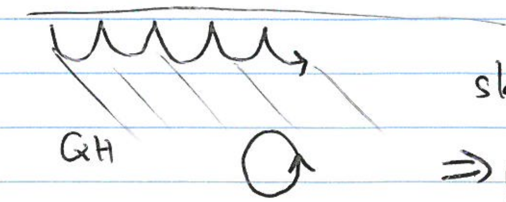
Edge States

Classical Picture

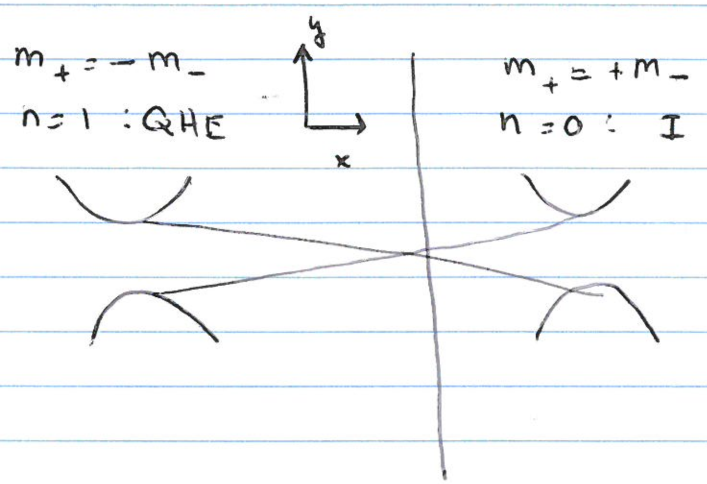


skipping orbit

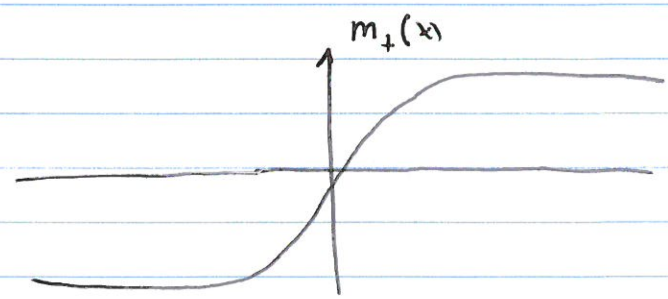
$\Rightarrow$ propagating state

"one way"

Topological boundary modes



band inversion

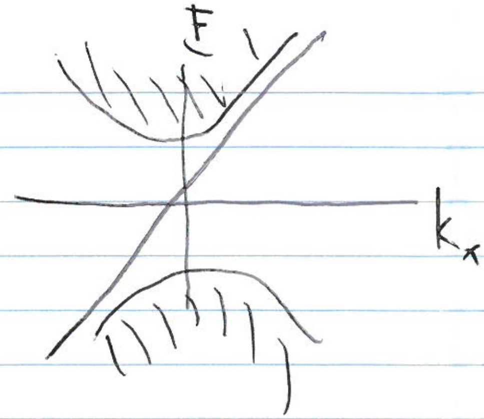


$$\mathcal{H} = v_F\left(-i\sigma_x \partial_x + k_y \sigma_y\right) + m_+(x)\,\sigma_z$$

Same as Jackiw-Rebbi for fixed k_y.

Zero mode $|\psi\rangle = e^{ik_y y}\, e^{-\int^{|x|} \frac{m_+(x)}{v}} \,|\sigma_y +\rangle$

$E|\psi\rangle = +v_F k_y |\psi\rangle$



Chiral Dirac fermion

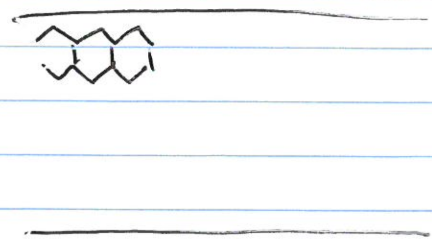
Other edge: $|\psi\rangle \propto |\sigma^y -\rangle$

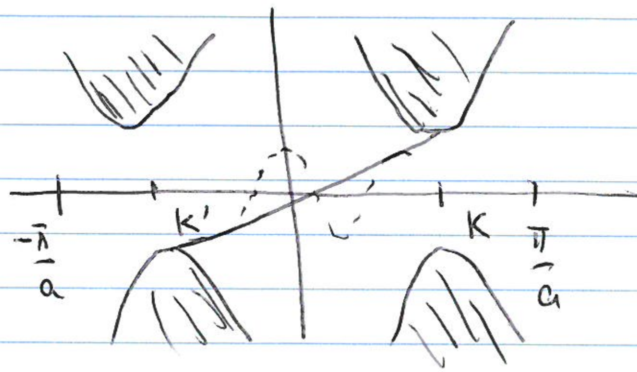
$E = -v_F k_y$

Chiral Dirac Fermion

1. One way: no choice but to go forward
2. Robust: Insensitive to disorder (no where to go)
   Impossible to localize
3. Impossible in purely 1D:
   Fermion doubling thm: What goes up must go down
   Evaded by spatially separating right & left movers.

Concrete model: Haldane model on a strip





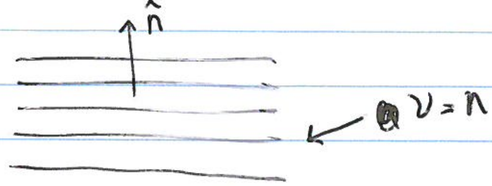
Chiral edge modes $\quad N_L - N_R$ is topo invariant

Bulk-Boundary Correspondence

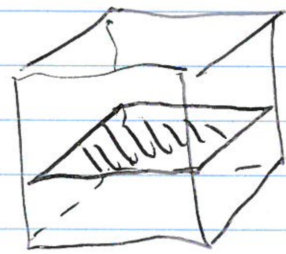
Boundary invariant		Δ Bulk invariant
$N_L - N_R$	$=$	Δn

Generalizations

1. 3D layered QH states (Halperin)



3D BZ



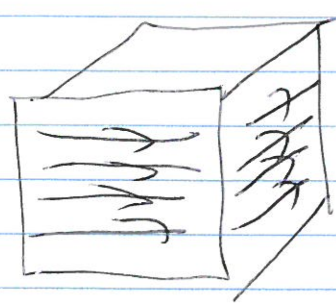
$$n_z = \frac{1}{2\pi}\int dk_x dk_y \, F(k_x, k_y, k_z)$$

indep of k_z

In general (n_x, n_y, n_z) define reciprocal lattice vector

$$G = \frac{2\pi}{a}(n_x, n_y, n_z)$$

Miller indices for lattice planes



Chiral surface states

$d=4$: 4D IQHE (Zhang, Hu '01)

$$A_{ij} = \langle u_i | \nabla_k | u_j \rangle \cdot dk$$ Non Abelian Berry Connection 1-form

$$F = dA + A \wedge A :$$ Non Abelian Berry Curvature 2-form

$$n = \frac{1}{8\pi^2} \int \mathrm{Tr}[F \wedge F] \in \mathbb{Z}$$ 2nd Chern Number Integral of 4 form over 4D BZ

Boundary States : 3+1D chiral Dirac Fermions (single Weyl point)

Higher D :

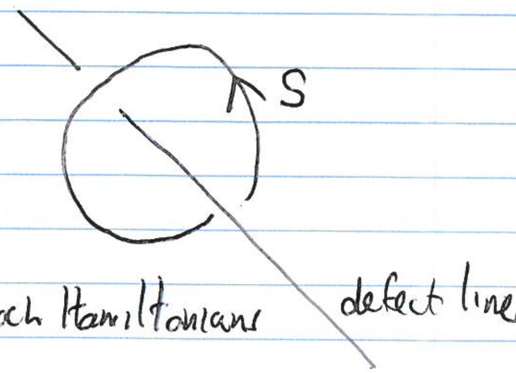
			d		
	1	2	3	4	......
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	
AIII	$\mathbb{Z}$	0	$\mathbb{Z}$	0	

"Bott Periodicity : $d \to d+2$"

Topological Defects

Imagine a Band Structure that varies slowly in real space.

$$H = H(\vec{k}, s)$$



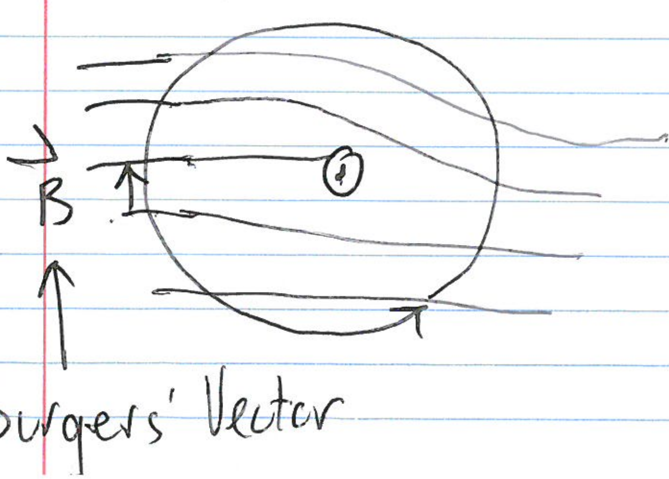
1 parameter family of 3D Bloch Hamiltonians

2nd Chern number: $$n = \frac{1}{8\pi^2}\int_{T^3 \times S^1} \mathrm{Tr}[F \wedge F]$$

Generalized bulk-boundary correspondence:

$n \leftrightarrows$ # chiral modes bound to defect line

Example 3D IQHE



$$n = \frac{1}{2\pi}\underset{\text{3D Chern\#}}{\vec{G}} \cdot \underset{\text{Burgers Vec.}}{\vec{B}}$$

Quantum Spin Hall Insulator

Energy Gaps in Graphene $\quad \mathcal{H} = v(\sigma^x \tau^z q_x + \sigma^y q_y) + V$

valley $\pm K$

1. $V = m_{CDW}\, \sigma^z$

$\Rightarrow I \quad$ (Break P)

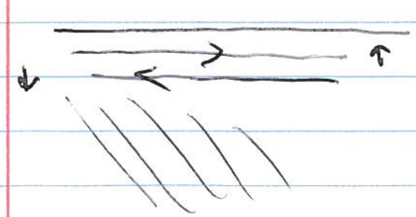
2. $V = m_H\, \sigma^z \tau^z$

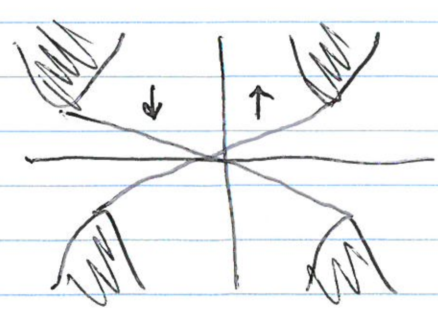
$\Rightarrow$ IQHE $\quad$ (Break T)

3. Intrinsic spin-orbit interaction

$V = m_{SO}\, \sigma^z \tau^z s^z \quad$ (Respects <u>all</u> symmetries)

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_\uparrow & 0 \\ 0 & \mathcal{H}_\downarrow \end{pmatrix} = \begin{pmatrix} H_{Haldane} & 0 \\ 0 & H^*_{Haldane} \end{pmatrix}$$





Is it an artifact of S_z conservation?

Time Reversal Symmetry $[\mathcal{H}, \Theta] = 0$

$$\Theta \psi = e^{i\pi S^y} \psi^*$$

Spin $\frac{1}{2}$: $\Theta \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \begin{pmatrix} \psi_\downarrow^* \\ -\psi_\uparrow^* \end{pmatrix}$

$\Theta^2 = -1$ (Same minus sign!)

Kramers' Thm:

For spin $\frac{1}{2}$ all states are at least 2-fold degenerate

Simple without spin-orbit, but non-trivial with spin-orbit.

Proof: If $|\chi\rangle$ is non degenerate, then

$$\Theta|\chi\rangle = c|\chi\rangle$$

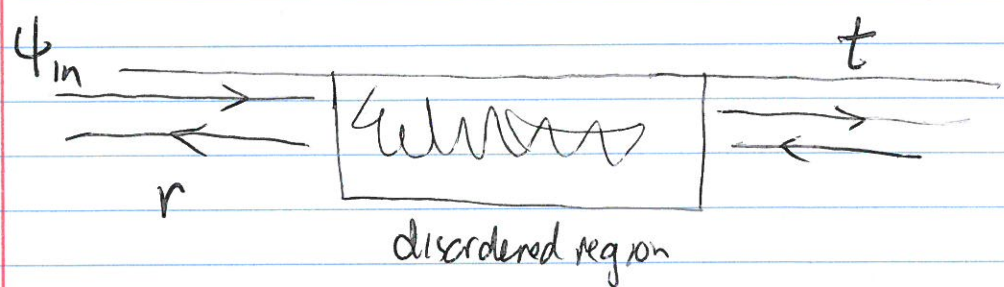
$$\Theta^2|\chi\rangle = c^* \Theta|\psi\rangle = |c|^2|\psi\rangle$$

$$\Theta^2 = |c|^2 \neq -1$$

Contradiction

Consequence for edge states

1. Crossing of edge state is protected
2. Absence of elastic backscattering
3. Absense of localization even for strong disorder



Under Θ $\quad r \rightarrow -r \implies r = 0$

$$|t| = 1$$

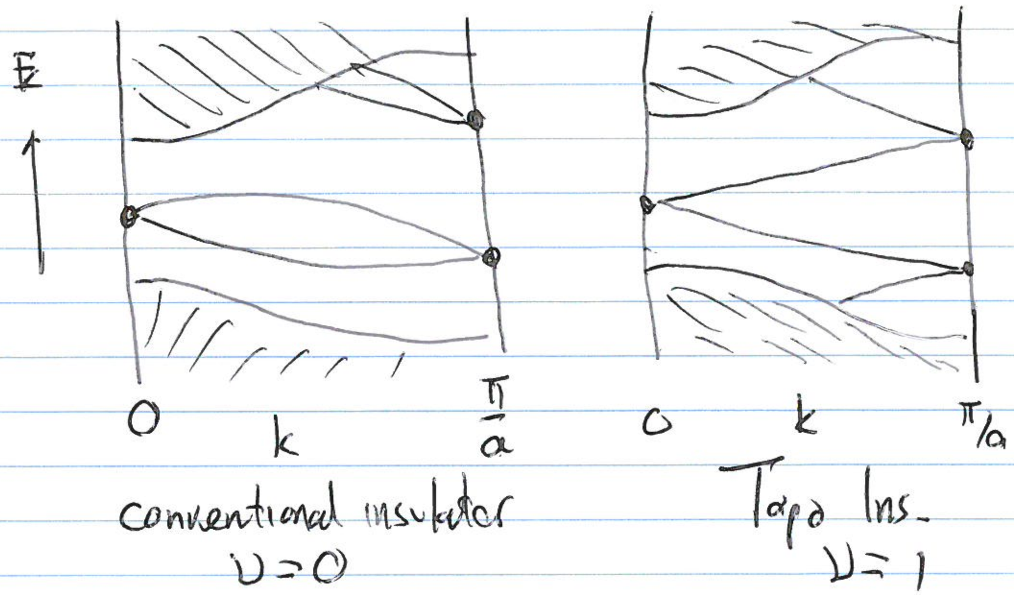
All eigenstates are extended, even for strong disorder.

What is the difference between QSHI and ordinary insulator?

- Chern number $n = 0$
- There is a new Z_2 invariant character $\nu = 0, 1$

$\mathcal{H}(k)$ s.t. $\mathcal{H}(-k) = \Theta \mathcal{H}(k) \Theta^{-1}$

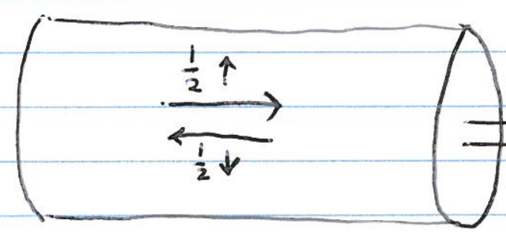
Show why there are 2 and only 2 states



There are two ways for kramers pairs to match.

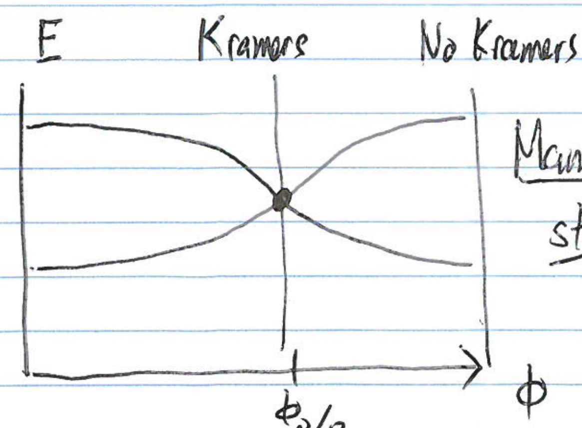
Physical Meaning of $\mathbb{Z}_2$ invariant

Modified Laughlin Argument



$$\Delta\Phi = \frac{\phi_0}{2}$$

Electron number parity at end changes
$\Rightarrow$ Kramers degeneracy changes



Many body states

Formula for Z_2 invariant

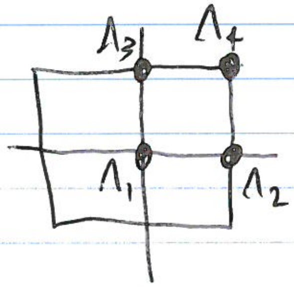
- Bloch WF's $\quad |u_n(k)\rangle \quad$ (N bands)
- T-reversal matrix

$$W_{mn}(k) = \langle u_m(k)|\Theta|u_n(-k)\rangle \in U(N)$$

- Antisymmetry

$$\Theta^2 = -1 \Rightarrow W(k) = -W^T(-k)$$

- T-invariant momenta $\quad \vec{k} = \Lambda_a = -\Lambda_a$



$$W(\Lambda_a) = -W^T(\Lambda_a)$$

- Pfaffian: $\quad \det[W(\Lambda_a)] = \left(\mathrm{Pf}[W(\Lambda_a)]\right)^2$

$\left(\text{ie} \quad \det\begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix} = z^2\right)$

- Λ_a-parity: $\quad \delta(\Lambda_a) \equiv \dfrac{\mathrm{Pf}[W(\Lambda_a)]}{\sqrt{\det[W(\Lambda_a)]}} = \pm 1$

- Gauge Dependent Product:

$$\delta(\Lambda_a)\,\delta(\Lambda_b)$$

Fixes $\sqrt{}$ ambiguity, but not invariant under large gauge transformation

(Analogous to polarization $\frac{e}{2\pi}\oint A\cdot dk$)

- Z_2 Invariant

$$(-1)^{\nu} = \prod_{a=1}^{4} \delta(\Lambda_a)$$

Gauge invariant, but requires globally continuous gauge.

ν is easier to compute if there is symmetry

1. S_z conserved: $n_\uparrow = -n_\downarrow \in \mathbb{Z}$

$$\nu = n_\uparrow \mod 2$$

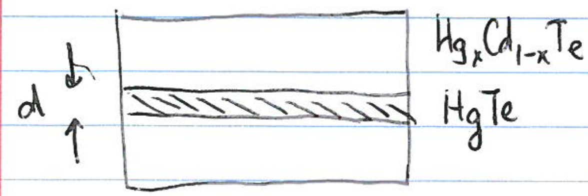
2. Inversion (P)

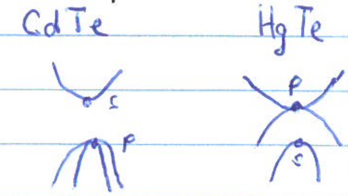
Parity Eigenvalues

$$(-1)^{\nu} = \prod_{a=1}^{4} \prod_{n} \xi_{2n}(\Lambda_a)$$

HgCdTe Quantum Wells

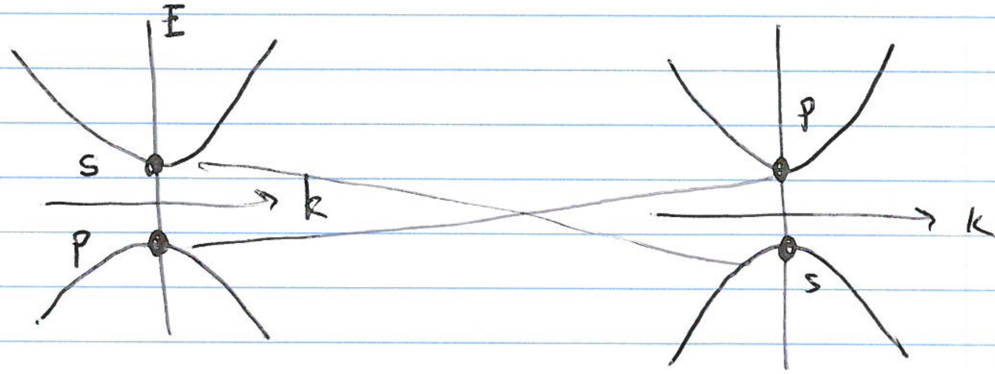
Bernevig, Hughes, Zhang '06, Molenkamp et al





$d < 6.3$ Normal band order

$d > 6.3\,nm$: Inverted



$\prod \xi = +1$

I

$\prod \xi = -1$

TI

BHZ model : describes band inversion

Symmetry allowed spin orbit term

$$\mathcal{H} = (m + k^2)\tau^z + V\tau^x \vec{\sigma}\cdot k$$

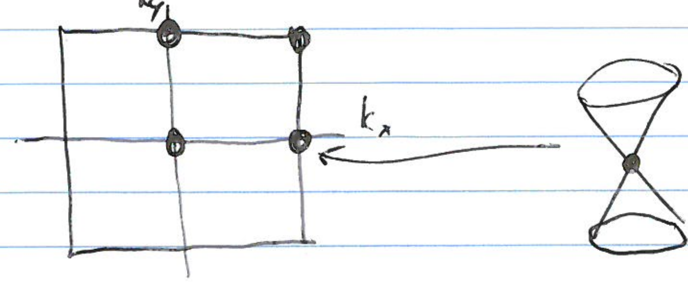
$$\tau^z = \begin{cases} +1 & s \\ -1 & p \end{cases}$$

$m > 0$: uninverted

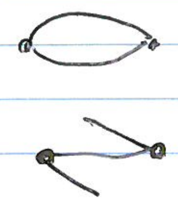
$m < 0$: inverted

3D TI

Consider surface BZ



Lots of Dirac Pts: but How do they connect

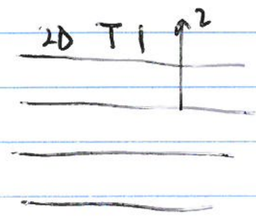


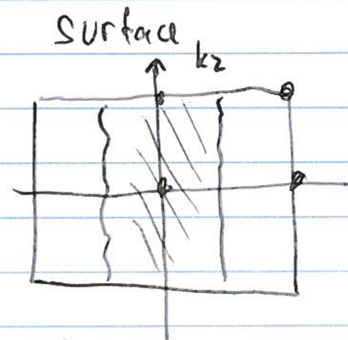
or

1. Trivial insulator:

(Possibly) no surface states

2. Weak TI



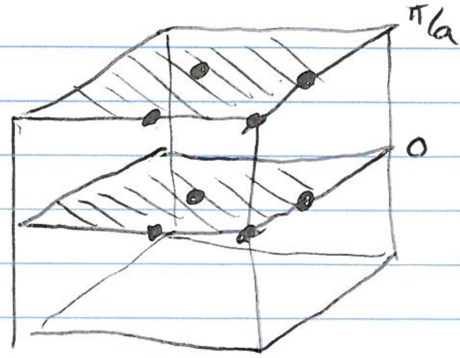


enclose 2/4 dirac pts.

Similar to layered 3D QH. 3 $\mathbb{Z}_2$ invariants

$$G = \frac{2\pi}{a}(\nu_x, \nu_y, \nu_z)$$ "Mod 2 reciprocal lattice vector"

Can be evaluated by considering 2D time reversal invariant planes in 3D B.Z.



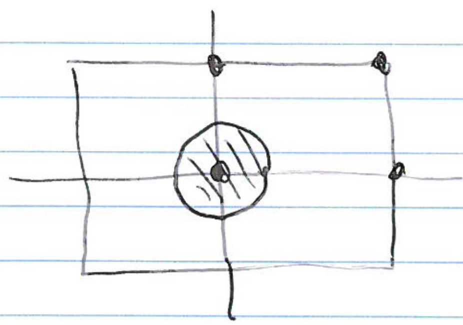
$$\nu(k_z=0) = \nu\left(k_z=\tfrac{\pi}{a}\right) = \nu_Z$$

$$\nu(k_z=0) = \prod_{\square} \delta \qquad \nu\left(k_z=\tfrac{\pi}{a}\right) = \prod_{\square} \delta$$

More interesting possibility: $\nu(k_z=0)\,\nu\left(k_z=\frac{\pi}{a}\right) \equiv \nu_0 = -1$

$$\nu_0 = \prod_{\text{cube}} \delta$$

Strong TI



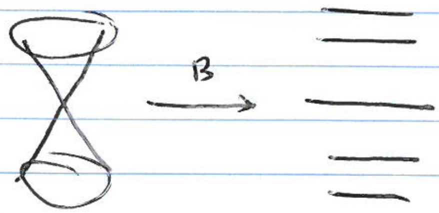
Surface Fermi surface encloses a single Dirac PT.

1. Protected by T
2. Impossible to localize
3. Violates doubling Thm:
   can't have single Dirac Pt. protected by T in 2D.

Dirac Pts are nice, but even more interesting when you kill them by lowering symmetry.

Break T:

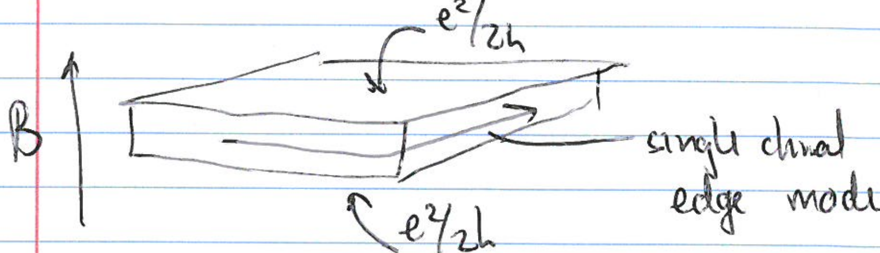
1. Surface QH effect (orbital field)



0 L.L. (non degenerate)

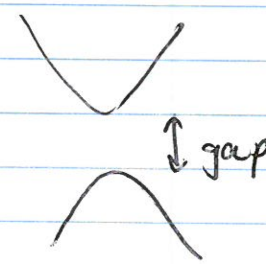
$\sigma_{xy} = \frac{e^2}{h}\left(n + \frac{1}{2}\right)$ Fractional IQHE?

Resolution: Surface can't have boundary

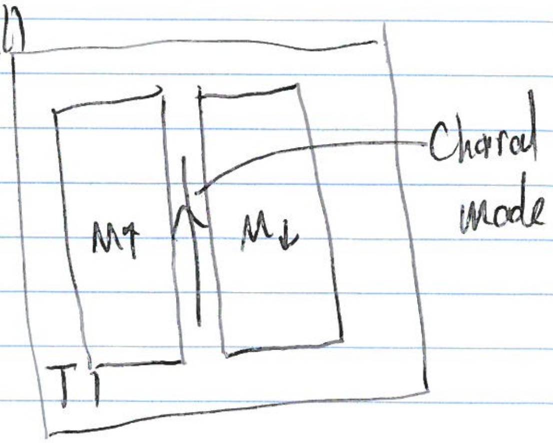


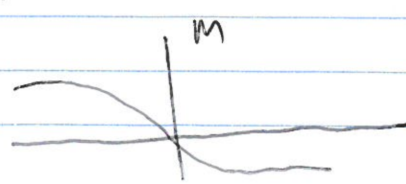
Zeeman Field

$\mathcal{H} = v_F \vec{\sigma} \cdot \vec{p} + m\sigma_z$



Domain Wall



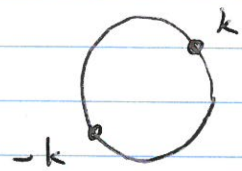


Break Gauge Symmetry $\Rightarrow$ S.C.

$$\mathcal{H}_{BdG} = \tau_z\left(v_F \vec{\sigma}\cdot p - \mu\right) + \tau_x \Delta_1 + \tau_y \Delta_2$$

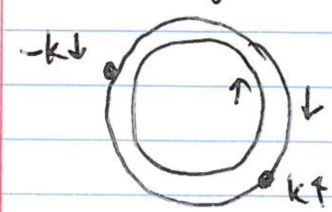
S.C. order parameter $\Delta = \Delta_1 + i\Delta_2 = |\Delta| e^{i\varphi}$

Similar to spinless p-wave



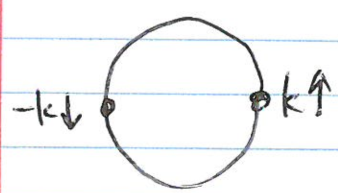
spinless : $\langle c_k c_{-k} \rangle \sim \Delta e^{i\varphi}(k_x + i k_y)$

ordinary S.C.



$\langle c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} \rangle = \Delta e^{i\varphi}$

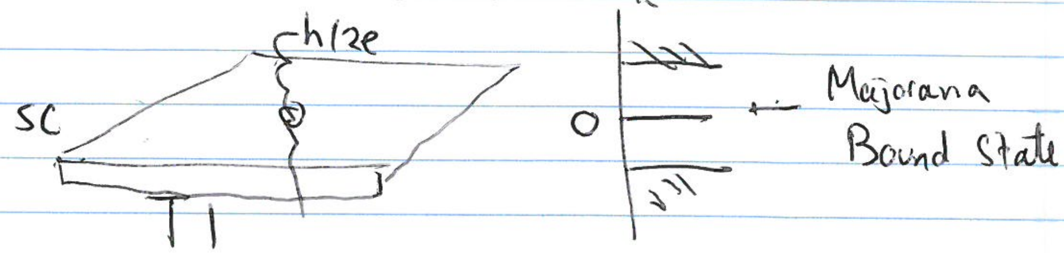
TI surface



$\cancel{\Delta e^{i\varphi}} \langle c_{k\uparrow} c_{-k\downarrow} \rangle = \Delta e^{i\varphi}$

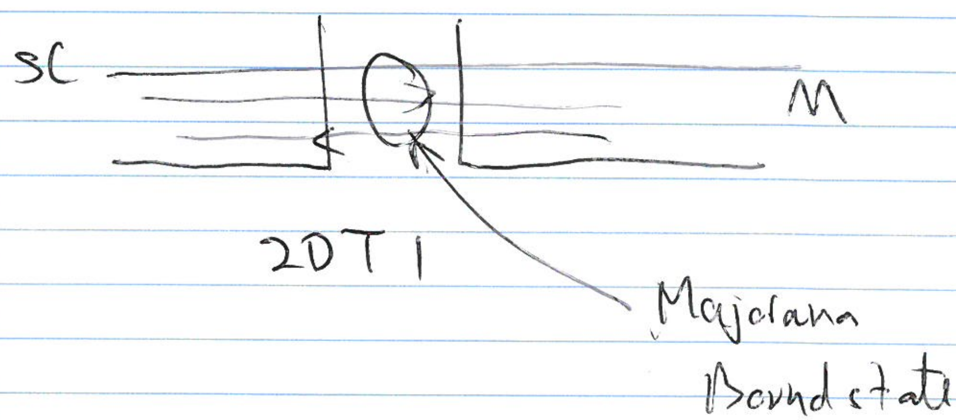
0D Majorana Bound States

1. Vortex on surface of 3D TI

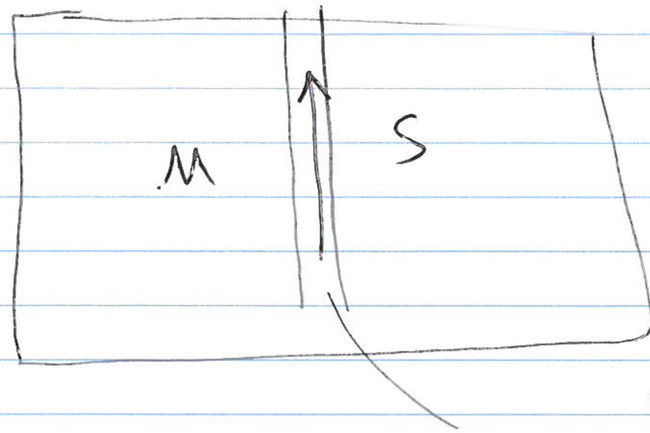


(Jackiw & Rossi '   ) for $\mu = 0$

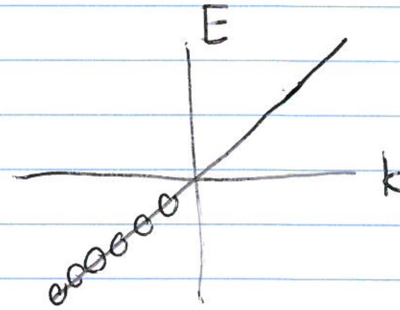
2. S.C. - Magnet on edge of 1D TI



3. 1D chiral Majorana on surface of 3D TI



(like edge of p+ip S.C.



Chiral Majorana mode

Role of vacuum in p+ip played by magnetic gap. T broken on outside instead of inside S.C.