Each "spin" can be in 2 states: $|0\rangle, |1\rangle$.

Convenient to use spin-1/2 notation:

$$|0\rangle = |\sigma^x = +1\rangle$$

$$|1\rangle = |\sigma^x = -1\rangle$$

$\underline{Q_v}$: 3 spin interaction

$$Q_v = \frac{1}{2}\left(1 + \prod \sigma_\ell^x\right)$$

B_p : 12 spin interaction

$$B_p = \frac{1}{2} P_p \left(1 - \prod \sigma_\ell^z \cdot \prod i^{\frac{1-\sigma_\ell^x}{2}}\right) P_p$$

where $$P_p = \prod_{v \in p} Q_v$$

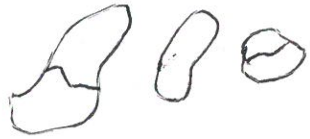
Model realizes "doubled semion phase"

4 QP's : $\{1, s, \bar{s}, s\bar{s}\} = \{1, s\} \times \{1, \bar{s}\}$ [redacted] = Sem x $\overline{\text{Sem}}$

$s, \bar{s}$ have exchange statistics $e^{i\theta_{ex}} = \pm i$, and trivial mutual statistics

Example 2: Doubled Fibonacci model

Data: $N=1$, branching



$$\tau = \frac{1+\sqrt{5}}{2}$$

need '+' to satisfy unitarity condition

Again use notation

$$|0\rangle = |\sigma^x = +1\rangle$$
$$|1\rangle = |\sigma^x = -1\rangle$$

$\underline{Q_v}$

$$Q_v \left| {}_{+}\curlywedge^{+}_{+} \right\rangle = \left| {}_{+}\curlywedge^{+}_{+} \right\rangle$$

$$Q_v \left| {}_{-}\curlywedge^{+}_{-} \right\rangle = \left| {}_{-}\curlywedge^{+}_{-} \right\rangle$$

$$Q_v \left| {}_{+}\curlywedge^{-}_{+} \right\rangle = 0$$

$\underline{B_p}$

Not illuminating to write down

Model realizes "doubled Fibonacci phase"

4 QP's: $\{1, \phi, \bar{\phi}, \phi\bar{\phi}\} = \{1, \phi\} \times \{1, \bar{\phi}\} = \text{Fib} \times \overline{\text{Fib}}$

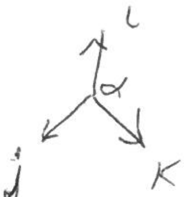
ϕ: non-abelian "Fibonacci" anyon with $\phi \times \phi = 1 + \phi$

$\bar{\phi}$: same but with opposite braiding statistics

General string-net models

So far, we've focused on a special case. More generally:

1. strings carry orientation: $\xrightarrow[i]{} \sim \xleftarrow[i^*]{}$

2. Additional degrees of freedom at vertices 

$\alpha = 1, 2, \ldots, \delta_{ijk}$ where δ_{ijk} can be bigger than 1.

3. More complicated rules for erasing vacuum string $\xrightarrow[0]{}$

" " " " reversing string orientation.

Need to decorate "symmetric" vertices with dot: 

(Lin, Levin, 2014)

See also (Kitaev, Kong, 2011), (Lan, Wen, 2013)

How general are these models?

1. Can realize any top. phase of form $T \times \overline{T}$

   [Doubled semion]
   [Doubled Fibonacci]

   To do this, let:

   string types $\Longleftrightarrow$ QP's in T

   Branching rules $\Longleftrightarrow$ Fusion rules in T

   F $\Longleftrightarrow$ F-symbol in T

   d_i $\Longleftrightarrow$ quantum dimension of particle i.

2. Can realize any gauge theory with finite gauge group G ("Quantum double" models)

   [Toric code]

3. Can realize any Dijkgraaf-Witten theory with finite gauge group G. ("Twisted quantum double" models)

   [Doubled semion]

Can realize other phases, too.

Mathematicians: Can realize any "Drinfeld center" $Z(\mathcal{C})$

Mathematics of string-net models

Input data $(F^{ijm}_{kln}, d_i, \delta_{ijk})$ = "unitary finite spherical fusion category $\mathcal{C}$"

Output = model for a top. phase with anyons $\mathcal{A} = \{a, b, c, \ldots\}$ and data N^c_{ab}, R^c_{ab}, F^d_{abc}, ...

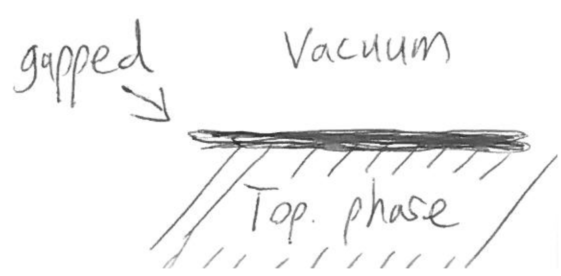
How is $\mathcal{A}$ (and associated braiding data) related to $\mathcal{C}$?

Answer: $\mathcal{A} = Z(\mathcal{C})$

In words: $\mathcal{A}$ is "Drinfeld center" of $\mathcal{C}$.

Physical Characterization

Conjecture 1: A (bosonic) topological phase can be realized by a string-net model if and only if it supports a gapped edge.



"only if" is easy: will explain later

"if" is hard $\Rightarrow$ only proven in Abelian case

Corollary: String-net models <u>cannot</u> realize topological phases with a nonzero chiral central charge: $c_- \neq 0$.

In particular, cannot realize Laughlin states.

Conjecture 2: String-net models can realize every (bosonic) top. phase that can be described by a commuting projector Hamiltonian.

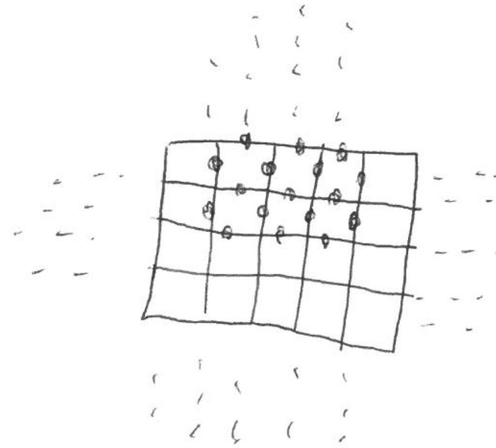
Application: Boundaries

Exactly solvable models can be used to study gapped boundaries of top. phases.

Example: Toric code

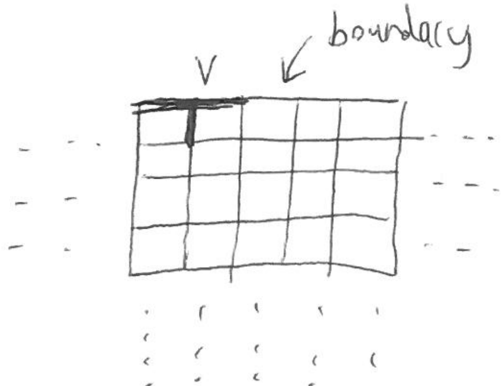
(Bravyi, Kitaev, 1998)

Use square lattice for this discussion



$$H_{bulk} = -\sum_v \underbrace{\prod_{+} \sigma_\ell^x}_{Q_v} - \sum_p \underbrace{\prod_{\square} \sigma_\ell^z}_{B_p}$$

Can we find Hedge such that $H = H_{bulk} + H_{edge}$ is gapped?



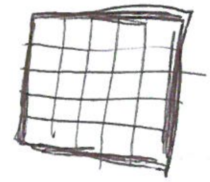
Yes: $H_{edge} = -\sum_{v \in bdry} \underbrace{\prod_{\top} \sigma_\ell^x}_{Q_v^{bdry}}$

$$H = -\sum_{v \in bulk} Q_v - \sum_{v \in bdry} Q_v^{bdry} - \sum_p B_p$$

Q_v, Q_v^{bdry}, B_p all commute. Let $\{q_v, q_v^{bdry}, b_p\}$ be eigenvalues.

Gd. state $= |q_v = q_v^{bdry} = b_p = 1\rangle$

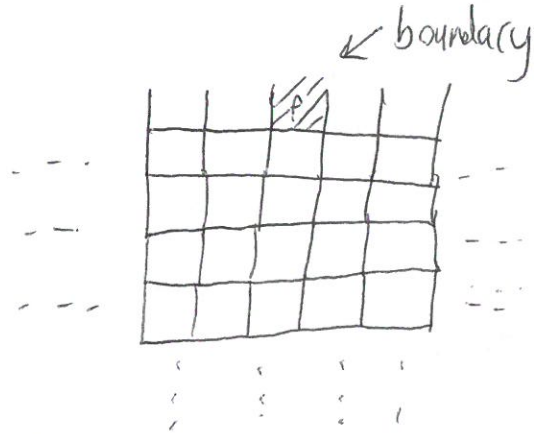
Can check Gd. state is unique in disk-like geometry:



Energy gap $= 2$

"smooth boundary"

Another type of gapped boundary:

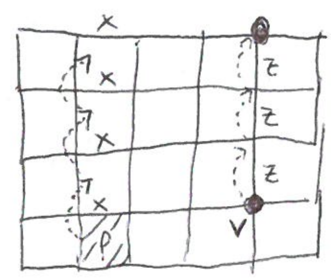


$$H_{edge} = -\sum_{p \in bdry} \underbrace{\prod_{\sqcup} \sigma_\ell^z}_{B_p^{bdry}}$$

"rough boundary"

What is physical distinction between 2 boundaries?

Fluxes can be annihilated at smooth bdry:



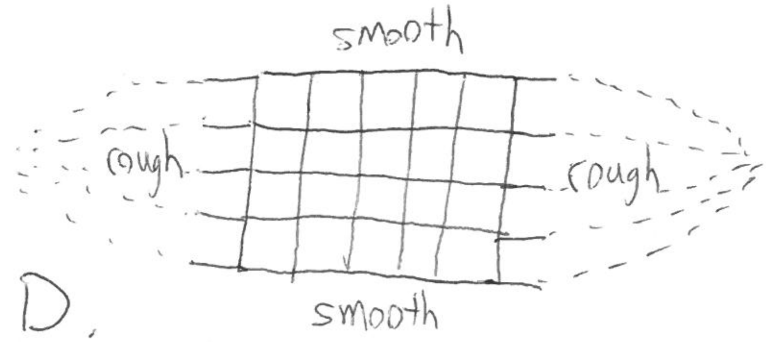
Charges cannot be annihilated

Opposite is true at rough boundary

$\therefore$ Sharp difference between 2 types of boundaries

$\Rightarrow$ cannot be adiabatically connected without (boundary) phase transition

Can also study junctions between boundaries:



Let's compute gd. state deg. D.

$$D = \mathrm{Tr}(P_{gs})$$

$$= \mathrm{Tr}\left[\prod_{p \in bulk}\left(\frac{1+B_p}{2}\right) \cdot \prod_{v \in bulk}\left(\frac{1+Q_v}{2}\right) \prod_{p \in bdcy}\left(\frac{1+B_p^{bdcy}}{2}\right) \prod_{v \in bdcy}\left(\frac{1+Q_v^{bdcy}}{2}\right)\right]$$

Expand out product. Only term with nonzero trace is $1 \cdot 1 \cdots 1$ since Pauli matrices are traceless.

$$D = 2^{-N_p^{bulk} - N_v^{bulk} - N_p^{bdcy} - N_v^{bdcy}} \cdot \mathrm{Tr}(\mathbb{1})$$

Let V, E, F be # vertices, edges, faces in above graph

$$N_p^{bulk} + N_p^{bdcy} = F$$

$$N_v^{bulk} + N_v^{bdcy} = V - 2$$

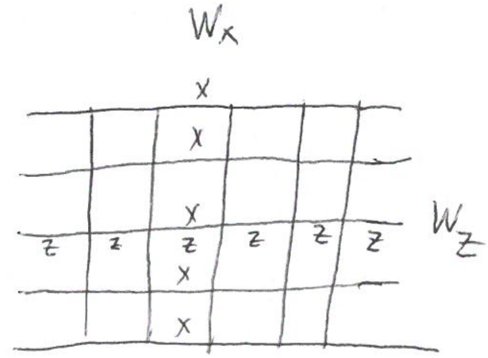
Also, $\mathrm{Tr}(\mathbb{1}) = 2^E$.

All together:

$$D = 2^{-F-V+2+E}$$

$$= 2 \qquad (\text{since } V-E+F=1)$$

Another way to derive 2-fold degeneracy:



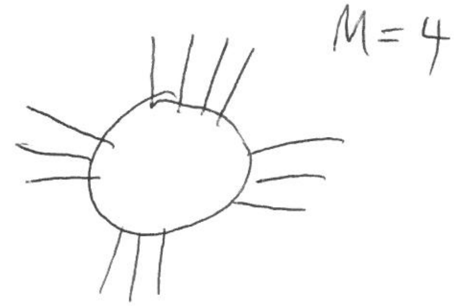
$$\left.\begin{array}{l} [W_x, H] = 0 \\ [W_z, H] = 0 \\ W_x W_z = -W_z W_x \end{array}\right\} \Rightarrow D \geq 2$$

second approach implies degeneracy is "protected":

If we perturb model, string operators W_x, W_z can be replaced by "dressed" string operators that move flux and charge excitations. New operators will still anticommute, due to mutual statistics of flux and charge.

Remarks

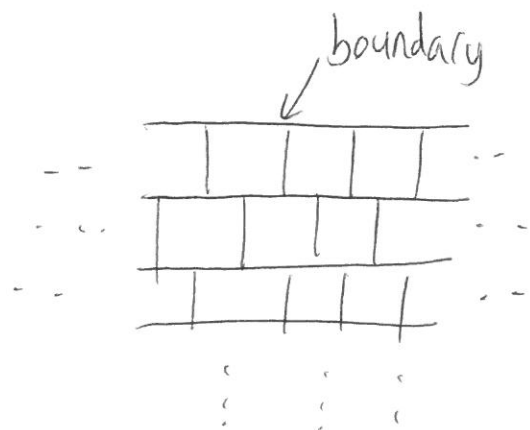
1. For M rough and M smooth, degeneracy is $D = 2^{M-1}$



Note similarity to:

(same degeneracy)

2. All string-net models support a gapped boundary similar to "smooth" boundary.



For general analysis of boundaries of string-net models see (Kitaev, Kong, 2011)