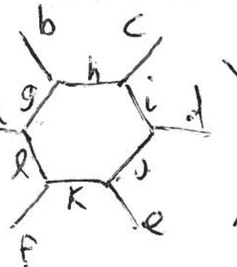
Defined by

$$B_p = \frac{1}{D}\sum_{s=0}^{N} d_s B_p^s \;, \qquad D = \sum_{s=0}^{N} d_s^2$$

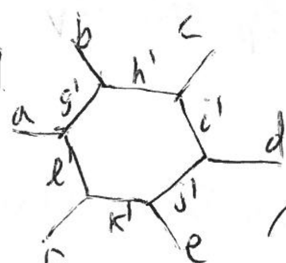
B_p^s is a 12 spin interaction;



B_p^s |



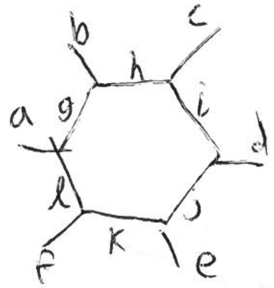
$$\rangle = \sum_{g'h'\cdots l'} F^{alg}_{sg'l'} F^{bgh}_{sh'g'} \cdots F^{fkl}_{sl'k'}$$ |



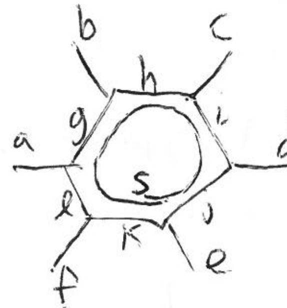
$\rangle$

Another, simpler def. for B_p^s:

B_p^s |



$\rangle$ = |



$\rangle$

B_p^s adds a type-s string in plaquette p.

Properties of Hamiltonian

Suppose $\left(F^{ijm}_{k\ell n}, d_i, \delta_{ijk}\right)$ satisfy (i)–(iii) and also

$$F^{ijm}_{k\ell n} = \left(F^{in\ell}_{kmj}\right)^* \quad \text{(unitarity condition)}$$

Then:

1. $\{B_p, Q_v\}$ all commute and are Hermitian
2. B_p and Q_v are projection operators

$\Rightarrow$ H is exactly solvable

3. Gd. state wave function is Φ (on lattice).

4. Model realizes a top. phase:

   (a) Gapped

   (b) QP's have [redacted] non-trivial braiding statistics

Example 1: Doubled semion model

Data: $N=1$, no branching

$$d_1 = F^{110}_{110} = -1$$

Each "spin" can be in 2 states: $|0\rangle, |1\rangle$.

Convenient to use spin-1/2 notation:

$$|0\rangle = |\sigma^x = +1\rangle$$

$$|1\rangle = |\sigma^x = -1\rangle$$

<u>Q_v</u>: 3 spin interaction

$$Q_v = \frac{1}{2}\left(1 + \prod_{\text{⅄}} \sigma_\ell^x\right)$$

B_p: 12 spin interaction

$$B_p = \frac{1}{2} P_p \left(1 + \prod_{\hexagon} \sigma_\ell^z \cdot \prod i^{\frac{1-\sigma_\ell^x}{2}}\right) P_p$$

where $$P_p = \prod_{v \in p} Q_v$$

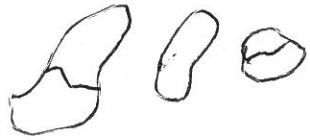
Model realizes "doubled semion phase"

4 QP's: $\{1, s, \bar{s}, s\bar{s}\} = \{1, s\} \times \{1, \bar{s}\}$ = Sem × $\overline{\text{Sem}}$

[redacted] [redacted]

$s, \bar{s}$ have exchange statistics $e^{i\theta_{ex}} = \pm i$, and trivial mutual statisti

Example 2: Doubled Fibonacci model



Data: $N=1$, branching

$$\tau = \frac{1+\sqrt{5}}{2}$$ need '+' to satisfy unitarity condition

Again use notation

$$|0\rangle = |\sigma^x = +1\rangle$$

$$|1\rangle = |\sigma^x = -1\rangle$$

Q_v

$$Q_v \,|\, {}_{+}\curlywedge^{+}_{+} \rangle = |\, {}_{+}\curlywedge^{+}_{+} \rangle$$

$$Q_v \,|\, {}_{-}\curlywedge^{+}_{-} \rangle = |\, {}_{-}\curlywedge^{+}_{-} \rangle$$

$$Q_v \,|\, {}_{+}\curlywedge^{-}_{+} \rangle = 0$$

B_p

Not illuminating to write down

Model realizes "doubled Fibonacci phase"

4 QP's: $\{1, \phi, \bar{\phi}, \phi\bar{\phi}\} = \{1, \phi\} \times \{1, \bar{\phi}\} = \text{Fib} \times \overline{\text{Fib}}$

ϕ: non-abelian "Fibonacci" anyon with $\phi \times \phi = 1 + \phi$

$\bar{\phi}$: similar but with opposite braiding statistics

General string-net models

So far, we've focused on a special case. More generally:

1. Strings carry orientation: $\xrightarrow[i]{} \sim \xleftarrow[i^*]{}$

2. Additional degrees of freedom at vertices: 

   $\alpha = 1, 2, \ldots, \delta_{ijk}$ where δ_{ijk} can be bigger than 1.

3. More complicated rules for vacuum string $\xrightarrow[0]{}$

   Need to decorate "symmetric" vertices with dot: 

(Kitaev, Kong, 2011)
(Lin, Levin, 2014), (Lan, Wen, 2013)

How general are these models?

1. Can realize any top. phase of form $T \times \bar{T}$

To do this, let:

string types $\Longleftrightarrow$ QP's in T

Branching rules $\Longleftrightarrow$ Fusion rules in T (i.e. $i \times j = \sum_k \delta_{ijk} \cdot k$)

$F \Longleftrightarrow$ F-symbol in T

$d_i \Longleftrightarrow$ quantum dimension of particle i.

2. Can realize any gauge theory with finite gauge group G

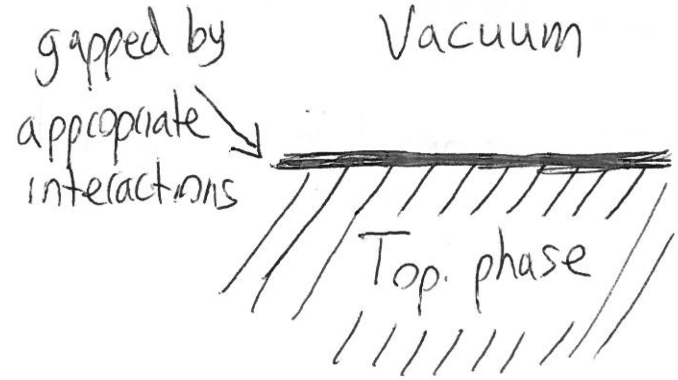
<u>Cannot</u> realize topological phases with a nonzero chiral central charge; $c_- \neq 0$

In particular, cannot realize Laughlin-like states.

<u>Conjecture</u>: A (bosonic) topological phase can be realized by a string-net model if and only if it supports a gapped edge.

"only if" is easy: will explain later

"if" is hard $\Rightarrow$ only proven in Abelian case



Mathematics of string-net models

Input data $(F^{ijm}_{k\ell n}, d_i, \delta_{ijk})$ = "unitary finite spherical fusion category $\mathcal{C}$"

Output = model for a topological phase with anyons $\mathcal{A} = \{a, b, c, \ldots\}$ with braiding data $N^c_{ab}, R^c_{ab}, F^d_{abc}, \ldots$

How is $\mathcal{A}$ (and associated braiding data) related to $\mathcal{C}$?

Answer: $\mathcal{A} = Z(\mathcal{C})$

In words: $\mathcal{A}$ is "Drinfeld center" of $\mathcal{C}$.

Also sometimes called "quantum double".