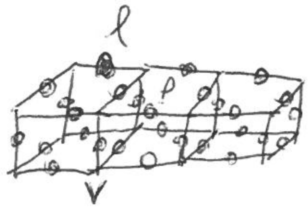
Exactly solvable models for 3D top. phases

3D toric code model



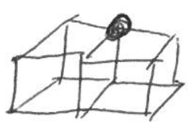
(Hamma, Zanardi, Wen, 2005)

$$H = -\underbrace{\sum_v \prod_{*} \sigma_l^x}_{Q_v} - \underbrace{\sum_p \prod_{\square} \sigma_l^z}_{B_p}$$

Eigenstates: $|\{b_p\}, \{q_v\}\rangle$, $b_p, q_v = \pm 1$

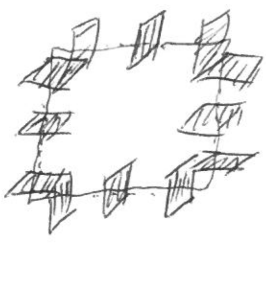
Gd. state: $|\{b_p = q_v = 1\}\rangle$

2 types of excitations:

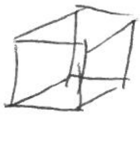


$q_v = -1$ for some $v \Rightarrow$ "charge" $E = 2$

$b_p = -1$ for some loop $\Rightarrow$ "flux loop" $E = 2(\text{length})$



Why does flux have to form closed loop?



$$\prod_{p \in C} B_p = 1 \Rightarrow \prod_{p \in C} b_p = 1$$

C: cube

$\Rightarrow b_p = -1$ plaquettes form closed loops

Braiding statistics:

Charges and flux loops have nontriv. mutual statistics:

$e^{i\theta} = -1$

Charges are bosons:  $e^{i\theta} = 1$

Visualizing gd. state $|\Phi\rangle$:

String picture: $\sigma_\ell^x = -1 \iff$ string on link ℓ

$\sigma_\ell^x = +1 \iff$ no string " " "

$$|\Phi\rangle = \sum_{\substack{X \text{ clsd.} \\ \text{string}}} |X\rangle,$$ i.e. 3D string condensate

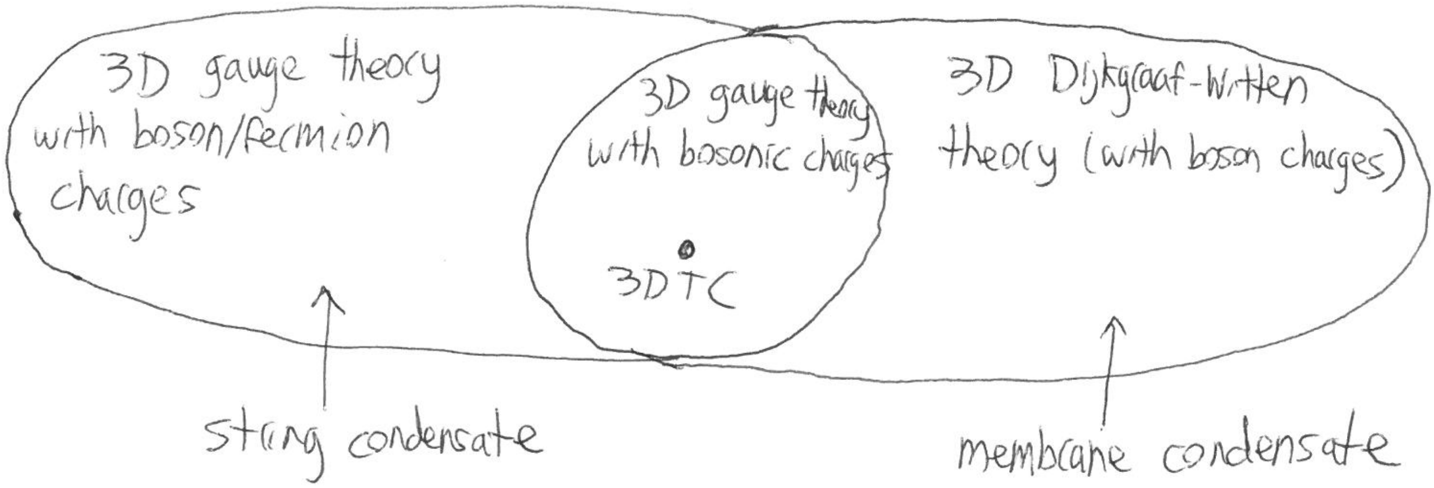
Alternate picture: $\sigma_\ell^z = -1 \iff$ membrane on dual plaquette $\hat{p}$

$\sigma_\ell^z = +1 \iff$ no membrane " "

$$|\Phi\rangle = \sum_{\substack{X \text{ clsd} \\ \text{membrane}}} |X\rangle,$$ i.e. 3D membrane condensate

Known 3D Topological phases

Restricting to "fully 3D" states: (not layered states, Haah code



Dijkgraaf - Witten models

(Dijkgraaf, Witten, 1990)

(Hu, Wan, Wu, 2012)

Models can be defined in any spatial dimension.

Start with 2D for simplicity.

Input: 1. Finite group G

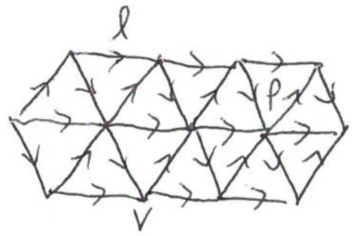
2. 3-cocycle α, i.e.

$\alpha: G \times G \times G \to U(1)$ obeying:

$$\frac{\alpha(g_2, g_3, g_4)\,\alpha(g_1, g_2, g_3)\,\alpha(g_1, g_2 g_3, g_4)}{\alpha(g_1 g_2, g_3, g_4)\,\alpha(g_1, g_2, g_3 g_4)} = 1$$

and $\alpha(1, g, h) = \alpha(g, 1, h) = \alpha(g, h, 1) = 1$

Hilbert space:



Each link can be in any state $|g\rangle$ where $g \in G$

Hamiltonian:

$$H = -\sum_v Q_v - \sum_p B_p$$

B_p: 3 spin interaction

$B_p \Big|$ 

$\Big\rangle = \delta_{ghk} \Big|$ 

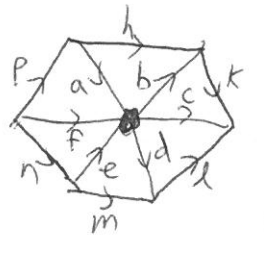
$\Big\rangle$

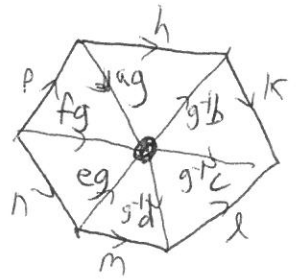
$$\delta_{ghk} = \begin{cases} 1 & \text{if } ghk^{-1} = 1 \\ 0 & \text{otherwise} \end{cases}$$

Favors configs. with vanishing flux.

Q_v: 12 spin interaction

$$Q_v = \frac{1}{|G|} \sum_{g \in G} Q_v^g$$

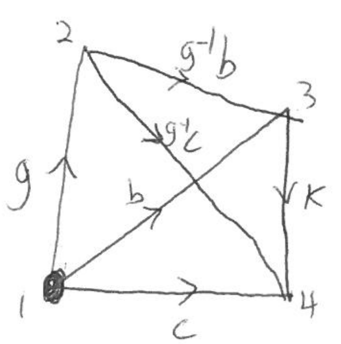
$Q_v^g \Big|$ 

$\Big\rangle = \prod_{\Delta} e^{i\theta^g(\Delta)} \Big|$ 

$\Big\rangle$

To compute $e^{i\theta^g\left(\right.}$ 

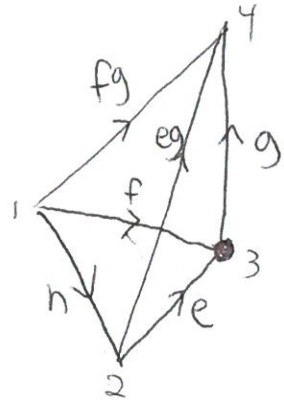
$\left.\right)$, draw tetrahedron:



$$i\theta^g\left(\triangle_{b,k,c}\right) = \alpha(g, g^{-1}b, k)^{-1}$$ ← because tetrahedron is left-handed.

imilarly:

$$i\theta^g\left(\triangledown_{f,n,e}\right) = \alpha(n,e,g)^{+1}$$



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emarks:

. When $\alpha = 1$, this is usual quantum double model, i.e. gauge theory with group G with bosonic charges.

2. For $G = \mathbb{Z}_2 = \{1, g\}$, there are 2 possibilities for α:

(a) $\alpha \equiv 1 \implies$ Toric code

(b) $\alpha(g,g,g) = -1$, $\alpha \equiv 1$ otherwise $\implies$ Doubled semion model

3. Dijkgraaf-Witten models $\equiv$ Gauged group cohomology models (models for SPT phases with unitary symmetry gp. G)

4. 3D generalization:

Triangulation of plane $\longrightarrow$ Triangulation of 3D space by tetrahedra

(Wan, Wang, He, 2

3-cocycle $\longrightarrow$ 4 cocycle

What is physical difference between 3D D-W models and usual 3D gauge theory (with bosonic charges)?

Focus on $G = \mathbb{Z}_2 \times \mathbb{Z}_2$; compare 2 types of models

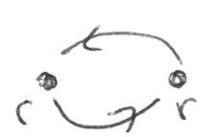
Excitations of $\mathbb{Z}_2 \times \mathbb{Z}_2$ gauge theory $[\equiv (3D\ TC) \times (3D\ TC)]$

Charges: $\{1, r, b, rb\}$

Flux loops: $\{$no flux, $r, b, rb\}$

Braiding statistics:

$e^{i\theta} = -1$



$e^{i\theta} = +1$

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$e^{i\theta} = +1$

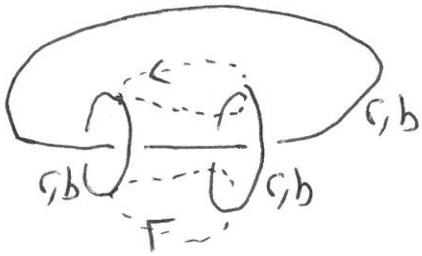
What about $\mathbb{Z}_2 \times \mathbb{Z}_2$ Dijkgraaf-Witten models?

$H^4(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2 \Rightarrow$ 3 non-trivial D-W models with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

Can check that all of these models have same excitations and statistics as above.

So are they actually different?

Yes!



3D TC × 3D TC : phase $= \pm 1$

DW models : Some phases $= \pm i$

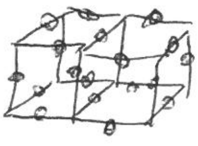
Sharp difference $\Rightarrow$ DW models describe distinct topological phases from usual gauge theory

(Wang, Levin, 2014)

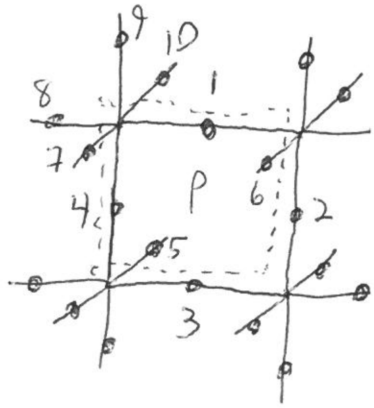
3D gauge theory with fermionic charges

Example: (Levin, Wen, 2005)

$$H = -\sum_{V} Q_V - \sum_{p} \tilde{B}_p$$



$$Q_V = \prod_{*} \sigma_l^{x}$$



To define $\tilde{B}_p$, choose projection of 3D lattice onto 2D plane.

$$\tilde{B}_p = \left(\sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z\right) \cdot \left(\sigma_4^x \sigma_5^x \sigma_6^x \sigma_7^x \sigma_8^x \sigma_9^x \sigma_{10}^x\right)$$

$\Rightarrow$ $\mathbb{Z}_2$ gauge theory with fermionic charges.

In string picture, gd. state $|\Phi\rangle$ is given by

$$\Phi(X) = (-1)^{N_{\text{intersections}}(X)}$$

$N_{int} = 3$

Can be generalized to arbitrary finite gauge group G.

What I didn't cover

1. Walker–Wang models (Walker, Wang, 2011)

2. Haah code (Haah, 2011)

3. Models with symmetry, especially group cohomology models (Chen, Gu, Liu, Wen, 2011)