Trial wfns in FQHE — Laughlin 1983

General wfn for N ptcles (eb) in LLL

$$\Psi(z_1 \ldots z_N) = f(z_1 \ldots z_N)\, e^{-\frac{1}{4}\sum_j |z_j|^2}$$

f is holomorphic in each z_j, symmetric (bosons) antisymm (fermions). Can use these if int

$$\underset{\text{matrix els}}{\text{typ. int. pot.}} \ll \hbar\omega_c$$

Laughlin proposed for $\nu = \frac{1}{3}$ FQHE (1983)

$$\Psi_{Laugh}(\{z_j\}) = \prod_{i<j} (z_i - z_j)^Q\, e^{-\frac{1}{4}\sum_j |z_j|^2}$$

Q positive integer; Q odd (F), even (B)

Highest power any z_i is

$$m_{max} = N_\phi = Q(N-1)$$

so if ptcle density is unif inside radius $\sqrt{2m_{max}}$ then filling factor as $N \to \infty$ is

$$\nu = 2\pi \bar{n} = \lim_{N\to\infty} \frac{N}{N_\phi} = \frac{1}{Q}$$

To study density, use plasma mapping

$$|\Psi_{Laugh}|^2 = \exp Q\left[\sum_{i<j} \ln|z_i - z_j|^2 - \frac{1}{2Q}\sum_j |z_j|^2 \right]$$

is Boltzmann weight of 2D plasma of ptcles of charge 1, w. uniform background density $-\frac{1}{2\pi Q}$, and temperature $1/Q$.

It is in a screening phase if $Q \lesssim 70$. Then av. ptcle dens must cancel background density (N large, inside disk) so uniform inside disk. (Correction decay exp'ly, screening length ~ 1 for $Q \gtrsim 1$.)

For $Q=1$, can see directly: wfn is Slater det (Vandermonde det times Gaussian), all states filled out to $m = m_{max}$.

There is a "special" LLL Ham for which Ψ_{Laugh} is exact ground state ($Q>1$) (Haldane 1983)

— wfn $\sim (z_i - z_j)^Q$ as $z_i \to z_j$,

not general form $(z_i - z_j)^{1,0}$ or $(z_i - z_j)$

Fractionally-charged qptles

A LLL state, Laughlin plus one "qhole"

$$\Psi_{\text{Laugh},+w} = \prod_j (z_j - w) \cdot \Psi_{\text{Laughlin}} \quad \text{(Laughlin 1983)}$$

- ptles avoid w, hole in density
- in plasma, "impurity" fixed at w, charge $\frac{1}{Q}$. By screening, deficit in ptle number (within a screening length) is <u>exactly</u> $1/Q$. "Missing" charge pushed to edge
- more qholes $\prod_{j,k} (z_j - w_k) \cdot \Psi_{\text{Laugh}}$
- can also construct "quasielectron" pos. frac. charge, no unique nice functions

These are qp's in our sense. Away from w, state resembles ground state.
Q qholes equiv to removing a ptle - a "real" hole, created by local operator. So <u>Q types of excitations</u>
[Say same in fermion case (Q odd): view el creation/dest op as local, even though anticommute. Works out but not entirely trivial. Still do not allow product of odd no of fermion fields in Ham. Ham must preserve "fermion parity".]

Nonzero expⁿ value of repulsive int energy (for well-separated qhole + qel) implies
$\exists$ <u>energy gap</u>, so we have top. phase when screening holds in plasma, so get FQHE.

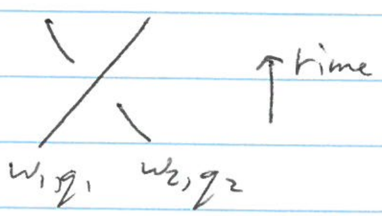
Fusion rules for qp's: just addition of fract. charge say q/Q ($q \in \mathbb{Z}$)

$$\phi_{q_1} \times \phi_{q_2} = \phi_{q_1+q_2 \,(\mathrm{mod}\, Q)}$$

(Abelian)

<u>Statistics of Laughlin qholes</u> Arovas et al, 1984

Calculate Berry phase for an exchange:



~~One qh: Ψ_{Laugh}, two above~~

Berry phase (Berry 1983):
suppose $|\Psi(s)\rangle$ is normalized, non-deg for each $s \in [0,1]$, and $|\Psi(1)\rangle = |\Psi(0)\rangle$

Let $\dfrac{d\gamma}{ds} = i\langle \Psi(s) | \dfrac{d\Psi}{ds}(s)\rangle$ Berry <u>connection</u> (ess. vector pot)

Then Berry phase picked up on adiabatic transport $s=0$ to 1 is

$$\exp i \oint ds \frac{d\gamma}{ds}$$

("dynamical" phase removed)

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One qh, suppose $w = w(s)$ is simple ~~closed~~ ~~loop~~ curve inside drop

$$\frac{d\Psi_{\text{Laugh},+w}}{ds} = -\frac{dw}{ds}\sum_i \frac{1}{z_i - w}\Psi_{\text{Laugh},+w}$$

$$= -\frac{dw}{ds}\int d^2z' \, \frac{n(z')}{z'-w}\Psi_{\text{Laugh},+w}$$

where $n(z) = \sum_i \delta(z_i - z)$ is number density

So

$$\frac{d\gamma}{ds} = -i\int d^2z' \, \frac{1}{2}\left(\frac{1}{z'-w}\frac{dw}{ds} - \frac{1}{\bar{z}'-\bar{w}}\frac{d\bar{w}}{ds}\right)\langle n(z')\rangle_{+w}$$

($\frac{1}{2}$ and subtraction because have to normalize $|\Psi(s)\rangle$)

— exercise

~~Integrate~~

$$\cancel{\gamma(1) - \gamma(0) = -i\oint dw \int d^2z' \, \frac{1}{2}\left(\frac{dw}{z'-w} - \frac{d\bar{w}}{\bar{z}'-\bar{w}}\right)\langle n(z')\rangle_{+w}}$$

$\langle n(z)\rangle_{+w}$ is fn of z and w, and because of screening $\to \bar{n}$ $|z-w| >$ screening length, and also rot. inv about w ~~z or w~~ (away from edge).

In each term in z' int, dw/ds ($d\bar{w}/ds$) is constant (ind of z'), ~~and by rot inv~~

Write $\langle n(z)\rangle_{+w} = \langle n(z)\rangle + \delta n_1(z) + \delta n_2(z)$

$\delta n_1 \neq 0$ for z near w and $\delta n_2 \neq 0$ at edge only. $\int(\delta n_1 + \delta n_2)d^2z = 0$

δn_1 for z near w is rot inv fn of $z-w$.

$$\int d^2z' \, \frac{\delta n_1(z')}{z'-w} = 0 \text{ by rot inv}$$

$\int d^2z'\, \delta n_z$
$\parallel$

$\int d^2 z' \frac{\delta n_z(z')}{z'-w}$. If w e.g. at center of drop, $\int dr\, \delta n_z \sim \frac{\cancel{q}/Q}{R}$ so $\int d\theta \cdot R \int dr\, \delta n_z = \frac{1}{Q}$

so $\left| \int d^2 z' \frac{\delta n_z}{z'-w} \right| \leq \int d^2 z' \frac{|\delta n_z|}{R} = O\left(\frac{1}{R}\right) \to 0$

R = radius of drop

even when not zero by symmetry

So replace $\langle n(z') \rangle_{+w}$ by $\langle n(z') \rangle$ in Berry connection, integrate

$$\gamma(1) - \gamma(0) = -i \oint \int d^2 z' \frac{1}{2}\left(\frac{dw}{z'-w} - \frac{d\bar{w}}{\bar{z}'-\bar{w}} \right) \langle n(z') \rangle$$

do w int first

$$= -2\pi \int_{\text{interior}} d^2 z' \langle n(z') \rangle$$

by Cauchy's Thm.

$\langle n(z') \rangle = \bar{n}$ inside drop.

Like Aharonov-Bohm phase. Path-dep part discussed earlier.

in fact, exactly that for ptcle of charge $-\frac{1}{Q}$ (in el. units) in field B

If the path encloses a second qhole, additional Berry phase

$$\Delta\gamma = 2\pi/Q$$

(~~...~~ $\langle n(z) \rangle$ due to missing ptcle no $\frac{1}{Q}$ inside. not constant in z)

Sim if path is actually exchange of two qholes, get half of above, ie and

$\boxed{\Theta = \pi/Q \pmod{2\pi}}$ for basic qholes (θ defined earlier for anyons)

$$\Delta\gamma = \pi \frac{q_1 q_2}{Q}$$

for qholes of charges q_1/Q, q_2/Q.

For $q_1 = q_2 = Q$, bosons ($\Delta\gamma \equiv 0 \bmod 2\pi$) if Q even
fermions ($\Delta\gamma \equiv \pi \bmod 2\pi$) if Q odd.

Calc for gen intractable analytically (no nice eqns), Statistics (ie $q_1 = q_2$) for ~~two~~ quasiel must be <u>same</u> as for qhole of opposite charge (antiparticle) for consistency within framework.

Conformal field theory construction

Moore, NR, 1991

Basics: massless scalar field in 2 dimensions (<u>not</u> 2+1)
Euclidean

~~L~~ Action ~~Ψ~~ $S = \int \frac{1}{2}(\nabla\varphi)^2$ (uncertain of a factor here)

~~Field~~ Solution can be split $\varphi(z,\bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z})$
"right and left"
not c.c. of φ

Use $\varphi(z)$ only, "holomorphic"

Correlator $\langle \varphi(z)\varphi(0)\rangle = -\ln z$ (defines norm^n

$\boxtimes$ [Note $-i\partial\varphi$ is a good 2D ~~..~~ conformal "primary" field]

Operator pic: Vacuum $|0\rangle$. Radial quantization and ordering

The log resembles ~~..~~ 2D Coulomb int - relation to plasma

Construct correlator ("time-ordered")

$$\langle ~~\varnothing~~ | \, O \prod_{i=1}^{N} a(z_i) \, ~~|0\rangle~~ \rangle$$

string theorists 60, see S. Coleman 1975, KT 1974

and also "normal ordering"]

Here $a(z_i) = e^{i\varphi(z)/\sqrt{\nu}}$, radial "time" ordering / implicit (if we think of operator pic)

"charge"

$$O = e^{-i\frac{\sqrt{\nu}}{2\pi}\int d^2z\,\varphi(z)}$$

Expand exp, use Wick's Thm, get

$$\langle e^{i\varphi/\sqrt{\nu}} e^{-i\varphi/\sqrt{\nu}} \rangle ~~\langle~~ \left(1 + \frac{i\varphi(z)}{\sqrt{\nu}} - \frac{1}{2\nu}\varphi^2(z) + \dots\right)\left(1 - \frac{i\varphi(0)}{\sqrt{\nu}} - \frac{1}{2\nu}\varphi^2(0) + \dots\right)\rangle$$

$$= 1 \mp \frac{1}{\nu}\ln z + \frac{2}{4\nu^2}(\ln z)^2 \mp \dots$$

if "ignore" self contractions $\langle\varphi(0)^2\rangle = \infty$

- role of normal ordering.

Combinatorics, find

$$= \frac{1}{z^{1/\nu}} .$$

Plasma Boltzmann weight / with only one +, one - charge

Extend to many charges q_i. Related to bosonization (Coleman)

Rule: total charge $= 0$, otherwise ~~result~~ correlator is zero

Our case, get (if $\int d^2z'$ over region containing charge N)

$$\prod_{i<j} (z_i - z_j)^{1/\nu} e^{-\frac{1}{4}\sum_j |z_j|^2} \times \text{singular phase (drop!)}$$

— Laughlin wfn Ψ_{Laugh} if set $\nu = \frac{1}{Q}$

Wfn for quasiholes: insert $\tau(w) = e^{i\sqrt{\nu}\varphi(w)}$ for each qh, as well as $\prod_i a(z_i)$. We get

$$\prod_{k<l} (w_k - w_l)^{1/Q} \cdot \prod_{k,j} (w_k - z_j) \cdot \prod_{i<j} (z_i - z_j)^Q \times e^{-\frac{1}{4}\sum_j |z_j|^2 - \frac{1}{4Q}\sum_k |w_k|^2}$$

not single valued! Called "monodromy" of function — have to include with Berry phase

Note coeffs in exponent were chosen so that (after removing singular phase!) the fn is single-valued in z's, ptcle coords, (and so in LLL).

~~Use~~. Above form of multi-qh wfn was proposed by Halperin (1984)! ~~while studying hi~~

Note prefactor involving w's only cannot affect ptcle correlations since we normalize wfns anyway, and change in phase is a "change of gauge"

~~Arg~~

This choice seems related to fractional statistics — changes by $e^{i\pi/Q}$ for exchange of two qholes, same as adiabatic

Extend the construction: (Moore, NR, 1991)

~~[illegible]~~ Use another CFT, "primary" field $\psi(z)$ (holo)

$$a(z) = e^{i\varphi(z)/\sqrt{2}}\,\psi(z)\,,\quad \mathcal{O} \text{ as before}$$

(don't change φ part, because it gives all basic FQHE stuff — "charge sector")

Primary fields in CFT have "fusion rules" as described above in different context. We assume ψ has Abelian fusion rules. In fact, should be "simple current".

Ex: ψ is Majorana (fermion) field

$$\langle \psi(z)\,\psi(0)\rangle = \frac{1}{z}$$

Then

$$\langle \prod_j a(z_j) \rangle$$

$$= \prod_{i<j} (z_i - z_j)^{1/2} \cdot \left(\frac{1}{z_1 - z_2}\,\frac{1}{z_3 - z_4} \cdots \frac{1}{z_{N-1} - z_N} \pm \text{perms} \right)$$

— Moore-Read state. N must be even. $\times\, e^{-\frac{1}{4}\sum_j |z_j|^2}$

The sum over distinct perms

$$\frac{1}{2^{N/2}(N/2)!} \sum_{\sigma \in S_N} \operatorname{sgn} \sigma \prod_{j=1}^{N/2} M_{\sigma(2j-1),\sigma(2j)} \equiv \operatorname{Pf} M$$

(Pfaffian), any antisym matrix M

Have $M_{ij} = \dfrac{1}{z_i - z_j}$

Form

$\mathrm{Pf}\, g(z_i - z_j)$ for fn g also arises from BCS thy for pairing spinless fermions ~~ptcles~~.

For $g(z_i - z_j) = \dfrac{1}{z_i - z_j}$, have $p - ip$ pairing.

Antisymmetry of Pf, and single-valued wfn ~~$1/\nu = Q$ should be~~ implies

$1/\nu = Q$ should be integer

$$Q = \begin{cases} \text{even} & (\text{particles are } \underline{\text{fermions}}) \\ \text{odd} & (\text{ptcles are bosons}) \end{cases}$$

— reverse of Laughlin. // (End of example)

Note ν is indeed the filling factor, assuming we get a uniform fluid state.

Quasiholes: $\tau(w) = e^{iq\varphi(w)/\sqrt{\nu}}\,\sigma(w)$

q = charge of qhole (not the q above), σ is primary field ("chiral vertex operator") in CFT

Condition to ensure ground and qhole wfns are single-valued in ptle coordinates:

Say $\psi = \psi_1$, it has Abelian fusion rules, so do its products.

In ope language (more detail than fusion rule)

$$\psi_1(z)\,\psi_1(0) \sim \frac{1}{z^{2h_1 - h_2}}\,\psi_2(0) + \cdots$$ asymptotically as $z \to 0$ in a correlator

and so on; generally

$$\psi_n(z)\,\psi_l(0) \sim \frac{c_{n,l}}{z^{h_n + h_l - h_{n+l}}}\,\psi_{n+l}(0) + \cdots$$ (descendants differ by integer power of z)

comes to fusion rule $\psi_n \times \psi_l = \psi_{n+l}$

Further ("simple current") $\sigma = \sigma_1$, $\psi_1 \times \sigma_1 = \sigma_2$, etc (single term on right), or corres ope.

Then $\left\langle \mathcal{O} \prod_{k=1}^{n} \tau(w_k) \prod_{j=1}^{N} a(z_j) \right\rangle$, by <u>careful choice of ν</u>, to cancel poles in $\psi_1\psi_1$, is single-valued fn of z_j's. (in fact, polynomial)

<u>Not</u> nec of w_k's, in fact can be multisheeted, not just in simple way like Abelian anyons — get different fn of z_j's also <u>non-Abelian</u> monodromy (<u>statistics</u>)

In MR example, CFT is Ising model critical thy, $\sigma(w)$ is "spin field" (part of Ising spin, $\sigma(z\bar{z}) = \sigma(z)\bar{\sigma}(\bar{z})$)

$$\psi(z)\,\sigma(0) \sim \frac{1}{z^{1/2}}\,\sigma(0) \quad , \ \psi \text{ is simple current}$$

Ex: with two qholes, Pf factor becomes

$$(w_1 - w_2)^{-\frac{1}{8}}\, \mathrm{Pf}\left(\frac{(z_i - w_1)(z_j - w_2) + (w_1 \leftrightarrow w_2)}{z_i - z_j} \right)$$

Four qholes: see Nayak + Wilczek 1996 (MR, 1991)
No of qholes n must be even, ~~#~~ unless allow "change in b.c." at ∞ (edge of disk) (also on sphere). (cf flux quant^n in paired state).

~~The number of (lin. ind) states (w/ qholes) produced is~~

$$\cancel{2^{n/2 - 1} \quad \text{if } N \text{ is fixed.}}$$

~~[$\sigma(w)$ not an ordinary operator !]~~

There is a special Ham for which grnd & qhole states are exact zero energy ~~e~~states (Greiter et al, 1991), NR, + Rezayi 1996). These states are degenerate, ind of qhole positions.

~~Fewer than 2 states per qhole~~
~~— info shared non-locally.~~

Fusion rules

Case $Q=1$, (i.e. bosons) Three qp types: $\mathbb{1}, \psi, \sigma$
charge: $0, 0, \frac{1}{2} \pmod 1$

(warning: not really same meaning of ψ, σ)

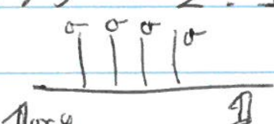
Can also add Laughlin qholes $\prod \left[\prod (z_i - w') \text{ or } e^{i\varphi(w')\sqrt{2}} \right]$
with charge 1, so these charges really mod 1.

Then fusion rules $(\mathbb{1} \times \text{anything} = \text{itself})$

$$\psi \times \psi = \mathbb{1}$$
$$\psi \times \sigma = \sigma$$
$$\sigma \times \sigma = \mathbb{1} + \psi$$

Multiplying, e.g. $n=4$ qholes,

$$(\sigma \times (\sigma \times (\sigma \times \sigma))) = 2 \cdot \mathbb{1} + 2\psi$$



For N even, must select final state $\mathbb{1}$, so in fact get 2 ~~$2^{n/2}$~~

In general, gives $2^{n/2-1}$ $(= 2 \text{ for } n=4)$

For general Q, have $3Q$ types, but same multiplicities for n qholes "containing" σ.

These can be understood using Majorana zero modes (also in Ising CFT).

Note: 1) Fewer than 2 states per qhole — info shared non-locally
2) $\sigma(w)$ not an ordinary operator

$p_x \pm ip_y$ paired (superconducting) states — NR + Green, PRB, 2000

at level of BCS mean field for pairing

Spinless or spin polarized ptcles

$$K_{eff} = \sum_{\underline{k}} \left[h_k \, c_{\underline{k}}^{+} c_{\underline{k}} + \tfrac{1}{2}\left(\bar{\Delta}_{\underline{k}} c_{-\underline{k}} c_{\underline{k}} + \Delta_{\underline{k}} c_{\underline{k}}^{+} c_{-\underline{k}}^{+} \right) \right]$$

$h_{\underline{k}} = \frac{k^2}{2m_{el}} - \mu \qquad \Delta_{\underline{k}} = (l=-1) \text{ estate}$

$\simeq \hat{\Delta}(k_x - ik_y)$ at small $\underline{k}$ i.e. $p - ip$

$\to 0$ as $\underline{k} \to \infty$

(free space, not lattice)

$$\{c_{\underline{k}}, c_{\underline{k}'}^{+}\} = \delta_{\underline{k}\underline{k}'}$$

Note $\Delta_{\underline{k}} = -\Delta_{-\underline{k}}$ required – odd parity

Solve by Bog:

$$|\Omega\rangle = \prod_{\underline{k}}{}' \left(u_{\underline{k}} + v_{\underline{k}} c_{\underline{k}}^{+} c_{-\underline{k}}^{+} \right) |0\rangle$$

each pair $(\underline{k}, -\underline{k})$ only once; $|0\rangle$ vac, $c_{\underline{k}}|0\rangle = 0$

$$|u_{\underline{k}}|^2 + |v_{\underline{k}}|^2 = 1 \qquad (\text{norm}^{n})$$

Can also write

$$K_{eff} = \frac{1}{2} \sum_{\underline{k}} \left(c_{\underline{k}}^{+} \;\; c_{-\underline{k}} \right) \mathcal{H}_{\underline{k}} \begin{pmatrix} c_{\underline{k}} \\ c_{-\underline{k}}^{+} \end{pmatrix}$$

$$\mathcal{H}_{\underline{k}} = \begin{pmatrix} h_{\underline{k}} & \Delta_{\underline{k}} \\ -\bar{\Delta}_{-\underline{k}} & -h_{-\underline{k}} \end{pmatrix}$$

obeys $\mathcal{H}_{\underline{k}}^{+} = \mathcal{H}_{\underline{k}}$, $\sigma_x \mathcal{H}_{\underline{k}} \sigma_x = -\bar{\mathcal{H}}_{-\underline{k}}$ ("class D")

Then $w_{\underline{k}} = \begin{pmatrix} u_{\underline{k}} \\ v_{\underline{k}} \end{pmatrix} \leftrightarrow \alpha_{\underline{k}} = u_{\underline{k}} c_{\underline{k}} + v_{\underline{k}} c^{+}_{-\underline{k}}$

$\alpha_{\underline{k}} |\Omega\rangle = 0$

impose, &

$$\mathcal{H}_{\underline{k}} w_{\underline{k}} = E_{\underline{k}} w_{\underline{k}}, \quad \text{suppose } E_{\underline{k}} \geq 0$$

"Bog.-de Gennes eq^{n}"

$$K_{eff} = \sum_{\underline{k}} E_{\underline{k}} \alpha^{+}_{\underline{k}} \alpha_{\underline{k}}$$

~~Because of "symmetry" $\sigma_x \overline{w_{-\underline{k}}}$ has eval $-E_{\underline{k}}$~~

$\sigma_x \overline{w_{\underline{k}}}$ (up to phase) has eval $-E_{-\underline{k}}$

Solve $E_{\underline{k}} = \sqrt{h_{\underline{k}}^2 + |\Delta_{\underline{k}}|^2}$

$g_{\underline{k}} \equiv v_{\underline{k}}/u_{\underline{k}} = (E_{\underline{k}} - h_{\underline{k}})/\overline{\Delta_{\underline{k}}}$, $\quad |u_{\underline{k}}|^2 = \frac{1}{2}\left(1 + \frac{h_{\underline{k}}}{E_{\underline{k}}}\right)$, $\quad |v_{\underline{k}}|^2 = \frac{1}{2}\left(1 - \frac{h_{\underline{k}}}{E_{\underline{k}}}\right)$

Can mult $\begin{pmatrix} u_{\underline{k}} \\ v_{\underline{k}} \end{pmatrix}$ by phase factor.

$\Delta_{\underline{k}} \to 0$ as $\underline{k} \to 0$, so $E_{\underline{k}} - |h_{\underline{k}}| \to 0$ as $\underline{k} \to 0$

Three possibilities:

1) $h_{\underline{k}} < 0$ at suff small $\underline{k}$, (usual weak coupling or $\mu > 0$)

$\Rightarrow g_{\underline{k}} \sim \dfrac{\text{const}}{\overline{\Delta_{\underline{k}}}} \to \infty$ as $\underline{k} \to 0$

i.e. $|v_{\underline{k}}| \to 1$, $u_{\underline{k}} \to 0$ "weak pairing"

2) $h_{\underline{k}} > 0$ " " (like strong coupling or $\mu < 0$)

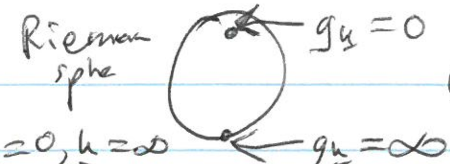
$\Rightarrow g \to 0$ ~~const~~ as $\underline{k} \to 0$ ("strong pairing")

$|u_{\underline{k}}| \to 1$, $v_{\underline{k}} \to 0$ ~~$|v_{\underline{k}}| \to 1$, $u_{\underline{k}} \to 0$~~

3) $h_{\underline{k}} \to 0$ as $\underline{k} \to 0$: $|u_{\underline{k}}|, |v_{\underline{k}}|$ both nonzero

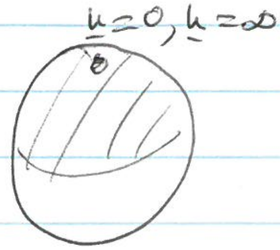
Also $|u_{\underline{k}}| \to 1$, $v_{\underline{k}} \to 0$ as $\underline{k} \to \infty$, all cases

Find ~~windin~~ winding of $w_{\underline{k}}$:



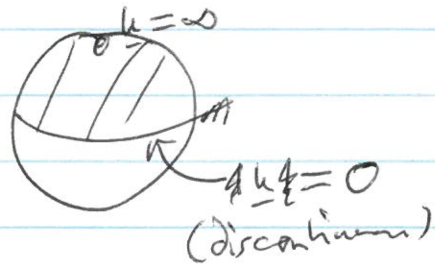
(or $\binom{u}{v}$ normalized, modulo phase ($= \mathbb{CP}^1$))

Strong pairing:

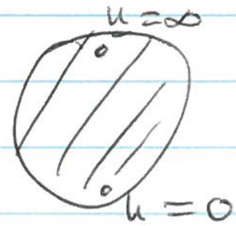


, degree $= 0$

~~was~~ Transition:



Weak pairing



, degree $= 1$ (or -1)

– change in topology in $\underline{k}$ space at transition.
(sim to Chern nos in band structure)

Real space behavior:

Wfn $\langle \underline{r}_1 \ldots \underline{r}_N | \Omega \rangle = \langle 0 | c_{\underline{r}_1} \ldots c_{\underline{r}_N} | \Omega \rangle$

$$\cancel{= \frac{1}{2^{N/2}(N/2)!} \sum_{\sigma \in S_N} \mathrm{sgn}\,\sigma}$$

$$= \mathrm{Pf}\, g(\underline{r}_i - \underline{r}_j)$$

$g(\underline{r}) = \int \frac{d^2k}{(2\pi)^2} e^{i\underline{k}\cdot\underline{r}} g_{\underline{k}}$

Note $g_{\underline{k}} = -g_{-\underline{k}} \Rightarrow g(\underline{r}) = -g(-\underline{r})$,

$g(\underline{r}_i - \underline{r}_j)$ is antisymm matrix.

Strong pairing: $g_{\underline{k}}$ real analytic at small $\underline{k}$, (if $h_{\underline{k}}$, $\Delta_{\underline{k}}$ are)

$$\text{so} \quad g(\underline{r}) \sim e^{-r/\xi}, \quad r \to \infty,$$

times power for "strong pairing"

Weak pairing: $g_{\underline{k}} \propto \frac{1}{k_x + ik_y} \Rightarrow \boxed{g(\underline{r}) \propto \frac{1}{x+iy}}$ $|\underline{r}| \to \infty$

"Long distance wfn" is

$$\text{Pf}\, \frac{1}{z_i - z_j}$$

as in MR state!

Transition: $g(\underline{r}) \propto \frac{1}{z|z|}$ — intermediate.

Edges, vortices

Small μ, small $\underline{k}$,

$$E_k = \sqrt{\mu^2 + |\hat{\Delta}|^2 \underline{k}^2} \quad \text{— relativistic}$$

BdG eq^n become Dirac eq: (real space, time)

$$i\frac{\partial u}{\partial t} = -\mu u + \bar{\hat{\Delta}}\, i\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) v$$

$$i\frac{\partial v}{\partial t} = \mu v + \hat{\Delta}\, i\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) u$$

These ~~allow~~ compatible with $u(\underline{r},t) = \overline{v(\underline{r},t)}$

— reality condition. Can transform so both comp are real

(Comes from $c^\dagger_{\underline{k}} = (c_{-\underline{k}})^\dagger$ fact have $c_{\underline{k}}, c^\dagger_{-\underline{k}}$)

— i.e. fermion is Majorana fermion (field here).

Correct, have ptcle excitation each $\underline{k}$, $E_{\underline{k}} > 0$, is own antiptcle (ie annihilate with $-\underline{k}$).

Contrast with Dirac: fermions of charge +1, −1 at each $\underline{k}$.

(This is the ~~high~~ ptcle physics usage & is what Majorana actually did — in 3+1 D)

Make μ depend on x (say, add a pot $V(\underline{r})$ to K_{eff})

Domain wall, eg in x given k_y ($\hat{\Delta}$ = real now)

$$E u = -\mu u + \hat{\Delta} i \left(\frac{\partial v}{\partial x} - k_y v \right)$$

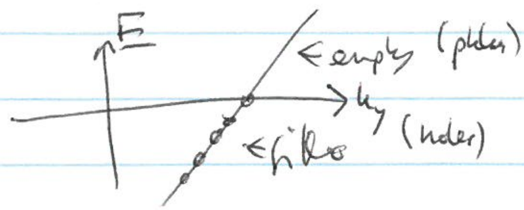
$$E v = \mu u + \hat{\Delta} i \left(\frac{\partial u}{\partial x} + k_y u \right)$$

Using Jackiw-Rebbi bound state, have zero mode for $k_y = 0$, chiral Majorana mode in general,

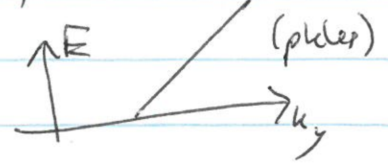
$$\underline{\underline{E = -\hat{\Delta} k_y}}$$

($v = iu$, collapses to one first order eq)

Dirac:



Majorana



Edge at outside: modes ang mom $\frac{1}{R}\left(m+\frac{1}{2}\right)$ (R = radius)

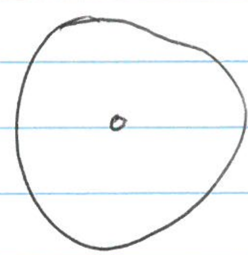
Vortex:

— circular edge <u>inside</u> fluid

— $\Delta(\underline{x})$ winds by 2π around vortex

⇒ find one zero mode, ie $E=0$ with ~~at~~ $u=\bar{v}$ "Majorana"
Corres operator is <u>self-adjoint</u> γ

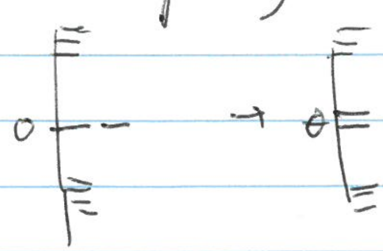
Vortex in circular drop — ~~another~~ edge modes now $\frac{1}{R} m$



another zero mode at edge

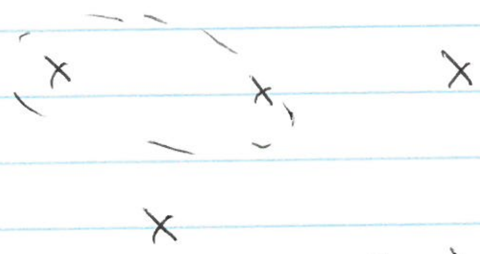
— <u>Zero modes come in pairs</u> (in finite system)

— $\exp^{ly}$ small splittings when well separated



n vortices (n even), $\{\gamma_i, \gamma_j\} = \delta_{ij}$
⇒ need $\underline{\underline{2^{n/2}}}$ states to realise algebra ~~(with well defined fermion parity)~~

So for two vortices (qp's), considered as one pair,



states are either 1 (no fermion) or ψ (fermion).
Hence fusion rules for vortices

$$\sigma \times \sigma = 1 + \psi$$

Overall fermion parity should be fixed $\rightarrow$ $2^{n/2-1}$ states