Each "spin" can be in 2 states: \( 10\rangle, 11\rangle. \n\)
Convenient to use spin-1/2 notation:

\[
10\rangle = 1 \sigma^X = +1\rangle \\
11\rangle = 1 \sigma^X = -1\rangle \\
\]

\( Q_V : \) 3 spin interaction

\[
Q_V = \frac{1}{2} \left( 1 + \prod \sigma^X \right) \\
\]

\( B_P : \) 12 spin interaction

\[
B_P = \frac{1}{2} P \left( 1 + \prod \sigma^Z \right) \prod i \left( \frac{1 - \sigma^X}{2} \right) P_P \\
\]

where

\[
P_P = \prod_{v \in P} Q_V \\
\]

Model realizes "doubled seamion phase"

4 \( Q_P 's : \) \( 1, s, \bar{s}, s \bar{s} \) = \( 1, s \bar{s} \times 1, \bar{s} s \) = \( s e m \times \bar{sem} \)

\( s, \bar{s} \) have exchange statistics \( e^{i \theta e x} = \pm 1 \) and trivial mutual statistics
Example 2: Doubled Fibonacci model

Data: \( N=1 \), branching

\[
\Lambda = \frac{1 + \sqrt{5}}{2}
\]

Need + to satisfy uncertainty condition

Again use notation

\[
|0\rangle = |\sigma^x = +\rangle
\]

\[
|1\rangle = |\sigma^x = -\rangle
\]

\[
Q_v |e^{+}_{1}\rangle = 1 |e^{+}_{1}\rangle
\]

\[
Q_v |e^{-}_{1}\rangle = 1 |e^{-}_{1}\rangle
\]

\[
Q_v |e^{+}_{-}\rangle = 0
\]

Not illuminating to write down
Model realizes "doubled Fibonacci phase"

4 QP's: $\{1, \phi, \bar{\phi}, \phi\bar{\phi}\} = \{1, \phi^3 \times \bar{1}, \bar{\phi}^3 = F_1 b \times F_1 \bar{b}

$\phi$: non-abelian "Fibonacci" anyon with $\phi \times \phi = 1 + \phi$

$\bar{\phi}$: same but with opposite braiding statistics

General string-net models

So far, we've focused on a special case. More generally:

1. Strings carry orientation: $\rightarrow i \sim \leftarrow i$

2. Additional degrees of freedom at vertices $a = 1, 2, \ldots, \delta_{ijk}$ where $\delta_{ijk}$ can be bigger than 1.

3. More complicated rules for erasing vacuum string $\rightarrow 0$

"\ldots" reversing string orientation.

Need to decorate "symmetric" vertices with dot:

$(\text{Lin, Levin, 2014})$

See also (Kitaev, Kong, 2011), (Lan, Wen, 2013)
How general are these models?

1. Can realize any top. phase of form $T \times T$

To do this, let:

- string types $\leftrightarrow$ QPs in $T$
- Branching rules $\leftrightarrow$ Fusion rules in $T$
- $F \leftrightarrow$ F-symbol in $T$
- $d_i \leftrightarrow$ quantum dimension of particle $i$

2. Can realize any gauge theory with finite gauge group $G$ ("Quantum double" models)

3. Can realize any Dijkgraaf-Witten theory with finite gauge group $G$. ("Twisted quantum double" models)

Can realize other phases, too.

Mathematicians: Can realize any "Drinfeld center" $Z(C)$
Mathematics of string-net models

Input data

\[(F_{ijm}, d_i, s_{jk})\]

Output

\[= \text{"unitary finite spherical fusion category } C\"

\[\text{model for a top. phase with anyons } A = \{a, b, c, \ldots\}\]

\[\text{and data } N_{ab}, R_{ab}, F_{abc}, \ldots\]

How is \(A\) (and associated braiding data) related to \(C\)?

Answer: \(A = Z(C)\)

In words: \(A\) is "Drinfeld center" of \(C\).

---

Physical Characterization

Conjecture I: A (bosonic) topological phase can be realized by a string-net model if and only if it supports a gapped edge.

\[\text{gapped } \Rightarrow \text{Vacuum}\]

\[\text{Top. phase}\]
"only if" is easy: will explain later
"if" is hard $\Rightarrow$ only proven in Abelian case

**Corollary:** String-net models cannot realize topological phases with a nonzero chiral central charge: $c \neq 0$.

In particular, cannot realize Laughlin states.

**Conjecture 2:** String-net models can realize every (bosonic) top. phase that can be described by a commuting projector Hamiltonian.
Application: Boundaries

Exactly solvable models can be used to study gapped boundaries of top. phases.

Example: Toric code

(Bravyi, Kitaev, 1998)

Use square lattice for this discussion

\[ H_{\text{bulk}} = -\sum_{v} \prod_{\mathcal{Q}_v} \sigma^x - \sum_{p} \prod_{\mathcal{B}_p} \sigma^z \]

Can we find a 
Hedge such that 
\[ H = H_{\text{bulk}} + H_{\text{edge}} \] is gapped?

Yes: 
\[ H_{\text{edge}} = -\sum_{\text{vebary}} \prod_{\mathcal{Q}_v} \sigma^x \]

\[ H = -\sum_{\text{vebulk}} Q_v - \sum_{\text{vebary}} Q_{\text{bary}} - \sum_{p} B_p \]
$Q_v, Q_v^{bary}, B_p$ all commute. Let $\{q_v, q_v^{bary}, b_p\}$ be eigenvalues.

Gd. state $= \left| q_v = q_v^{bary} = b_p = 1 \right>$

Can check Gd. state is unique in disk-like geometry:

Energy gap = 2

Another type of gapped boundary:

$H_{edge} = - \sum_{p \in bary} \prod_{x} \sigma_x^{bary} \frac{1}{B_p}$

“smooth boundary”

“rough boundary”

What is physical distinction between 2 boundaries?

Fluxes can be annihilated at smooth bary:

Charges cannot be annihilated

Opposite is true at rough boundary

Sharp difference between 2 types of boundaries

$\Rightarrow$ cannot be adiabatically connected without (boundary) phase transition
Can also study junctions between boundaries.

Let’s compute gd.-state deg., $D$.

$$D = \text{Tr} \left( P_{gs} \right)$$

$$= \text{Tr} \left[ \prod_{P_{\text{bulk}}} \left( \frac{1 + B_p}{2} \right) \prod_{V_{\text{bulk}}} \left( \frac{1 + Q_v}{2} \right) \prod_{P_{\text{bdry}}} \left( \frac{1 + B_{p_{\text{bdry}}}}{2} \right) \prod_{V_{\text{bdry}}} \left( \frac{1 + Q_{v_{\text{bdry}}}}{2} \right) \right]$$

Expand out product. Only term with nonzero trace is 1, since Pauli matrices are traceless.

$$D = 2^{-N_{p_{\text{bulk}}} - N_{v_{\text{bulk}}} - N_{p_{\text{bdry}}} - N_{v_{\text{bdry}}}} \times \text{Tr}(1)$$

Let $V, E, F$ be # vertices, edges, faces in above graph

$$N_{p_{\text{bulk}}} + N_{p_{\text{bdry}}} = F$$

$$N_{v_{\text{bulk}}} + N_{v_{\text{bdry}}} = V - 2$$

Also, $\text{Tr}(1) = 2^E$. 
All together:
\[
D = 2 - F - V + 2 + E
\]
\[
= 2
\]
(since \( V - E + F = 1 \))

Another way to derive 2-fold degeneracy:
\[
\begin{align*}
\{ [W_x, H] = 0, [W_z, H] = 0 \} & \Rightarrow D \geq 2 \\
W_x W_z &= -W_z W_x
\end{align*}
\]

Second approach implies degeneracy is "protected":
If we perturb model, strong operators \( W_x, W_z \) can be replaced by "dressed"^\text{strong} operators that move flux and charge excitations. New operators will still anticommute, due to mutual statistics of flux and charge.
Remarks

1. For $M$ rough and $M$ smooth, degeneracy is $D = 2^{M-1}$.

Note similarity to:

2. All string-net models support a gapped boundary similar to "smooth" boundary.

For general analysis of boundaries of string-net models see (Kitaev, Kong, 2011)