Edge States

Classical Picture

skipping orbit

\[ \text{gh} \quad \Rightarrow \text{propagating state} \quad \text{"one way"} \]

Topological boundary modes

\[
m_+ = -m_-
\]

\[ n = 1 : \text{QHE} \quad x \]

\[ m_+ = +m_- \quad n = 0 : I \]

\[ \text{band inversion} \]

\[ m_+(x) \]

\[ h = \nu_F \left( -i \sigma_x d_x + k_y \sigma_y \right) + m_+(x) \sigma_z \]

Same as Jackiw-Rebbi for fixed \( k_y \).

Zero mode \[ \psi \] of \[ e^{i \frac{k_y y}{E} - \int_{m(x)}^{0} \frac{m(x)}{v} \text{d}x} \]

\[ |\sigma_i \rangle \]
\[ E \left| \psi \right> = +v_F k_y \left| \psi \right> \]

**Chiral Dirac Fermion**

Other edge: \[ \frac{1}{2} \left( \left| \psi^+ \right> - \left| \psi^- \right> \right) \]

\[ E = -v_F k_y \]

**Chiral Dirac Fermion**

1. One way: no choice but to go forward
2. Robust: Insensitive to disorder (nowhere to go)
   Impossible to localize
3. Impossible in purely 1D:
   Fermion doubling theorem: What goes up must go down
   Evaded by spatially separating right & left movers.
Concrete model: Haldane model on a strip

Chiral edge modes $N_L - N_R$ is topo invariant

Bulk-Boundary Correspondence

Boundary invariant $\Delta$ Bulk invariant

$N_L - N_R = \Delta N$
Generalizations

1. 3D layered unit cell states (Halperin)

\[ n_z = \int \frac{dk_x dk_y}{k^2} F(k_x, k_y, k_z) \]

In general, \((n_x, n_y, n_z)\) define reciprocal lattice vector

\[ G = x \hat{a} + y \hat{b} + z \hat{c} = \frac{2\pi}{a} (n_x, n_y, n_z) \]

Miller indices for lattice planes

Chiral surface states
\[ d = 4 : \quad \text{4D IQHE (Zhang, Hu '01)} \]

\[ A_{ij} = \langle u_i | \nabla_{ij} | u_j \rangle \, dk \]

- **Non-Abelian Berry Connection 1-form**

\[ F = dA + A \wedge A : \quad \text{Non-Abelian Berry Curvature 2-form} \]

\[ n = \frac{1}{8\pi^2} \int \text{Tr} [F^2] \in \mathbb{Z} \]

- **2nd Chern Number**

- **Integral of 4-form over 4D BZ**

**Boundary States**: 3+1D chiral Dirac Fermions
- (single Weyl point)

**Higher D:**

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & \cdots \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

- **A**

**All**

\[ Z \, \, \, \, \, \, \, \, Z \, \, \, \, \, \, \, Z \]

- **``Bolt Periodicity: \( d \rightarrow d + 2 \)``**
Topological Defects

Imagine a Band Structure that varies slowly in real space.

\[ H = H \left( \frac{1}{k}, s \right) \]

1 parameter family of 3D Bloch Hamiltonians

2nd Chern number: \[ n = \frac{1}{8\pi^2} \int_{T^3 \times S^1} Tr \left[ F \wedge F \right] \]

Generalized bulk-boundary correspondence:

\[ n \rightarrow \# \text{chiral modes bound to defect line} \]

Example: 3D IQHE

\[ n = \frac{1}{2\pi^2} G \cdot B \]

Burgers' Vector

3D Chern # Burgers' Vec.
Quantum Spin Hall Insulator

Energy Gaps in Graphene $\mathcal{H} = \epsilon_{\sigma} \sigma_x q_x + \sigma_y q_y + V$

1. $V = m_{CPW} \sigma_z$
   $\Rightarrow$ IQHE (Break P)

2. $V = m_{H} \sigma_z \tau_z$
   $\Rightarrow$ IQHE (Break T)

3. Intrinsic spin-orbit interaction
   $V = m_{SO} \sigma_z \tau_z S_z$ (Respects all symmetries)

\[
\mathcal{H} = \begin{pmatrix}
\mathcal{H}_{\uparrow} & 0 \\
0 & \mathcal{H}_{\downarrow}
\end{pmatrix} = \begin{pmatrix}
\mathcal{H}_{\text{Haldane}} & 0 \\
0 & \mathcal{H}_{\text{Haldane}}^\ast
\end{pmatrix}
\]

Is it an artifact of $S_z$ conservation?
Time Reversal Symmetry \[ [H, \Theta] = 0 \]

\[ \Theta \psi = e^{i \pi S_y} \psi^* \]

\[ \text{Spin } \frac{1}{2}: \quad \Theta \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \psi_+ \\ -\psi_- \end{pmatrix} \]

\[ \Theta^2 = -1 \quad \text{(Same minus sign \text{!})} \]

Kramers' Theorem:

For spin \( \frac{1}{2} \) all states are at least 2-fold degenerate.

Simple without spin-orbit, but non-trivial with spin-orbit.

Proof: If \( |\psi\rangle \) is non-degenerate, then

\[ \Theta |\psi\rangle = c |\psi\rangle \]

\[ \Theta^2 |\psi\rangle = c^* \Theta |\psi\rangle = |c^2 |\psi\rangle \]

\[ \Theta^2 \Theta = |c^2 \neq -1 \]

Contradiction
Consequence for edge states

1. Crossing of edge state is protected
2. Absence of elastic backscattering
3. Absence of localization even for strong disorder

\[
\begin{align*}
\Psi_{in} \quad \text{disordered region} \quad \Psi_{out}
\end{align*}
\]

Under \( \psi \): \( r \rightarrow -r \Rightarrow r = 0 \)

\[ |t| = 1 \]

All eigenstates are extended, even for strong disorder.

What is the difference between QSHI and ordinary insulator?

- Chern number \( n = 0 \)
- There is a new \( \mathbb{Z}_2 \) invariant character \( \nu = 0 \),

\[ H(k) \text{ s.t. } H(-k) = \Theta H(k) \Theta^{-1} \]
Show why there are 2 and only 2 states

conventional insulator
\[ U = 0 \]

Topo Ins.
\[ U = 1 \]

There are two ways for Kramers pairs to match.

Physical Meaning of $\mathbb{Z}_2$ Invariant Argument

Electron number parity at end changes

$\Rightarrow$ Kramers degeneracy changes

$\Delta \Phi = \frac{\Phi_0}{2}$

Many-body states
**Formula for $\mathbb{Z}_2$ invariant**

- Block WF's  \[ U_n(k) \] (\( N \) bands)
- $T$-reversal matrix
  \[ W_{mn}(k) = \langle u_m(k) | \Theta | u_n(-k) \rangle \in U(N) \]
- Antisymmetry
  \[ \Theta^2 = -1 \Rightarrow W(k) = -W^T(-k) \]
- $T$-invariant momenta \( \vec{k} = \Lambda_a = -\Lambda_a \)

\[ W(\Lambda_a) = -W^T(\Lambda_a) \]

- Pfaffian: \[ \text{det} [w(\Lambda_a)] = (\text{Pf}[w(\Lambda_a)])^2 \]
  (i.e. \[ \text{det} \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = z^2 \])

- $\Lambda_a$-parity: \[ \frac{\text{Pf}[w(\Lambda_a)]}{\sqrt{\text{det}[w(\Lambda_a)]}} = \mp \]
  \[ \delta(\Lambda_a) \]
Gauge Dependent Product:

\[ S(\Lambda_a) S(\Lambda_b) \]

Fixes \( \sqrt{ } \) ambiguity, but not invariant under large gauge transformations

(Anaologous to polarization \( \epsilon^a \phi^a \phi^a \))

\[ Z_2 \text{ Invariant} \]

\[ (-1)^{2^n} = \prod_{a=1}^{\infty} S(\Lambda_a) \]

Gauge invariant, but requires globally continuous gauge.

\( Z_2 \) is easier to compute if there is symmetry.

1. \( Z_2 \) conserved: \( n_\uparrow = -n_\downarrow \in \mathbb{Z} \)

\[ \nu = n_\uparrow \mod 2 \]

2. Inversion (P):

\[ (-1)^\nu = \prod_{a=1}^{\infty} \prod_n S_{2n}(\Lambda_a) \]

\( \nu \) Parity

\( \in \) Eigenvalues
HgCdTe Quantum Wells

Bernien, Hughes, Chang '96, Molenkamp et al

\[
\begin{array}{c}
\text{Hg}_x\text{Cd}_{1-x}\text{Te} \\
\text{HgTe}
\end{array}
\]

\[ \begin{array}{c}
\text{CdTe} \\
\text{HgTe}
\end{array} \]

\( d < 6.3 \text{ nm: Normal band order} \quad d > 6.3 \text{ nm: Inverted} \)

\[ \begin{array}{c}
E \\
p \\
s
\end{array} \]

\[ \begin{array}{c}
p \\
s \\
E
\end{array} \]

\( \Pi \xi = 1 \)

\( \Pi \xi = -1 \)

\[ \begin{array}{c}
I \\
T
\end{array} \]

**BHJ model** describes band inversion. Symmetry allowed spin-orbital

\[ H = (m + k^2) \tau^2 + V T \tau \frac{1}{\delta} \cdot \mathbf{k} \]

\[ \tau^2 = \begin{cases} 
  +1 & \text{if } m > 0 \text{ (uninverted)} \\
  -1 & \text{if } m < 0 \text{ (inverted)}
\end{cases} \]
3D TI

Consider surface BZ

Lots of Dirac Pts: but How do they connect?

1. Trivial insulator:
   (Possibly) no surface states

2. Weak TI

   2D TI $p^2$

   Enclose 2/4 dirac pts.

Similar to layered 3D QH. 3 $\mathbb{Z}_2$ invariants

$$G = \frac{2\pi}{a} \left( \hat{x}_1 \hat{x}_2 \hat{y}_1 \hat{y}_2 \right) \text{ "Mod 2 reciprocal lattice vector"}$$

Can be evaluated by considering 2D time reversal invariant planes in 3D B.Z.
\[ \nu (k_z = 0) \equiv \nu (k_z = \frac{\pi}{a}) = \nu \equiv \nu_0 = -1 \]

**Strong T**

Surface Fermi surface encloses a single Dirac PT.

1. Protected by \( T \)

2. Impossible to localize

3. Violates doubling Thm:

   can't have single Dirac Pt. protected by \( T \) in 2D.

Dirac Pts are nice, but even more interesting when you kill them by lowering symmetry.
Break T:

1. Surface QH effect (orbital field)

\[ B \rightarrow \text{O.L.L. (non-degenerate)} \]

\[ \sigma_{xy} = \frac{e^2}{h} \left( n + \frac{1}{2} \right) \] Fractional IQHE?

Resolution: Surface can't have boundary

Zeeman Field

\[ H = H_{\text{Zeeman}} + m \mathbf{\sigma}_z \]

Domain Wall

\[ \begin{array}{c}
\text{Chiral mode}
\end{array} \]
Barel Gauge Symmetry $\Rightarrow$ S.C.

$$H_{\text{BdG}} = \tau_z \left( \vec{v}_f \cdot \vec{p} - \mu \right) + \tau_x \Delta_1 + \tau_y \Delta_2$$

S.C. order parameter $\Delta = \Delta_1 + \Delta_2 = \left| \Delta \right| e^{i\phi}$

Similar to spinless p-wave

$$\text{spinless: } \langle c_{k} c_{-k} \rangle \sim \Delta e^{i\phi} \left( k_x + i k_y \right)$$

Ordinary S.C.

$$\langle c_{k\downarrow}^\dagger c_{k\downarrow} \rangle = \Delta e^{i\phi}$$

TI surface

$$\langle c_{k\uparrow}^\dagger c_{-k\uparrow} \rangle = \Delta e^{i\phi}$$
0D Majorana Bound States

1. Vortex on surface of $\mathcal{F}$ 3D $T_1$

\[ \text{SC} \downarrow \text{ch}12e \uparrow 0 \rightarrow \text{Majorana Bound State} \]

(Jackiw & Rossi) for $\mu=0$

2. S.C. - Magnet on edge of 1D $T_1$

\[ \text{SC} \downarrow \text{M} \uparrow \text{2DT}_1 \rightarrow \text{Majorana Bound State} \]
3. 1D chiral Majorana on surface of 3D Ti

Like edge of ptip S.C.

Role of vacuum in ptip played by magnetic gap. T broken on outside instead of inside S.C.