Exactly solvable models for 3D top. phases

3D toric code model

(Hamma, Zanardi, Wen, 2005)

\[ H = - \sum_v \prod_{x} \sigma^x_v - \sum_p \prod_{Q_v \supseteq \partial p} \sigma^z_v \]

Eigenstates: \( |b_p \uparrow \rangle, |q_v \uparrow \rangle \), \( b_p, q_v = \pm 1 \)

Ground state: \( |\uparrow b_p = q_v = 13 \rangle \)

2 types of excitations:

- \( q_v = -1 \) for some \( v \) \( \Rightarrow \) "charge" \( E = 2 \)
- \( b_p = -1 \) for some loop \( \Rightarrow \) "flux loop" \( E = 2 \) (length)

Why does flux have to form closed loop?

\[ \prod_{p \in C} b_p = 1 \ \Rightarrow \prod_{p \in C} b_p = 1 \]

\( \Rightarrow b_p = -1 \) plaquettes form closed loops
Charges and flux loops have non-trivial mutual statistics:

\[ e^{i\theta} = -1 \]

Charges are bosons:

\[ e^{i\theta} = 1 \]

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**Visualizing gd. state \( |\Phi\rangle \):**

**String picture:**

\[ \sigma^x_\ell = -1 \iff \text{string on link } \ell \]

\[ \sigma^x_\ell = +1 \iff \text{no string} \]

\[ |\Phi\rangle = \sum_{x \text{ clsd. string}} |x\rangle \], i.e. 3D string condensate

**Alternate picture:**

\[ \sigma^z_\ell = -1 \iff \text{membrane on dual plaquette } \hat{p} \]

\[ \sigma^z_\ell = +1 \iff \text{no membrane} \]

\[ |\Phi\rangle = \sum_{x \text{ clsd. membrane}} |x\rangle \], i.e. 3D membrane condensate
Known 3D Topological phases

Restricting to "fully 3D" states (not layered states, Hoah code)

3D gauge theory with boson/fermion charges

3D gauge theory with bosonic charges

3D Dijkgraaf-Witten theory (with boson charges)

String condensate

Membrane condensate

Dijkgraaf-Witten models

(Dijkgraaf, Witten, 1990)

(Hu, Wan, Wu, 2012)

Models can be defined in any spatial dimension.

Start with 2D for simplicity.
Input: 1. Finite group $G$

2. 3-cocycle $\alpha$, i.e.

\[ \alpha : G \times G \times G \rightarrow U(1) \text{ obeying:} \]

\[ \alpha(g_2, g_3, g_4) \alpha(g_1, g_2, g_3) \alpha(g_1, g_2 g_3, g_4) = 1 \]

and \[ \alpha(1, g, h) = \alpha(g, 1, h) = \alpha(g, h, 1) = 1 \]

Hilbert space:

Each link can be in any state $|g\rangle$ where $g \in G$

Hamiltonian:

\[ H = -\sum_v Q_v - \sum_p B_p \]
$B_p: 3$ spin interaction

$$B_p | g \Delta h \rangle = \delta_{ghk} | g \Delta h \rangle$$

$$\delta_{ghk} = \begin{cases} 
1 & \text{if } ghk^{-1} = 1 \\
0 & \text{otherwise}
\end{cases}$$

Favors configs. with vanishing flux.

$Q_V: 12$ spin interaction

$$Q_V = \frac{1}{|G|} \sum_{g \in G} Q_V^g$$

$$Q_V^g | \Delta \rangle = \prod_{\Delta} e^{i \Theta^g(\Delta)} | \Delta \rangle$$

To compute $e^{i \Theta^g(\Delta)}$, draw tetrahedron.
\[ \mathbf{A}^g(\Delta^b_{\kappa}) = \alpha(g, g^{-1}b, \kappa)^{-1} \quad \text{because tetrahedron is left-handed} \]

Similarly:
\[ \mathbf{A}^g(\Delta^f_{\kappa}) = \alpha(n, e, g)^+ \]

Remarks:

When \( \alpha = 1 \), this is usual quantum double model, i.e., gauge theory with group \( G \) with bosonic charges.

1. For \( G = \mathbb{Z}_2 \), there are 2 possibilities for \( \alpha \):
   
   (a) \( \alpha = 1 \) \( \implies \) Toric code

   \[ \alpha(g, g, g) = -1 \]

   \( \alpha = 1 \) otherwise \( \implies \) Doubled semion model

2. Dijkgraaf-Witten models \( \cong \) Gauged group cohomology models (models for SPT phases with unitary symmetry go \( G \))
4. 3D generalization: Triangulation of plane $\rightarrow$ Triangulation of 3D space by tetrahedra

3-cocycle $\rightarrow$ 4 cocycle

What is physical difference between 3D D-W models and usual 3D gauge theory (with bosonic charges)?

Focus on $G = \mathbb{Z}_2 \times \mathbb{Z}_2$; compare 2 types of models

Excitations of $\mathbb{Z}_2 \times \mathbb{Z}_2$ gauge theory $[\equiv (3D \text{ TC}) \times (3D \text{ TC})]$

Charges: $\{1, c, b, cb^3\}$

Flux loops: $\{\text{no flux, } c/b, cb^3\}$

Braiding statistics:

\[
\begin{align*}
& c \quad e^{i\theta} = -1 \\
& b \quad e^{i\theta} = +1
\end{align*}
\]
What about $\mathbb{Z}_2 \times \mathbb{Z}_2$ Dijkgraaf-Witten models?

$H^4(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2 \Rightarrow 3$ non-trivial D-W models with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

Can check that all of these models have same excitations and statistics as above.

So are they actually different?

Yes!

![Diagram]

3D TC x 3D TC: phase = ±1

D-W models: Some phases = ±i

Huge difference $\Rightarrow$ D-W models describe distinct topological phases from usual gauge theory

(Wang, Levin, 2014)
3D gauge theory with fermionic charges

Example: (Levin, Wen, 2005)

\[ H = -\sum_{V} Q_{V} - \sum_{P} \hat{B}_{P} \]

\[ Q_{V} = \prod \sigma_{x}^{V} \]

To define \( \hat{B}_{P} \), choose projection of 3D lattice onto 2D plane.

\[ \hat{B}_{P} = (\sigma_{1}^{2} \sigma_{2}^{2} \sigma_{3} \sigma_{4}^{2}), (\sigma_{5} \sigma_{5}^{2} \sigma_{6}^{2} \sigma_{7} \sigma_{8}^{2} \sigma_{9} \sigma_{10}^{2}) \]

\( \Rightarrow \mathbb{Z}_{2} \) gauge theory with fermionic charges.

In string picture, ground state \( \underline{\Phi} \) is given by

\[ \underline{\Phi}(X) = (-1)^{N_{\text{intersections}}(X)} \]

Can be generalized to arbitrary finite gauge group \( G \).
What I didn't cover

2. Haah code (Haah, 2011)
3. Models with symmetry, especially group cohomology models (Chen, Gu, Liu, Wen, 2011)