

First Day

Entanglement, Reason for studying.

Real interest in physics is the dynamics of things, not the ground state!

But the ground state can give information about excited states

(for example, solids are rigid because of regular arrangement of atoms in them, in the ground state)

There are examples of this beyond ~~old~~ symmetry breaking in quantum systems.

How is it possible? If $|G\rangle$ is the ground state, $H = \frac{E_0|G\rangle\langle G| + \sum E_n|n\rangle\langle n|}{\text{the Hamiltonian which has an arbitrary spectrum}}$ can be considered spin chains.

Spin-1/2 antiferromagnets

$$H = \sum S_j \cdot S_{j+1}$$

The ground state has long-range correlations

$$S_i \cdot S_j \sim \frac{(-1)^{i+j}}{|i-j|^\alpha}$$

But for spin 1 $S_i \cdot S_j \sim e^{-|i-j|/\xi}$

This implies the spin 1/2 chain has no gap in its excitation spectrum.

Analogously Hastings's theorem:

If H is a Hamiltonian with local interactions & there is a gap, then the correlations ~~have a~~ decay exponentially.

Proof (sort of): You can get the ground state by applying e^{-HT} to an initial ~~state~~ state.

After a time τ you get, you get within $e^{-\Delta\epsilon\tau}$ of the ground state, so τ doesn't have to be very large to get close to the ground state.

But in a finite length of time, ~~correlations propagate only over a certain~~ ~~cannot~~ spins that are far apart cannot become correlated. [Robinson-Lieb bound]

For

Also: Insulators have short-range correlations.

Metals have power-law correlations
 $\langle n(x) n(y) \rangle \sim \frac{\cos 2k_F|x-y|}{|x-y|^2}$

This is another example of Hastings's theorem.

To understand correlations better, specifically focusing on quantum behavior, let us think about

Entanglement

& the Einstein-Podolsky-Rosen paradox.

Entanglement of two spins

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

Does this describe a definite state or a statistical one?

If it is a statistical distribution in a classical sense.

If you measure the first spin's S_z and it is up, the second spin is down.

The spins can't influence each other, so this must have been true in advance.

$$\text{Now } |\downarrow\rangle = (|\rightarrow\rangle - |\leftarrow\rangle)/\sqrt{2}$$

So there should be a 50% chance of finding the second spin pointing left or right.

Now say you measure first spin, whether it points right or left.

$$|\Psi\rangle = (|\leftarrow\rangle|\rightarrow\rangle - |\rightarrow\rangle|\leftarrow\rangle)/\sqrt{2}$$

If first spin points right, then the second points left definitely.

So changing what you measure about first spin changes what the possibilities for the second spin are.

An ~~example~~ Measuring 1st spin's $S_z \Rightarrow$ second spin is \uparrow or \downarrow
Measuring 1st spin's $S_x \Rightarrow$ second spin is \rightarrow or \leftarrow .

Correct resolution: you can measure only ~~one~~ operators, you get the same B_x or S_z in each ~~case~~ ~~not~~ wave-functions. The possible outcomes of S_z are actually the same in both cases (if you don't know what 1st person saw)

~~But this is very quantum mechanical because it assumes~~

But can you resolve it better?

Maybe an electron already knows which way it will point if you measure each component of the spins $|\uparrow\rangle$ and $|\rightarrow\rangle$ and just describe part of the state.

Ex 2.4

Representation of singlet state

x	z	for particle A	x	z	for particle B	
+	+		-	-		25% chance
+	-		-	+		25% chance
-	-		+	+		25% chance
-	+		+	-		25% chance

Notice if you measure S_x or S_z in both cases, they are opposite

† if you measure S_x for one particle + S_z for the other particle they are not at all correlated.

Entanglement Hamiltonian

You can make it more precise and also more intuitive with the "entanglement Hamiltonian."

If $|\psi\rangle = \sum \lambda_i |i\rangle_A |i\rangle_B$

Then $\rho_A = \sum \lambda_i^2 |i\rangle_A \langle i|_A$ which is the same density matrix you would have

for the Hamiltonian

H_A with energies $E_i = -\ln \lambda_i^2$ at $T=1$.

• H_A = "entanglement Hamiltonian"

Formal Formula:

$$H_A = -\log \rho_A$$

E.g.

$$\frac{1}{\sqrt{3}} |\uparrow \downarrow\rangle + \frac{\sqrt{2}}{\sqrt{3}} |\downarrow \uparrow\rangle$$

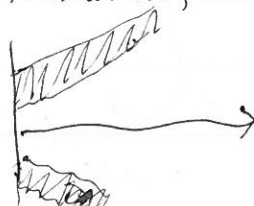
$$H_A = \begin{pmatrix} \ln 3 & \leftarrow \\ & \ln 3/2 \end{pmatrix}$$

$|\uparrow\rangle$ has a higher "energy" \leftrightarrow less likely component of wave function

Short-range correlations

\leftrightarrow H_A is inhomogeneous and it grows as you move away from the boundary.

E.g. for an insulator, the bands of H_A look like



16 + .09
25

↑↑↑

↓↑↑ + ↑↓↑

↑ ↓↑ + ε ↑↑↓

↓↑↑

+ ↓↑↓ + ε ↓↓↑

$a^2 + b^2 = 2c^2$

$(1+i)(1-i)$

$\sqrt{2}$

$7^2 \times$

7

General theory of entanglement:
Let $|\psi\rangle$ be a

Entanglement in a system with a gap has a short range. ~~What does it mean? Can you build up all the correlations by looking at pairs of nearest neighbors?~~

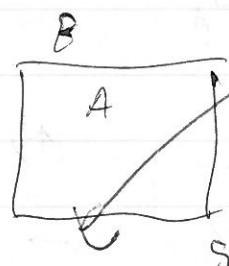
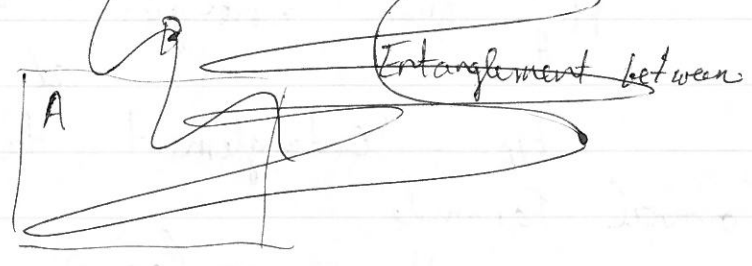
Example $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

Now try to get second spin entangled with third spin

$\uparrow(\downarrow\uparrow - \uparrow\downarrow) - \downarrow(\uparrow\downarrow - \downarrow\uparrow)$

but this changes the state of the first spin to $\uparrow + \downarrow$.

It did not work very well.



Entanglement between regions A and B is proportional to length of boundary because only spins within ξ of the boundary are correlated.

What does this mean? Can you build up entanglement by looking at pairs of adjacent spins?

Say you have a state like

$0.7|\uparrow\downarrow\uparrow\rangle + 0.1|\uparrow\uparrow\downarrow\rangle + 0.7|\downarrow\uparrow\downarrow\rangle + 0.1|\downarrow\downarrow\uparrow\rangle$

You can't say that ~~both~~ both the first pair + second pair of spins is entangled. ~~Since~~ the first pair is entangled you can't also entangle it with the second pair.

So all the most likely states are states with excitations near the boundary.

~~Entangled terms of Schmidt states terms do not just involve the spins on the edge, but these are the Schmidt decomposition.~~
Application: Describing 1D many body states.

(The Schmidt terms for region A are really entangled among all the spins, but only the spins near the edge vary from state to state.

The bulk of the low-energy terms is the same as in the ground state wave function.)

~~Area~~ Consequence: area law: $S(A) \propto$ length of boundary between A and B.

This is true only for states with a gap.

~~Gapless states have~~

Fines classification of phases:

1) 1D critical points with conformal symmetry
 $S = c \log L$
Specific heat

1) 1D critical points (with conformal symmetry)

If the specific heat is $\propto T$

& the wave-speed is v , the entanglement entropy is $\frac{c v}{\pi} \log L$.

(One usually defines the coefficient $\frac{c v}{\pi}$ to be $\frac{c}{3}$ where c is the central charge.)

2. Fermi Conformal points in higher dimensions
 $\text{const} \times (\text{size of boundary}),$

~~length~~ [↑] same as for
phases with a gap.

3. Fermi liquids eg. in ~~2D~~ have more entanglement.

Eg. in 3D

$$S \propto A \log A \text{ where}$$

A is the area of the boundary.

4. States with a spontaneously broken continuous symmetry.

$$S = \text{const} N^{2/3} + \log N$$

(for a system of N spins divided
into halves)

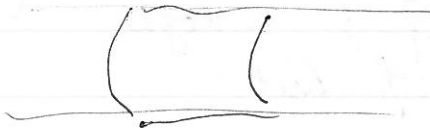
5. Topologically ordered states:



$$S = c_1 L - \gamma$$

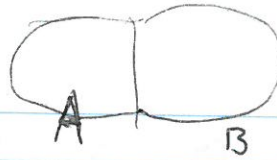
~~It is~~ ~~now~~ This is a little
bit incorrect because the
edge may not be uniform.

^N You can look at a cylinder though



(1+2A) (1+2)

↑



Break a system in two parts

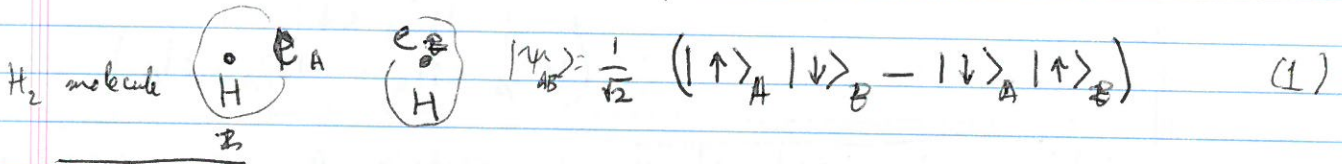
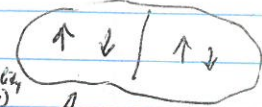
$|\psi_1\rangle + |\psi_2\rangle$
A pure state $|\psi_{AB}\rangle$ of a partitioned system does not imply that the parts are pure states.

Many worlds
as interference
of particles states
can't be brought
together again

Disappearance of
interference
due to measurement

Quantum Information vs. Classical Information

Classical ~~Systems~~: $S_{AB} \geq S_A$
(proof $\sum p_{ab} \ln p_{ab} \geq \sum p_a \ln p_a$)
Quantum Systems: Not true. At zero temperature the state of the full system is definite, pure state yet parts fluctuate.



Relation ~~between~~ subsystems of pure states having impure behavior
Correlations between parts
Case where subsystem is in a pure state the system.

Example: If $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ is a product state, then A is in the definite state $|\psi_A\rangle$.

Spin 1/2 example: $|\uparrow\rangle_A |\uparrow\rangle_B$
{Most states are not product states, e.g. (1)}
The S=0 triplet state is also entangled, even though it's in a different multiplet than the singlet.

Reduced Density Matrix

In Non-product states, always describe subsystems are described by density matrices:

Say $|\psi_{AB}\rangle = \sum_{a,b} c_{ab} |a\rangle_A |b\rangle_B$
Reduced density matrix ρ_A for A: $\rho_{AB} = |\psi_{AB}\rangle \langle \psi_{AB}|$

$\rho_A = \text{tr}_B \rho_{AB}$ "Partial Trace"

Explicitly $\langle a | \rho_A | a' \rangle = \sum_b \langle a | \rho_{AB} | a' b \rangle$

$= \sum_b \langle a | \rho_{AB} | a' b \rangle \langle \psi_{AB} | a' b \rangle$
 $= \sum_b c_{a'b} c_{ab}$ (Compare to (1))

Justification for E.g., for singlet:
Why ρ_A is the right description of the subsystem:
 $\langle \psi_{AB} | \rho_A | \psi_{AB} \rangle = \text{tr} \rho_A^2$

$$\begin{aligned}
 \rho_A &= \text{tr}_B \frac{1}{2} (|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B) (\langle\uparrow|_A \langle\downarrow|_B - \langle\downarrow|_A \langle\uparrow|_B) \\
 &= \text{tr}_B \frac{1}{2} |\uparrow\rangle_A \langle\uparrow|_A \langle\downarrow|_B \langle\downarrow|_B - \frac{1}{2} |\downarrow\rangle_A \langle\uparrow|_A \langle\uparrow|_B \langle\downarrow|_B \\
 &\quad - \frac{1}{2} |\uparrow\rangle_A \langle\downarrow|_A \langle\downarrow|_B \langle\uparrow|_B + \frac{1}{2} |\downarrow\rangle_A \langle\downarrow|_A \langle\uparrow|_B \langle\downarrow|_B \\
 &= \frac{1}{2} |\uparrow\rangle_A \langle\uparrow|_A \cdot 1 - \frac{1}{2} |\downarrow\rangle_A \langle\uparrow|_A \cdot 0 - \frac{1}{2} |\uparrow\rangle_A \langle\downarrow|_A \cdot 0 + \frac{1}{2} |\downarrow\rangle_A \langle\downarrow|_A \cdot 1 \\
 &= \frac{1}{2} (|\uparrow\rangle_A \langle\uparrow|_A + |\downarrow\rangle_A \langle\downarrow|_A)
 \end{aligned}$$

Density Matrix describes all the observable properties of the subsystem.

Why the reduced density matrix is a good description of the subsystem. How do you define the expectation value of an operator in a D.M. $\langle \chi \rangle_{AB} = \text{tr} \rho_A \chi$ for any operator on part A of the system.

then this ~~is~~ Singlet case $\langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = 0$, so it makes sense to describe left spin as randomly polarized. Look like a spin at infinite temperature.

Quantum Entropy Also - if subsystem is disconnected from environment, its reduced density matrix evolves by $U(t)_A U(t)_B$

Entropy of a subsystem measures inseparability: correlations. Entanglement departure from product state.

$$\begin{aligned}
 S(\rho_A) &= -\text{tr} \rho_A \ln \rho_A \\
 &= S(\rho_B) = -\text{tr} \rho_B \ln \rho_B
 \end{aligned}$$

"Entanglement Entropy"

[When state factors, $S=0$, so the entropy should get bigger the more inseparable it is. Actually somewhat familiar from classical stuff - if two systems A + B are correlated then they are not in a definite state (e.g. weather + clothes in Hawaii not correlated)]

Quantum Uncertainty Interesting fact: $S(\rho_A) = -\text{tr} \rho_A \ln \rho_A = -\text{tr} \rho_B \ln \rho_B = S(\rho_B)$

Entanglement S is a symmetrical measure of correlations.

Question: Does a classical system behave this way? (think of a volume of gas)



Derivation: ~~Using linear algebra~~ Schmidt Decomposition
 Possible to make a Schmidt decomposition:

$$|\Psi_{AB}\rangle = \sum_{i=1}^n \lambda_i |i\rangle_A |i\rangle_B$$

$\left\{ |i\rangle_A \quad 1 \leq i \leq d_A \right\}$ is an orthonormal basis
 $\left\{ |i\rangle_B \quad 1 \leq i \leq d_B \right\}$

λ_i is ~~not~~ the Schmidt and real; the number n of non-zero λ_i 's is the Schmidt #.

Thus

λ_i^2 are the eigenvalues of both ρ_A and ρ_B :

$$\rho_A = \text{tr}_B |\Psi_{AB}\rangle \langle \Psi_{AB}|$$

$$= \sum_{i,j} \lambda_i \lambda_j |i\rangle_A \langle j|_A \text{tr}_B |i\rangle_B \langle j|_B$$

$$= \sum_{i=1}^n \lambda_i^2 |i\rangle_A \langle i|_A$$

Sim., $\rho_B = \text{tr}_A |\Psi_{AB}\rangle \langle \Psi_{AB}|$

$$= \sum_{i=1}^n \lambda_i^2 |i\rangle_B \langle i|_B$$

Non-zero eigenvalues are the same

What is the ~~entropy~~ Entropy $S_A = -\text{tr} \rho_A \ln \rho_A = -\sum \lambda_i^2 \ln \lambda_i^2 = S_B$
 Derivation of Schmidt Decomposition of an electron spin in the Hydrogen Molecule?

$$|\Psi_{AB}\rangle = \sum_{a,b} c_{ab} |a\rangle_A |b\rangle_B$$

Singular value decomposition:

$$C = UDV \quad (\text{How is this the special diagonalizing a matrix a special case?})$$

U, V unitary
 D diagonal, non-negative d_B

$$d_A \begin{pmatrix} d_1 & & & & & \\ & d_2 & & & & \\ & & d_3 & & & \\ & & & d_4 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

$$|\Psi_{AB}\rangle = \sum_{a,b,i} u_{ai} d_i v_{ib} |a\rangle_A |b\rangle_B$$

$$= \sum_i d_i \left(\sum_a u_{ai} |a\rangle_A \right) \left(\sum_b v_{bi} |b\rangle_B \right)$$

U, V unitary $\Rightarrow |i\rangle_A, |i\rangle_B$ are each orthonormal bases

like to give hints/keywords to do calculations, etc.

Better to show interesting big ideas or present things logically and coherently?

Entanglement and measurement

Will Argue that: Quantum mechanics isn't weird until you have many-body systems.

Spin 1/2 particle $|\uparrow\rangle, |\downarrow\rangle$ one basis
 $|\rightarrow\rangle, |\leftarrow\rangle$
 $\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ $\frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$

Usual argument about $\Delta p \Delta x$ uncertainty: ~~doesn't have a well defined~~ Shows \uparrow
 $|\rightarrow\rangle$ is neither ~~up~~ nor ~~down~~, but both (even though if you measure S_x you get $\pm 1/2$)
 Evidence let $U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$.

Compare two procedures (use more general variables to make the point clear)

later

Start with $|\rightarrow\rangle = \alpha|\uparrow\rangle + \alpha|\downarrow\rangle$
 Apply U. Measure whether ~~up or down~~
 left or right, up or down

~~100%~~ \uparrow ~~0%~~ \downarrow
 100% 0%

$$U|\rightarrow\rangle = \sum_{s_1, s_2} U_{s_1 s_2} \alpha_{s_2} |s_1\rangle$$

$$P(\uparrow) = \left| \sum_{s_1} U_{\uparrow s_1} \alpha_{s_1} \right|^2 = \left(\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \right)^2 = 1$$

$$P(\downarrow) = \left| \sum_{s_1} U_{\downarrow s_1} \alpha_{s_1} \right|^2 = \left(\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \right)^2 = 0$$

Start with $|\uparrow\rangle$ or $|\downarrow\rangle$, with ~~50%~~ \uparrow probability, $|\alpha_{\uparrow}|^2, |\alpha_{\downarrow}|^2$
 Apply U. Measure whether left or right.

\uparrow \downarrow
 50% 50%

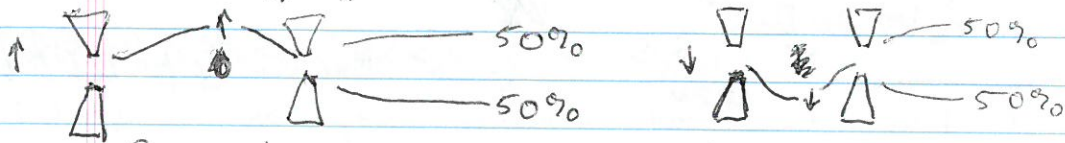
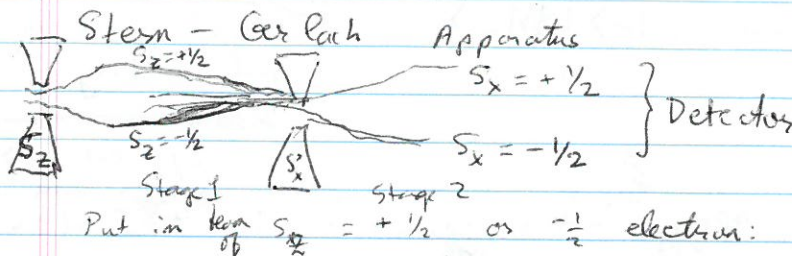
$$P(\uparrow) = \sum_s |\alpha_s|^2 P(\uparrow \text{ measured, } \uparrow \text{ prep}) + |\alpha_{\downarrow}|^2 P(\uparrow \text{ measured, } \downarrow \text{ prep})$$

$$= 2|\alpha_{\uparrow}|^2 \frac{1}{2} + |\alpha_{\downarrow}|^2 \frac{1}{2}$$

$$= \sum_s |\alpha_s|^2 |U_{\uparrow s}|^2 = \frac{1}{2} \frac{1}{2} + \frac{1}{2} = \frac{1}{2}$$

Difference: cross terms $U_{\uparrow s}^* U_{\downarrow s'} \alpha_s \alpha_{s'}^*$ which reflect electron being in either state.

DFB



$S_x = +\frac{1}{2}$ electron is neither up nor down but both.



Not the same as the 50-50 average of the above two experiments.

Paradox: How is it possible? ~~But~~ Dim light clearly shows electrons in both beams.

What about a single electron? If you measure

trajectory of a single electron

But measuring the beam where a single

Accurate measurement \Rightarrow Wave function collapse.

Electron Behaves as classically expected.

To see why: $\psi \rightarrow \sum U_{nm} \alpha_m$

Intermediate state $\psi \rightarrow \sum \alpha_n |n\rangle$

Second SG apparatus transforms into

$$U_{SG} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

(e.g. spin $\uparrow \rightarrow$)

So for input $|1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$

$$P(1) = \left| \sum_m U_{1m} \alpha_m \right|^2 = \left(\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \right)^2 = 1$$

$$P(2) = \left| \sum_m U_{2m} \alpha_m \right|^2 = \left(\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \right)^2 = 0$$

On the other hand, if you do compare to weighted average of S_z eigenstates

$$P(1) = \sum_m |\alpha_m|^2 P(1 \text{ measured} | m \text{ input})$$

$$= \sum_m |\alpha_m|^2 |U_{1m}|^2 = \frac{1}{2} \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(2) = \sum_m |\alpha_m|^2 |U_{2m}|^2 = \frac{1}{2} \frac{1}{4}$$

Nothing's really weird. Statistical about the coherent behavior (e.g. polarized light) - what's the apparatus says it's not a wave.

How does measuring apparatus $\psi \rightarrow \sum \alpha_m |m\rangle$ Does the wave-function really collapse?



Before

After

$|S\rangle, |M_1\rangle, |M_2\rangle$

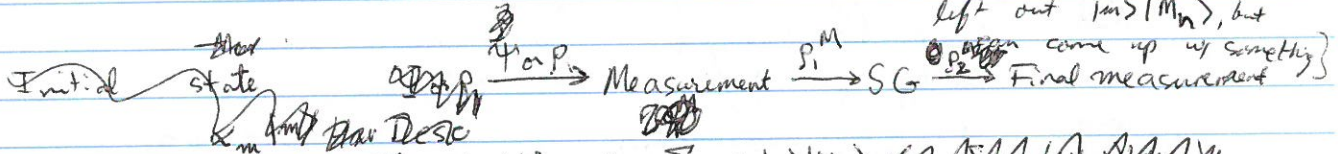
Shouldn't Schrödinger's eq. describe electron + apparatus?

How far does quantum mechanics extend?

Unitary evolution of you ~~and the system~~ consider entire system $H = H_e \oplus H_{Apparatus}$

$$U_M : |m\rangle |S\rangle \rightarrow |m\rangle |M_m\rangle$$

Haven't given the full unitary transformation because wavefunction span left out $|m\rangle |M_m\rangle$, but ρ_e come up w/ something



The final measurement measures only the state of the electron, so it is determined by the reduced density matrix ρ_e .

$$\rho_e = \sum_m \alpha_m |m\rangle \langle m| U^\dagger |M_m\rangle \langle M_m| U = \sum_m |\alpha_m|^2 U |m\rangle \langle m| U^\dagger$$

Cross terms have disappeared; state is impure, so get no interference between the two trajectories.

Does wave function really collapse? It is still $\sum_m \alpha_m |m\rangle |M_m\rangle$.

If you could apply U_M^{-1} , you'd see the interference again. But the measurement apparatus!

Difficult for macroscopic measurement apparatus!

See Many Perspectives on Schrodinger's cat: Cat is dead and alive, but it's also both

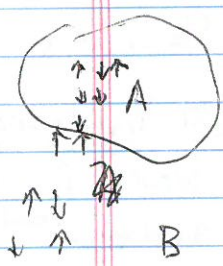
$$\frac{1}{\sqrt{2}} (|Dead Cat\rangle |Sad Me\rangle + |Alive Cat\rangle |Happy Me\rangle)$$

But I'm not aware of the other case - my belief always agrees with the state of the cat.

Zurek

Another topic: Area Law & Black Hole Entropy. Zero temperature system

J. Mark Srednicki; PRL 71,6 Bombelli et al PRD 34,373 1986



$$S_A = S_B \text{ at zero temperature (as proved since net state = pure state)}$$

Suggests entanglement

So S is not proportional to the volume of the region!

Possible that S is proportional to the area since A & B have the same area.

Density matrix of Measurement decoheres the electron's density matrix.

$$\rho_1 = \begin{pmatrix} |\alpha_1|^2 & \alpha_1 \alpha_2^* \\ \alpha_2 \alpha_1^* & |\alpha_2|^2 \end{pmatrix}$$

$$\rho_1^M = \begin{pmatrix} |\alpha_1|^2 & 0 \\ 0 & |\alpha_2|^2 \end{pmatrix}$$

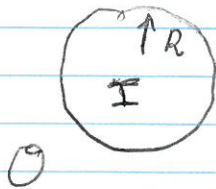
In general M(S) = $\sum_k |\psi_k\rangle \langle \psi_k|$ where k, k' are measured states. $\rho_2 = U \rho_1^M U^\dagger$

$$\rho_2 = U \rho_1^M U^\dagger$$

Short Range Correlations Explain Area Law

Justification: short-range correlations (e.g. spins) on opposite sides of interface.

Similar to Black Holes:



Nothing can escape from BH. Black Holes supposedly have a Hawking temperature. So exterior should be described by $\rho = \text{tr}_I |\Psi\rangle\langle\Psi|$. Even if universe is in a pure state the observable particles would seem to be in a thermal state.

So makes sense that

$$S_{BH} = \left(\frac{c^3 \pi R^2}{G \hbar} \right) \text{ is proportional to the area}$$

What about Finite Temperature Systems?

How can you explain the entropy of a finite temperature system? It is proportional to the volume.

Ergodicity Principle

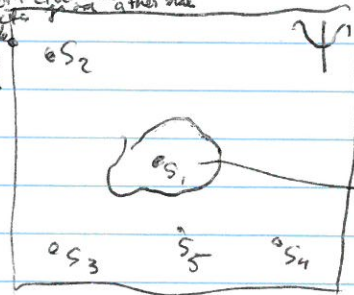
A single energy eigenstate of an entire system has the correlations one expects from another.

Refs

Pal, Haack arXiv 1003.2613 and refs 13-16 in it.

(e.g. Tasaki PRL 80, 1373)

Maybe system isn't described by a pure state. Schrodinger people have argued that it can be described that way. Partial entanglement entropy? Entropy = Entanglement Entropy?



Don't need a heat bath. If system \gg subsystem, it is its own heat bath, so a pure state should work. entropy be proportional to volume?

Paradox of Correlations

- 1) Must be true only when subsystem is much smaller than whole.
- 2) There must be very long-range correlations! I thought at higher temps, correlations disappear! But the correlations are very complicated: e.g., the 5 spins S_1, S_2, S_3, S_4, S_5 are spin down (or have an eigenvalue of +1) under the operators $S_1 S_2, S_2 S_3, S_3 S_4, S_4 S_5$.

Small groups of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

of the same species

Information that system A gives about B:

$$I(A, B) = S(A) + S(B) - S(AB)$$

Example: If my roommate & I have a red bag & a blue bag which we ~~share~~ take turns using, the possibilities are

1:	I have	Roommate has	
	Red bag	Blue bag	$S(A) = \ln 2$
			$S(B) = \ln 2$
2:	I have	Roommate has	
	blue bag	red bag	$S(AB) = \ln 2$
	\uparrow A	\uparrow B	So $I(A, B) = \ln 2$

~~In a classical sy~~

Entanglement

pure state $(|\downarrow_A \uparrow_B\rangle - |\uparrow_A \downarrow_B\rangle) / \sqrt{2}$. Entropy = $\ln 2$

The reduced density matrix for A is

$$\frac{1}{2} (|\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow|)$$

which doesn't describe a definite state.

$$\text{Entropy}(A) = \ln 2$$

For classical systems

$$S(A) \leq S(AB) \text{ always, but for}$$

quantum systems -

When a system is pure: $S(A) = S(B) = -\sum p_i \ln p_i$

and all the fluctuations in A are due to correlations with B.

So $S(A)$ is a measure of entanglement.

$$I(A, B) = 2S(A)$$

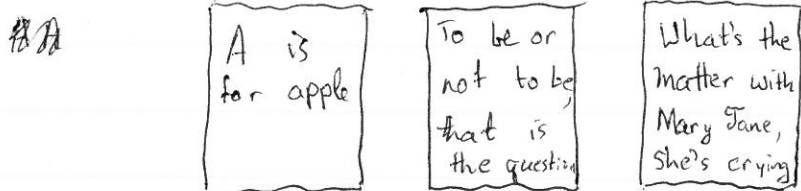
But information isn't the same as entanglement (consider the example of roommates with bags)

Entanglement is what ~~that~~ means that there ~~are~~ ^{are} some ~~are~~ extra correlations that aren't possible quantum mechanically.

Relationship between Entropy & Information

~~A better~~

Entropy is defined by listing the possible states



If there are many possibilities, S is big, and there is also a lot you can learn.

Example: Compare ~~adventure stories~~ to ~~the stories in adventure stories~~ ~~the hero always~~

~~good and boring books~~
~~Reading books always have the same story~~
~~a person gets trapped somewhere~~
~~with a dam about to break a dam and~~
 the stock market articles to the arts articles in a Newspaper:

Stock market articles:

~~Today the stock market went up because (..) announced a new (...)~~

Today the stock market went up increasing the president's popularity and the chance that ~~the Democrat~~ he will be able to ~~save~~ the United States reform the United States.

(1)

Entropy = $\ln 2$

~~Today the stock market went down because (... turns out to cause (...))~~

Today the stock market went down increasing the chance that the United States will be ripped up by rioting

(2)

5/1/18

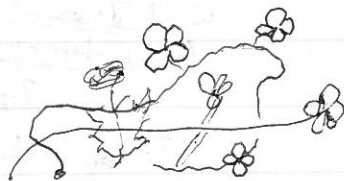
Information ~~that~~ that one set of atoms (A) knows about another set (B) =

$$I(A, B) = S(A) + S(B) - S(AB)$$

~~Then~~ Why is entropy related to information?

~~From the literature~~

A pile of weeds seems to have a lot of entropy but not much information:



~~But~~ one says it does have a lot of information in it because that if

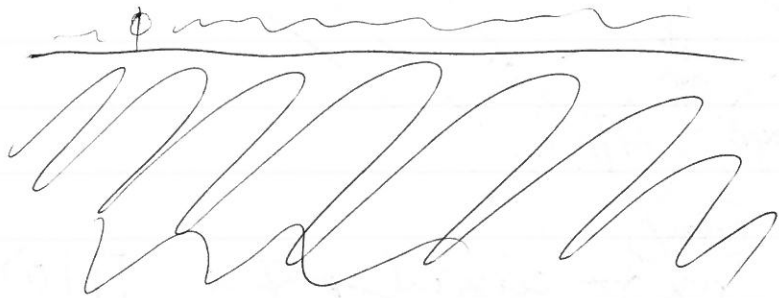
you look through them, there is a lot to learn (which types of flowers there are and how they are tangled up & whether their roots are attached or not).

Second Day

Calculating entanglement Hamiltonians

1. Horizon argument

2. $S = \int \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla\phi)^2 - m^2 \phi^2 dx dt$



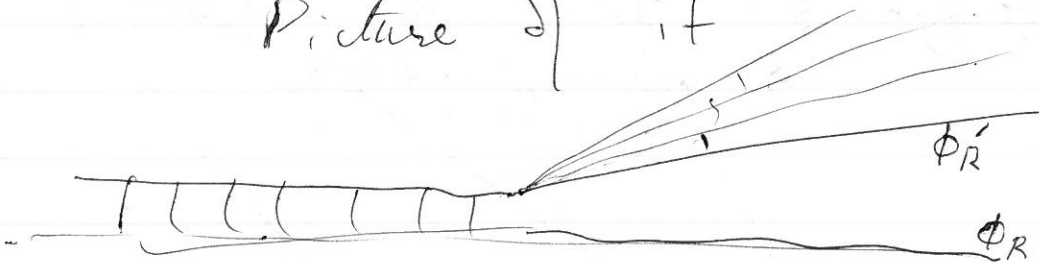
$$GS = \Psi[\phi(x)]$$

$$= \int_{\tau < 0} e^{iS[\phi(x, \tau)]} \Delta\phi$$

$\phi(x, 0) = \phi(x)$

Density matrix $\rho_{\phi_R \phi_R} = \sum_{\phi_L} \Psi^*(\phi_L, \phi_R) \Psi(\phi_L, \phi_R)$

Picture of it



Integrate $e^{-S[\phi]}$ in this region. Want to write it as $e^{-H_{ent} = (1 - \frac{H_{ent}}{n})}$

Do it one piece at a time.

n-pieces:

$$\text{Diagram} = 1 - \frac{H_{ent}}{n}$$

Why is Lorentz-symmetry important?

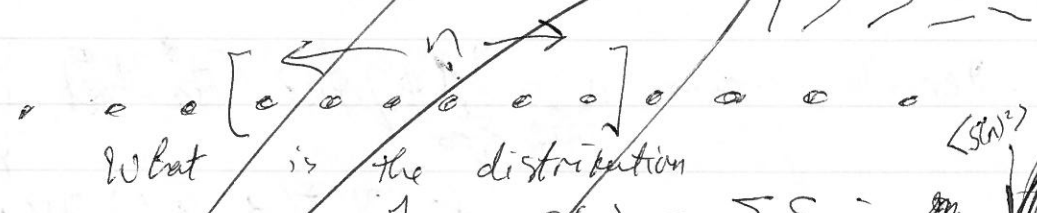
Entanglement Hamiltonian & Particle Number / Spin fluctuations

$$\sum \cos \theta S_i \cdot S_{i+1} + \sin \theta (S_i \cdot S_{i+1})^2$$

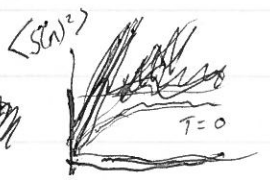


classical picture $|10\rangle$

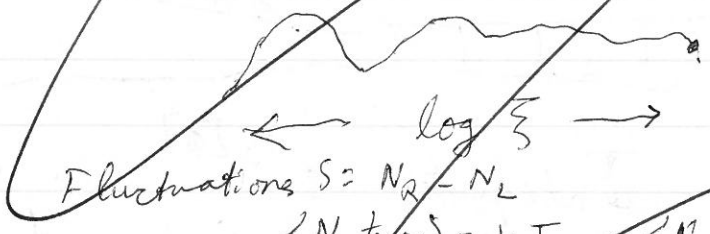
$\hat{n} \parallel \hat{z}$ axis



What is the distribution of $S(n) = \sum S_z \hat{z}_i$ or the variance?



Hamiltonian $S = \int dx p_n^2 + \partial_x \hat{n}(x)^2$
 \Rightarrow physical system entanglement
 Approximate to 2D



$\langle N_R \hbar \omega \rangle = kT$ $\langle N_R \rangle = \frac{kT}{\hbar \omega} \sim \log \xi$
 Variance $\sim (\log \xi)^2$

Exactly right answer from $\omega = \frac{2\pi c (\nu - \frac{1}{2})}{\log \xi}$
 speed of light

$$S = \int \frac{d\phi}{dT} = \frac{2\pi}{3V} \int_0^\infty \frac{vk}{e^{vk/T} - 1} = \frac{1}{\pi} \frac{T^2}{v} \int_0^\infty \frac{x^2}{e^x - 1} = \frac{\pi}{6} \frac{T^2}{v}$$

Small piece = $\int e^{-S[\phi]}$
 $\phi(r, 0 \leq \theta < 2\pi)$

$$Z = \int e^{-\int \left[\frac{1}{2} \pi(r)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial \theta} \right)^2 + \frac{1}{2} m^2 \phi^2 \right] r dr d\theta}$$

Now think of θ as the time, since you are integrating between $\theta = \theta_i$ and $\theta = \theta_f$

This is then $\left(\frac{\partial \phi}{\partial \theta} \right)^2 = \text{K.E.}$ $\frac{\partial \phi}{\partial r} = \text{rest} = \text{potential energy}$

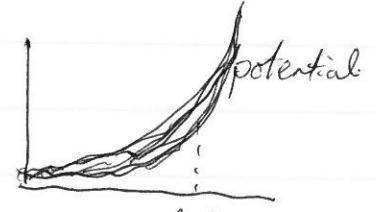
So $\int_{\text{integral}} = e^{-\frac{2\pi}{L} \int \left[\frac{1}{2} r \pi(r)^2 + \frac{1}{2} r \left(\frac{\partial \phi}{\partial r} \right)^2 + \frac{1}{2} m^2 r \phi^2 \right]}$

Number 1: Local.
 Number 2: Low energy is restricted to boundary

$$H = 2\pi \int r \left[\frac{1}{2} \pi(r)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial r} \right)^2 + m^2 \phi^2 \right] dr$$

Even clearer if $r = e^u$; $u = \ln r$

$$H = 2\pi \int_{\ln a}^\infty \left[\frac{1}{2} \pi(u)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial u} \right)^2 + \frac{1}{2} m^2 e^{2u} \phi^2 \right] du$$



This seems to be a hard wall. finite $u = \ln(1/m) = \ln 5$ finite string.

E.g.

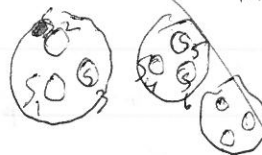
Entanglement entropy = thermal entropy of this entanglement Hamiltonian = thermal entropy of mass $\frac{\pi T}{3v} \cdot L = \frac{\pi}{3 \cdot 2\pi} \ln 3 = \frac{1}{6} \ln 3$

For any CFT near critical point: variables because u is taken $H_{\text{ent}} = 2\pi H_{\text{phys}}$ in u \rightarrow ~~XXXXXXXXXX~~ in a conformal theory

IF ν_{spn} So always $S_{\text{ent}} = \frac{\nu}{2\pi} \left(\frac{S_{\text{physical}}}{T} \right) \log 2$

Tensor network States:

How to see that states of the entanglement spectrum differ only at the edges.
 (and also useful for numerics as in Steven White's Density Matrix Renormalization Group, which Steven White will apply to Spin liquids)
 Example



Atoms made of 3 electrons. Eigenstates of the atoms don't have well-defined states of the electrons, but the pairs of adjacent electrons may want to have anti-aligned electrons.

To describe this wave-function, take

$$+6 \sum_{s_3, s_4} \psi_{i_1}(s_1, s_2, s_3) \psi_{i_2}(s_4, s_5, s_6)$$

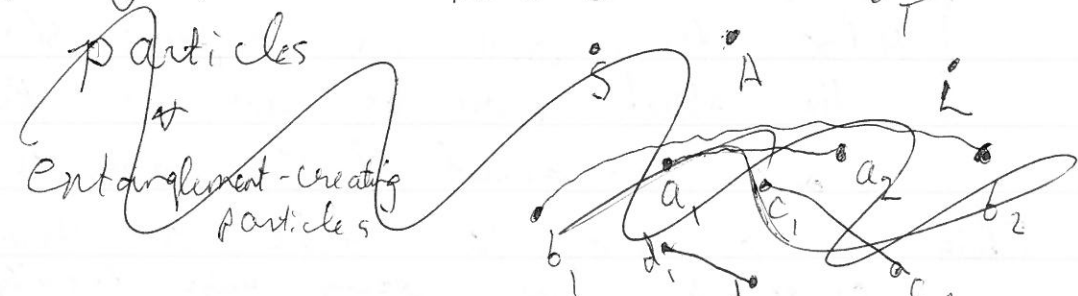
gives the weight of having $s_3 + s_4$ opposite in this state

Way to make things entangled:

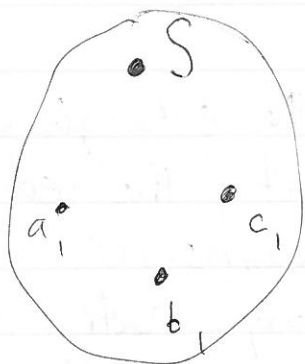
Start with a wave-function for the states



$$\sum_i c_i |i\rangle \rightarrow \sum_i c_i F_i |i\rangle$$



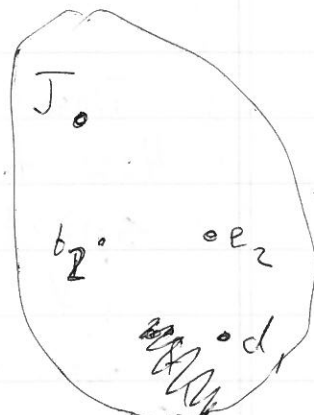
particles
 Entanglement-creating particles



Start with these particles in an entangled state



and also these



Now look at the particles labelled $c_1 + c_2$ (or any other pair)

Suppose they are the same type of particle, i.e. Hilbert spaces have the same dimension (d_c)

Now measure whether they ~~have the same~~ state are d_c in the state

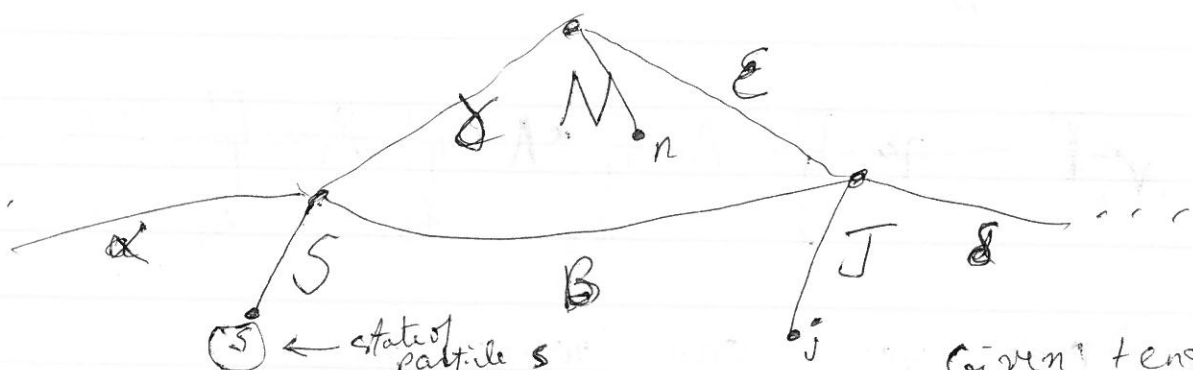
$$\sum_{j=1}^{d_c} |j\rangle_{c_1} |j\rangle_{c_2} \frac{1}{\sqrt{d_c}}$$

If the answer is yes in every case, then this ~~wave-function~~ causes S, N, J, \dots to become entangled with each other.

[Also they are in a pure state because the other particles are in their pure states on account of the answer being yes]

This is a way to make a many-body state by just doing things locally, although

Maybe an easier way to think about this:



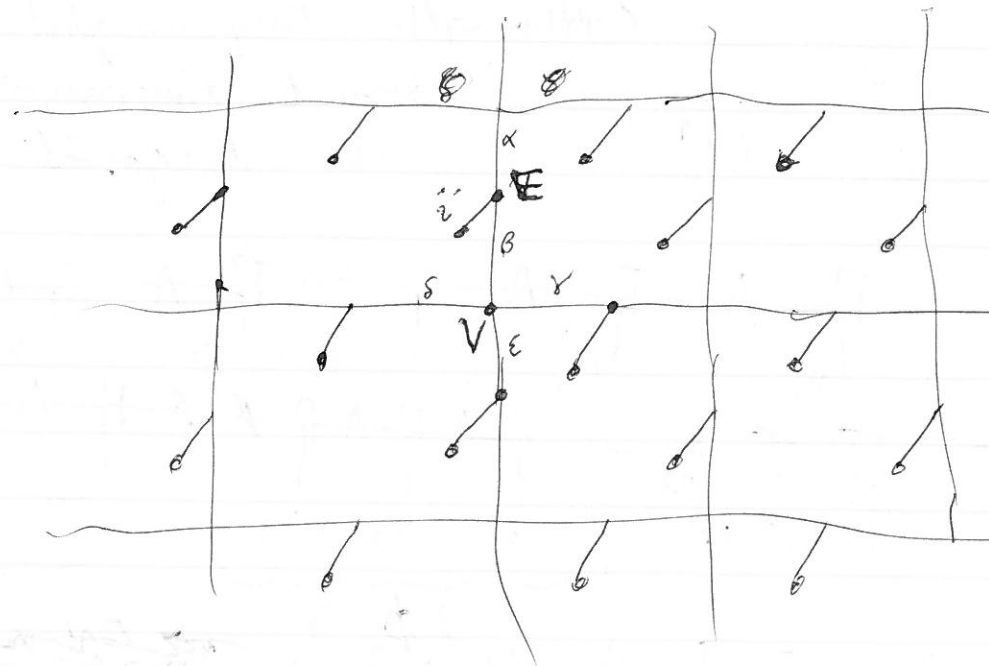
We want a wave-function $\Psi(s, n, j, \dots)$. Given tensors $S_{s\alpha\beta\gamma}$, $N_{n\gamma\epsilon}$, etc. for each vertex

Label each vertex

Write s, n, j on the ~~lower~~ twigs growing out. Replace each vertex by the tensor & sum over the variables on each edge

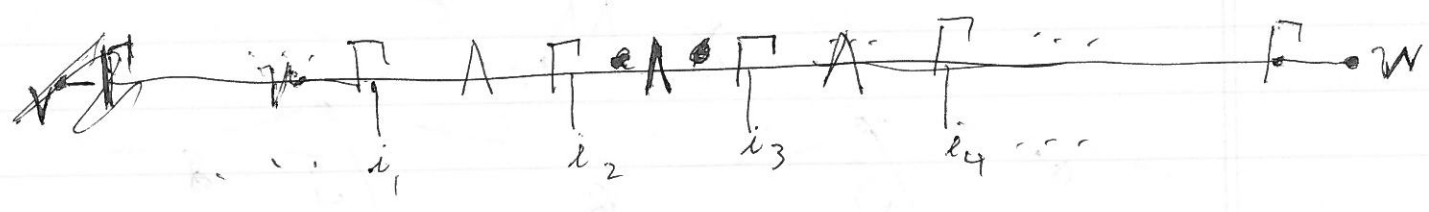
$$\Psi(s, n, j, \dots) = \sum_{\beta, \gamma, \epsilon, \dots} S_{s\alpha\beta\gamma} N_{n\gamma\epsilon} J_{j\beta\delta\epsilon} \dots$$

Example: the toric code model.



$i =$ state of spin on the edge = up or down
 $E_{i\alpha\beta} = 1$ if $i_{\alpha} = i_{\beta}$
 0 if not
 $V_{\beta\gamma\delta\epsilon} = 1$ if an even number of $\beta, \gamma, \delta, \epsilon = \downarrow$
 0 if not

1D wave-functions can be represented by simple tensor states



Note: you can write this as $v^T \cdot \Pi_{i_1} \Lambda \Pi_{i_2} \Lambda \Pi_{i_3} \Lambda \Pi_{i_4} \dots w$ where $\Pi_{i \alpha \beta}$ matrix is called a matrix product in $\alpha + \beta$ state.

1. Notice you can represent an infinite state w/ translational symmetry by just one set of matrices,

e.g. $\Pi_{\uparrow}, \Pi_{\downarrow}, \Lambda$ for a chain of spin 1/2 particles (although they might have infinite dimensions!)

2. If Λ is diagonal:

$$\begin{aligned}
 \sum_i & \text{---} \Pi_{i_1} \Lambda \Pi_{i_2} \Lambda \Pi_{i_3} \Lambda \Pi_{i_4} \Lambda \Pi_{i_5} \Lambda \Pi_{i_6} \text{---} \dots i_1 i_2 \dots i_6 \\
 &= \sum_i \sum_{\alpha, \beta} \lambda_{\alpha\beta} \text{---} \underbrace{\Pi_{i_1} \Lambda \Pi_{i_2} \Lambda \Pi_{i_3}}_{|\phi_\alpha\rangle} \underbrace{\Lambda \Pi_{i_4} \Lambda \Pi_{i_5} \Lambda \Pi_{i_6}}_{|\chi_\beta\rangle} \text{---} \\
 &= \sum_i \sum_{\alpha, \beta} \lambda_{\alpha\beta} |\phi_\alpha\rangle |\chi_\beta\rangle
 \end{aligned}$$

Entanglement can be found if Λ is diagonal + abs

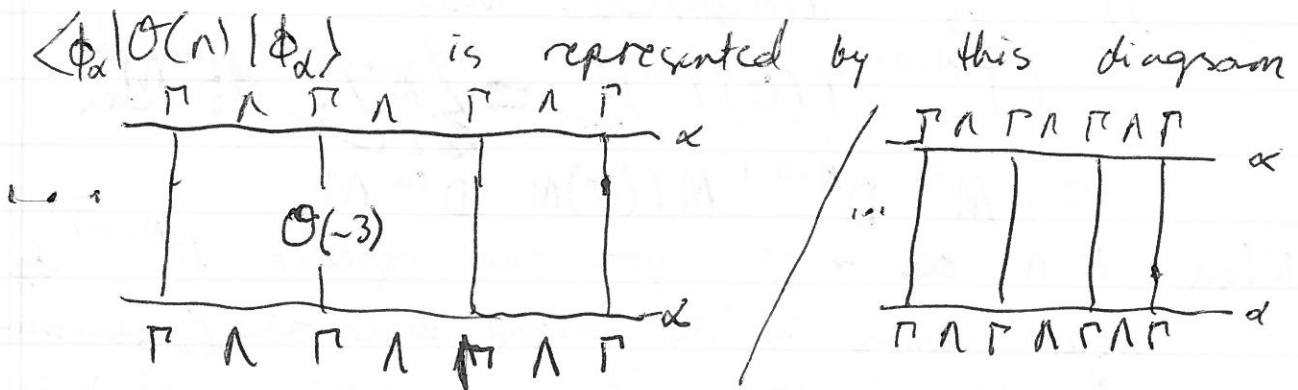


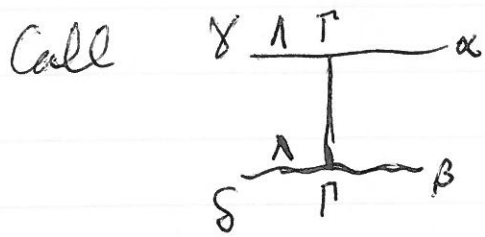
If the $\Lambda_{\alpha\beta}$ matrix is diagonal & the Γ 's satisfy special properties ("a canonical matrix product state") then you can show that the ϕ 's and χ 's are orthonormal so this is just the Schmidt decomposition.

Now let us see that the "low-energy" degrees of freedom in the entanglement spectrum are near the boundary.

Ex. this means that the only differences between different Schmidt states are near the boundary. That is

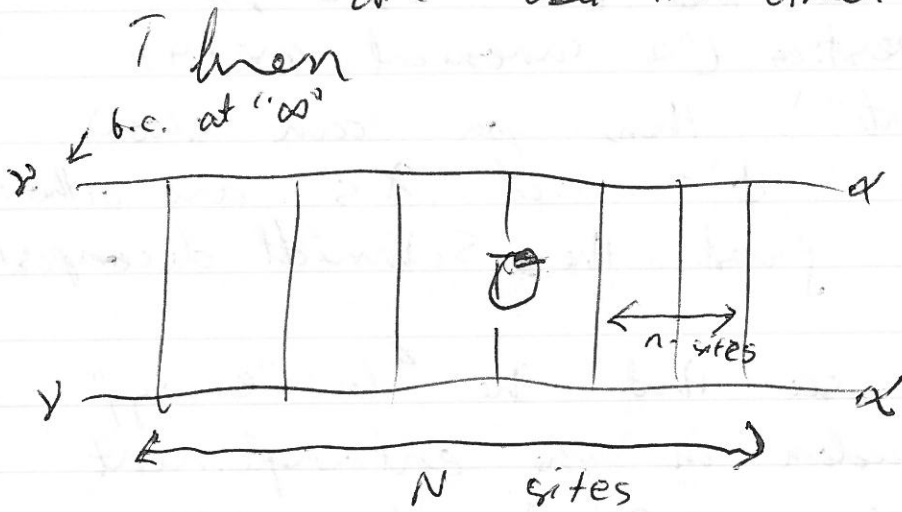
$$\langle \phi_\alpha | \mathcal{O}(n) | \phi_\alpha \rangle \rightarrow \text{something independent of } \alpha \text{ when } n \rightarrow -\infty$$





" T $\delta\delta$ "
 $\alpha\beta$

you can think of this as a matrix where $\alpha + \beta$ are combined into one index $\gamma + \delta$ are the other index.



$$= (T^{N-n-1} T(\theta) T^n)_{\alpha\alpha}^{\gamma\gamma} \quad \text{where } T(\theta) = \begin{array}{c} \frac{1 \quad T}{T} \\ \theta \\ \frac{T}{1 \quad T} \end{array}$$

Now diagonalize T : $T = M D M^{-1}$ where D is diagonal, thus

$$(T^{N-n-1} T(\theta) T^n)_{\alpha\alpha}^{\gamma\gamma} \Rightarrow \sum_i (M^{-1})_{\gamma\gamma}^i d_i^{N-n-1} M_{\alpha\alpha}^i$$

$$= M^{-1} D^{N-n-1} M T(\theta) M^{-1} D^n M$$

When $N-n \rightarrow \infty$ you can replace D^{N-n-1} by $d_{\max}^{N-n-1} \begin{pmatrix} 1 & & \\ & 0 & \\ & & \dots \end{pmatrix}$

where d_{\max} is its largest eigenvalue (assuming the largest eigenvalue is unique, which is true for physical states).

So $\langle \theta(n) \rangle \approx$

$$\frac{d_{\max}^{N-n-1} (M^{-1})_{\gamma\gamma}^1 M_{\alpha\beta}^1 T(\theta)_{\delta\delta}^1 (M^{-1})_{\delta\delta}^1 M_{\alpha\alpha}^1}{d_{\max}^N (M^{-1})_{\gamma\gamma}^1 M_{\alpha\alpha}^1}$$

$$\frac{d_{\max}^{N-n-1} (M^{-1})^{\nu\nu} (M^T(\theta)M^{-1}D^nM)_{\alpha\alpha}^i}{d_{\max}^N (M^{-1})^{\nu\nu} M_{\alpha\alpha}^i}$$

$$= \frac{1}{d_{\max}^{n+1}} \frac{(M^T(\theta)M^{-1}D^nM)_{\alpha\alpha}^i}{M_{\alpha\alpha}^i}$$

Note that the boundary condition at " ∞ " does not matter.

This is equal to

$$\langle \theta(-n) \rangle = \sum_i [M^T(\theta)M^{-1}]_i^i \frac{d_i^n}{d_{\max}^{n+1}} \frac{M_{\alpha\alpha}^i}{M_{\alpha\alpha}^i}$$

So the correlation expectation value is the sum of some terms that each decay exponentially, except the one with $d_i = d_{\max}$.

So

$$\langle \theta(-n) \rangle = \text{const} + \text{const} e^{-\left(\ln \frac{d_{\max}}{d'}\right)n}$$

where d' is the next largest eigenvalue after d_{\max} .

[Actually there can be many eigenvalues close to d_{\max} , which gives an extra factor $\left(\frac{d_{\max}}{d'}\right)^n$ in front of the second term, when you sum them all together]

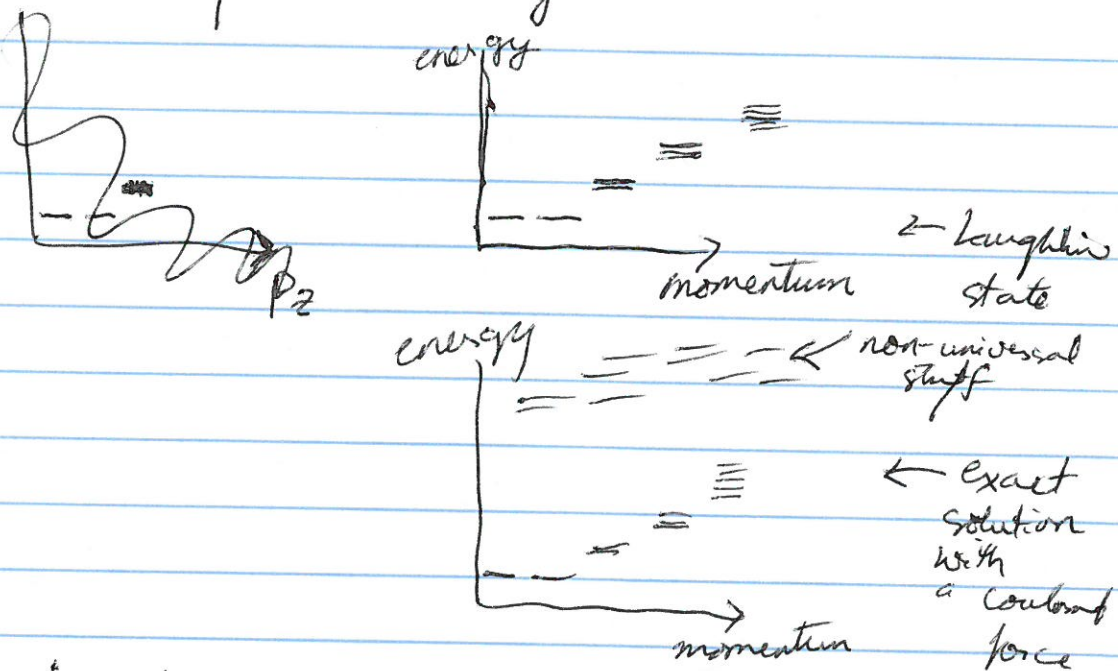
Why is the Entanglement Spectrum useful for topological phases?

A topological phase (like a topological insulator or the toric code state or quantum Hall state) is ~~characterized~~ characterized by special excitations: e.g. chiral modes on the edge or anyons. After a phase transition to a normal phase, these excitations disappear, which suggests ~~But those are~~ all properties of the excited state. On the boundary between a normal phase & a topological phase, the correlation function of the ground state goes to infinity, so it sounds like something about the ground state changes. ~~even if~~ What is it?

1. It is not an order parameter because there is no long-range correlation in these phases.

~~2. Maybe it is something about~~
2. The only non-trivial thing about the ground states is therefore how the different parts of the system are entangled with one another (idea emphasized by Wen)

3. How can you measure this?
 Li + Haldane suggested looking at the spectrum of the entanglement Hamiltonian. They tried this out for the quantum Hall effect. For example they found (numerically) that ~~any two~~ all the states in one phase have similar entanglement spectra at low "energies" + they match the physical spectrum of the edge.

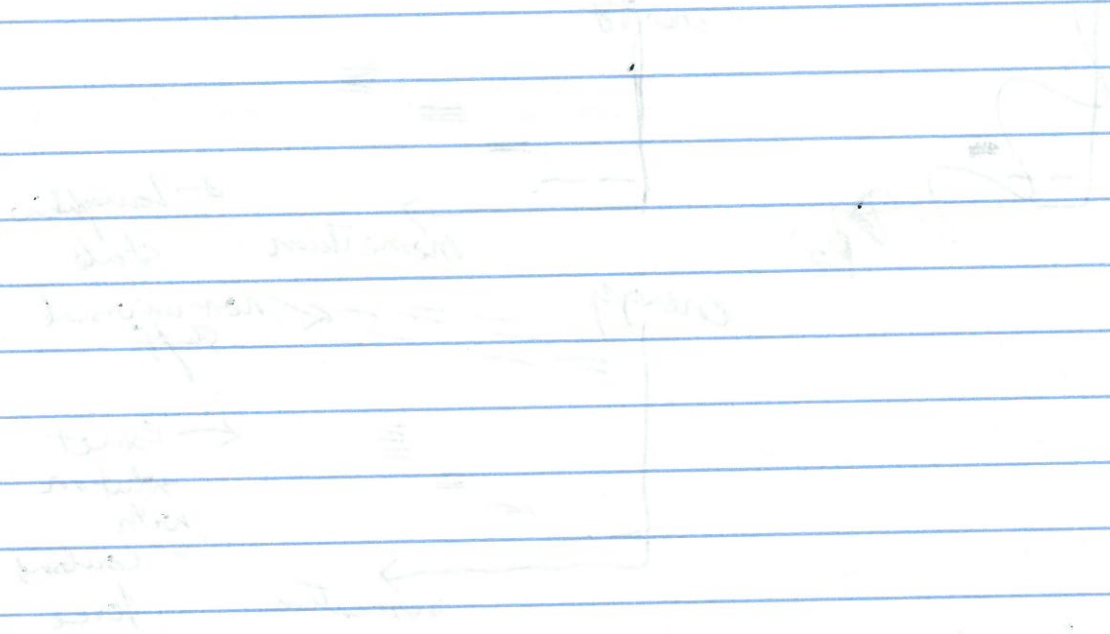


~~This has to~~

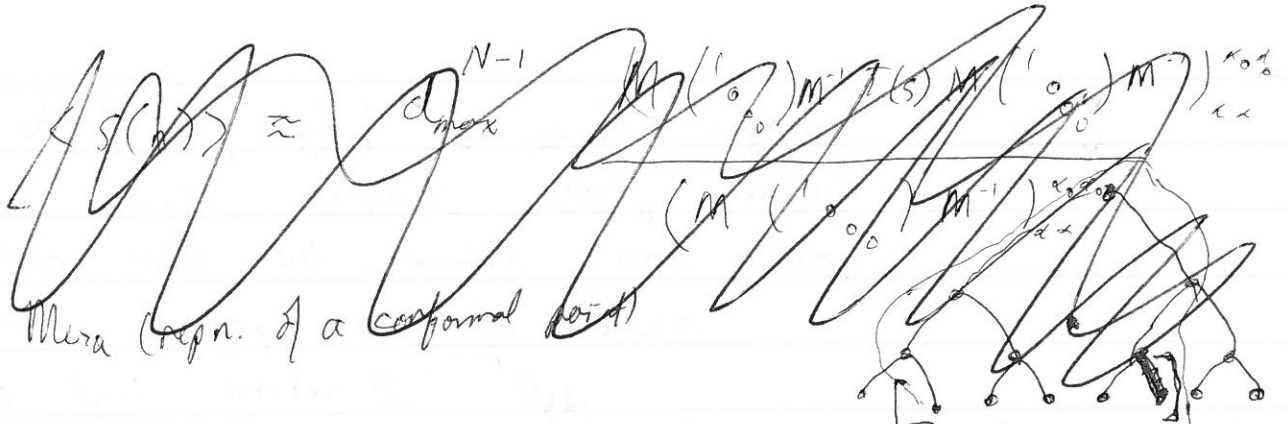
~~You can use it~~

You can ~~use~~ ^{generalize} this idea to understand check whether the ground state of ~~is~~ a certain spin model is a topological phase of a certain type + to recognize the statistical phases of ~~angles~~ ^{state} of the anyons using just the ground state wave functions. ~~state~~

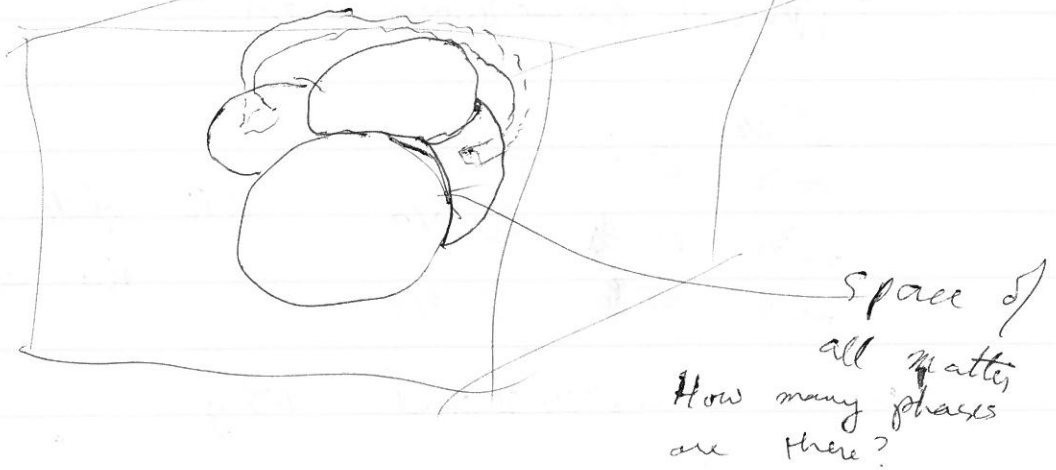
The first step in the process of...
 is to identify the...
 and then...
 the...
 and...
 the...
 and...
 the...
 and...



The...
 and...
 the...
 and...
 the...
 and...
 the...
 and...



Classifying Spin chains Topological Phases.



In 1D you can show using Fractionalized States (fractional edge modes strongly under the symmetries) long-ranged-Entangled states Anyons

In 1D, there are no long-range-entangled states. you need symmetries for the fractional states.

Strong proof with tensor states can combine the T 's from

Method: Entanglement Spectroscopy (Li + Halperin's idea)

Roughly why it works.

Spin chains: $S = \text{integers}$

$H = J \vec{S}_i \cdot \vec{S}_{i+1}$

← edge $S/2$

Nerves has a broken symmetry in bulk $S=0$

$S=1$

$S=2$

$S=3$

$U S_{iz}^2 + J S_i \cdot S_{i+1}$

← ordinary phase

← Haldane

0 same phase

$1/2$

1

$3/2$ same phase

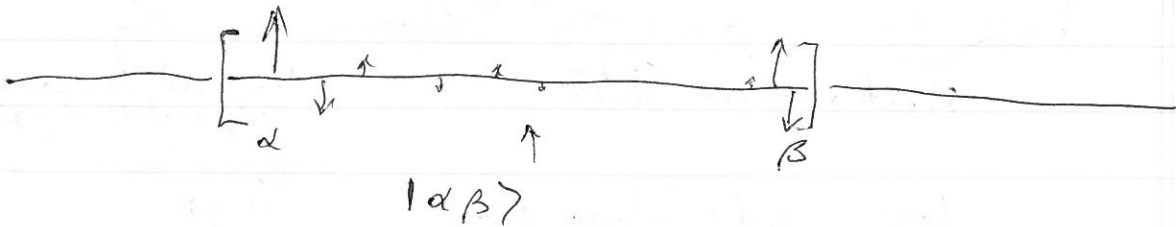
discontinuity in spin

[You can imagine these all being part of the same phase diagram 'cause you can assume the sites are made of 10 spin 1 atoms with diff interactions that couple them into spin 0, 1, 2, ...]

Typical entanglement spectra

$\equiv 5/2$	$\equiv 2$	$\equiv 1/2$	All states have the same symmetry!
$\equiv 1/2$	$\equiv 3/2$	$\equiv 5/2$	
$\equiv 3/2$	$\equiv 1/2$	$\equiv 3/2$	
$\equiv 1/2$	$\equiv 3/2$	$\equiv 1/2$	

Let's understand why



Interior is same as bulk state so it is symmetrical.

$$R |\alpha\beta\rangle = U_i U_j U_k |\alpha\beta\rangle$$

$$R_i R_j = R_k \Rightarrow U_i U_j = e^{i\phi_{ij}} U_k$$

called a proj. repn.

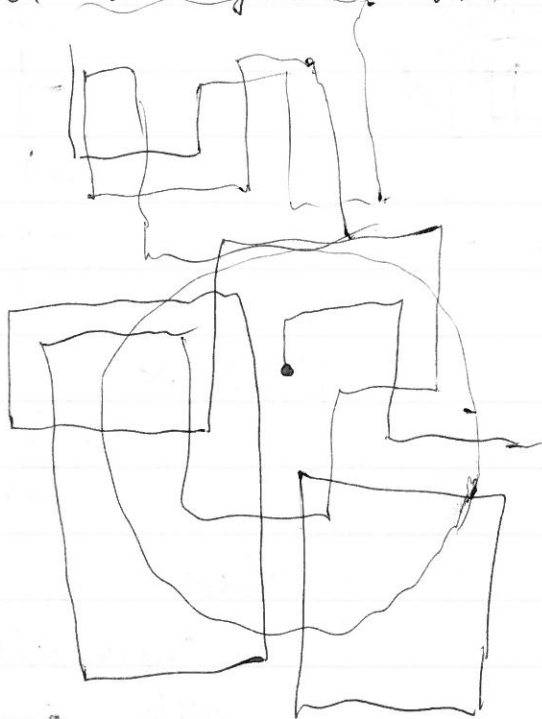
Example of 1/2 integer spin.

Example with less symmetry $\sum J_x S_{ix} S_{i+1,x} + J_y S_{iy} S_{i+1,y}$

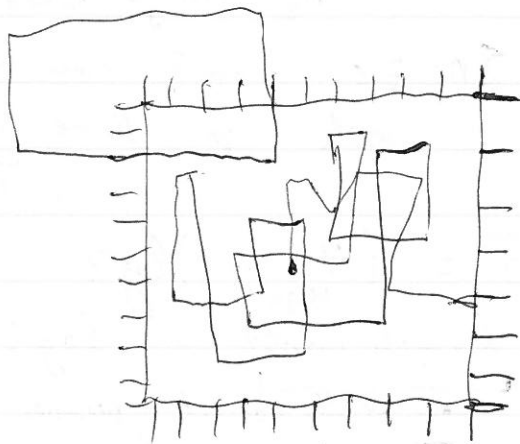
Didn't Get to this part

pg 9

Topological Entanglement entropy. Can you tell whether a system has anyon-excited states by looking at ground states?



Gauss's Law



$$\prod_{\text{loop}} \sigma_z = (-1)^{\# \text{ of } E \text{ anyons inside}}$$

Remember $W_M = \prod_a \sigma_z$ is an operator that moves an M particle from a to b .

So phase factor = $(-1)^{\# \text{ of } E\text{-anyons inside}}$

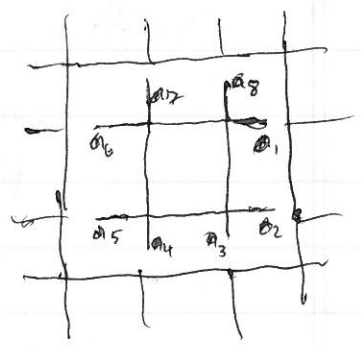
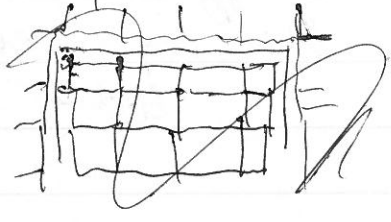
Now: It must always be like this in any anyon theory.

[Focus on abelian case] W_i moves anyon of type i

$W_i(\text{loop}) |GS\rangle = |GS\rangle$ but W_i is a non-trivial operator because it is not always the identity.

Note W_E creates a loop, so there must be fluctuating loops everywhere for this condition to be satisfied.

ES of toric code model



$$|\Psi\rangle \approx \sum_{\substack{n_i \equiv 0 \\ 2 \text{ or } 1 \\ \text{mod } 2}} |n_1, \dots, n_8\rangle_{\text{inside}} |n_1, \dots, n_8\rangle_{\text{outside}}$$

where $|n_1, \dots, n_8\rangle$ is the superposition of all states with ~~an odd~~ ~~or even~~ # of n_i strings mod 2 ending at i .

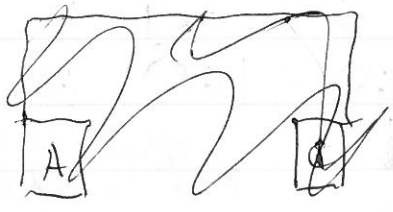
Now $\sum n_i \equiv 0 \pmod{2}$ since there are no anyons inside so there are actually only 2^7 independent variables terms.

$$S = (N_{\text{legs}} - 1) \log 2$$

Generally $S = \underbrace{\text{length-const}}_{\text{not really correct}}$ $-\log 2$.

$$= \sum_{\text{local contributions}} S_i - \log 2$$

because entropy could depend on slopes or be extra at corners which you can generalize further to more complicated regions

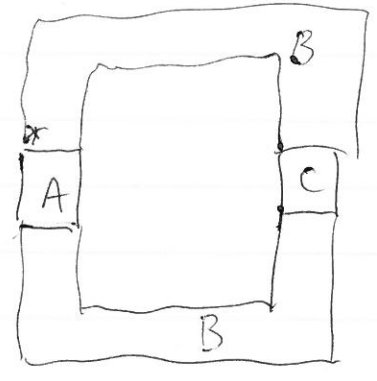


local



$$S = S_{\text{local}} - \log N_{\text{hd}}$$

Make local contributions cancel:



Consider

$$S_{AB} + S_{BC} - S_B - S_{ABC} \text{ (Levin + Wen)}$$

All local contributions cancel.

eg. ~~lower left corner~~

upper part of P appears in all 4 terms.
~~lower left~~ * has a different contribution

in $S_{AB} + S_B$, but

it ~~cancel~~ $S_{AB} + S_{ABC}$

and between S_{BC} and S_B .

This equals

$$-\log 2 - 1 - \log 2 - 1 + \log 2 \cdot 2 + \log \cdot 2 = 2 \log 2.$$

Interpretation: it is $S_{AB} - S_A - S_B + S$

$$[S_A + S_{BC} - S_{ABC}] - [S_A + S_B - S_{AB}]$$

$$= I(A, BC) - I(A, B) \text{ which is}$$

surprising to be non-zero. ~~cancel~~

Generalization to other phases:

This is always topological, i.e. independent of any local details.
~~It is also positive by strong subadditivity.~~

You can show that any region's entanglement entropy can be written in terms of

$$\text{equals } S_{\text{local}} \neq S_{\text{top}} - N \ln d$$

(it doesn't matter whether bdies are inside or outside of one another)

Apply this to Levin + Wen's case:

$$S_{AB} + S_{BC} - S_B - S_{ABC} = -2S_{\text{top}}$$

But this is ≥ 0 by strong subadditivity.

$$S_{\text{top}} \text{ is always } -\ln \sqrt{d_1^2 + d_2^2 + \dots + d_n^2}$$

← quantum dimension

e.g. for Fibonacci $S_{1,0} = -\ln \sqrt{5+15}$

17th
11/11

Mathematics: Introduction to Algebra

What is algebra? It is the study of mathematical symbols and the rules for manipulating these symbols. Algebra is a generalization of arithmetic.



Algebra is used in many fields, including science, engineering, and economics. It helps us solve problems involving unknown quantities.

One of the basic operations in algebra is solving equations. For example, if we have the equation $x + 5 = 10$, we can solve for x by subtracting 5 from both sides, giving us $x = 5$.

Another important concept in algebra is the use of variables. Variables are symbols that represent unknown values. They allow us to write general rules and solve a wide range of problems.

Algebra is a fundamental part of mathematics and is essential for understanding more advanced topics like calculus and physics.

By learning algebra, you are developing critical thinking and problem-solving skills that are valuable in many areas of life.