Boulder 2016 1

Fractional Quantum Hall Effect and Non-Abelian Statistics

7/7/16

Plan: Top. phases

G.Fateleg? Quasiparticles, fusion, statistics

Qu. Hall basics

Laughlin states, plasma mapping

Qholes, ~~statistics by adiab~~

Adiabatic transport, statistics

MR construction from CFT

Nonabelian stats, calc^ns. Edge ~~st~~ excitations

BCS thy, Majorana zero modes

CF thy? Other geometries.

[RR states. Ground state deg; modular properties; Verlinde formula]

[Hall viscosity, central charge]

Entanglement spectrum

Background reading:

NR Physics Today (2012)

Phys Rev B 79, 045308 (2009)

" " B 61, 10267 (2000), Sec II.

What is an (equilibrium) phase of matter?

Given some matter (either expt or theory) we would like to

- characterize the state independently of details of Hamiltonian, or of thermodynamic parameters
  - or even of constituents (e.g. which chemical element)

- be able to decide when phases count as same, when different
  - distinction should be sharp, not just quantitative

So phases remain invariant under continuous change of Hamiltonian or parameters, until some boundary is crossed, ~~then~~ at which point another phase is entered

E.g. liquid vs gas in a given substance — surely distinct, because transition (boiling)?

- but can continuously connect them without transition, at high pressure & temp
  - so same phase: "fluid"

Symmetry paradigm (Landau)

Assumptions (throughout): microscopic degrees of freedom (many ptcle system, or spins etc) are local in space; Hamiltonians involve only short-range hopping & interactions. No disorder.

Suppose there is a symmetry of Hamiltonian, throughout the parameter space.
There may be parameter regions (phases) where symmetry is spontaneously broken in the infinite-size ("thermodynamic") limit, or broken in different ways.

"Thm": these cannot be continuously connected without encountering a boundary at which symmetry changes.

Ex: liquid-solid transition. Hamiltonian for ptcles in cts space is translation & rotation inv.

Symmetry preserved in liquid, broken in solid (crystal).

[May be distinct ways to break, ie distinct solid (also liquid crystal) phases — e.g. ice, about 20 solid phases under pressure]

Used (?) to be widespread belief in converse Thm (?)

"if phases cannot be cts^ly connected w.o. crossing boundary, there must be a difference in ~~their~~ symmetry ~~#~~ breaking"

Now know this is <u>wrong</u>

<u>Challenges to paradigm</u> (70s – 80s):

- Kosterlitz–Thouless transition & low T phase of XY model in two dims
  - no symmetry breaking (but differences in correlations)
- spin glasses, metal-insulator transition, ...
- biggest: <u>integer & fractional quantum Hall effect</u> (1980, 1982)
  - <u>quantum</u> phases of matter: phases at zero temp, dominated by QM (no phase transition at T > 0)
  - no symmetry breaking; phases distinguished by Hall conductivity

In response to QH and to other exotic states, e.g. from high T_c / antiferromagnets, theorists developed concept of

Topological phases

Def: a qu. system at $T=0$ is in a top. phase if there is an energy gap above ground state(s) for bulk excitations in the thermo (i.e. infinite size) limit.
(hereafter, call it "gap")

— could be gapless excitation at edge

Folklore(?): an energy gap does not close under pert of Ham by suff small pert by local (short-range) terms, so in same phase.

At a transition, bulk gap must collapse
(could be 1st or 2nd order)

Note: 1) def^n looks trivial, and allows a top. phase to be a (the) trivial one. Not oversight!

Distinguish trivial from non-trivial separately.

2) other "topological" or robust effects not covered (e.g. gapless edges). Interesting, but we need a sharp definition.

Def^n: <u>topological properties</u> — properties unchanged throughout a topological phase

Examples:

0) existence (not magnitude) of bulk energy gap (but part of definition of top. phase)

1) multiplicity ~~of~~ of ground state of Ham in the phase when constructed on space of non-trivial topology: ~~sphere~~, torus, ... (Wen, Niu 1990)

2) existence of quasiparticle excitations with non-trivial statistics (Moore, NR 1991)

3) robust gapless edge excitations (Wen 1990)

4) quantized transport properties, such as Hall ~~conductivity~~ conductance. (Laughlin 1980)

In practise, all known non-trivial top. phases possess <u>one or more</u> of last ~~four~~ as well as 0) — they can be used to distinguish phases

Additionally, non-trivial top. phases involve non-trivial <u>entanglement</u>

Most natural way to understand top. properties is by formulating an <u>effective field theory</u> (ie as an RG fixed point)

- ie low energy ~~long wavelength~~ description of response of ground state to ext probes and of quasiparticle properties

- bulk part will consist of local terms and "top. inv" will hold because of "mass" gap & short-range Ham. Below gap, no local ordinary local excitation can be created

- closely connected with some "top. qu. field thy" (Witten 1989)

<u>Basic notions</u>

Begin with Top. degenerate (= equal energy) ground states $|\alpha\rangle$ if occur (e.g. on torus):

- ~~any~~ local op O_x - ie acting only on ptcles within some bounded distance of a point $\underline{x}$ (as identity elsewhere) Commute when well separated - so <u>bosonic</u> (fermion ops "not local")

- any local $O_{\underline{x}}$ must have $\langle\alpha|O_{\underline{x}}|\beta\rangle \propto \delta_{\alpha\beta}$

Otherwise could add

$$\lambda \int d^d x \; \mathcal{O}_{\underline{x}}$$

to Ham and split degeneracy — not top^i.
Assume non-top deg "already split" ("accidental")

$\boxed{\text{Deg. ground states indistinguishable by local probes}}$

(can do for deg due to symmetries etc) — disposes of e.g. spon. brea of discrete symmetry

Note: all such statements are up to exp^{ly} small corrections — exp in system size, separation, etc

Energy gap implies corr^{ns} (of local ops) decay exp^{ly} with separation, so there is a corr length $< \infty$, enters exp correct^{ns}

Quasiparticles

Assume space two dimensional

From ground state, may be able to "twist" around a point to make an intrinsic defect or qptle — a state in same Hilbert space (not necc energy eigenstate) ~~that created by a local operation, if qptle is isolated~~ — far away from point, state still looks like ground state (but not near the point; qptles can be detected and moved around by local ops)
Change in energy is finite

— case of interest is when an isolated such object cannot be created/destroyed by any local op (change state at all distance). It cannot disappear during time evolution

— identify as same "type" qptles that can be mapped to one another by applying some local operator, ~~or~~ inc by transporting in position. Type can be inferred from local measurements. Usually a finite number of types.

— identity or "do nothing" on ground state is honorary qptle, identity or trivial type

— ~~poss~~ antiptle exists for each type. (use inverse trick)

— possible to create qptle & anti-q.p. within a disk by using operator in disk

— more generally, two qptle of type α, β in a region can be viewed afrom far away as a single q.p., type γ, say

— fusion

— a state containing several well-separated q.p's can be degenerate. (topological, like for ground states)
 — multiplicity ind of positions
 — which of these deg states it is cannot be discerned or changed by a local operator — non-local info storage

- ~~{~~Hence interest for quantum information.~~}~~

  - errors/decoherence are due to local terms in Ham, so state (within degenerate subspace) is <u>top. protected</u> against errors.

Statements again have caveats: "local" involves a scale; an op spread over ~~the~~ scale of separation of two quasiparticles <u>can</u> change state.

Really about <u>order of limits</u>, and with exp^{ly} small corrections, as before.

<u>Calculating degeneracy</u>

Fusion described by <u>fusion rules</u>: if α, β can fuse to γ in $N_{\alpha\beta}^{\gamma}$ ~~t~~ distinct "ways", ($N_{\alpha\beta}^{\gamma}$ can be 0, 1 or > 1) can encode this as a formula:

ϕ_α represents qp of type α, then

$$\phi_\alpha \times \phi_\beta = \sum_\gamma N_{\alpha\beta}^{\gamma} \phi_\gamma \quad = \phi_\beta \times \phi_\alpha$$

formally says same thing (for all types α, β)
(Just formal; <u>not</u> ops on a Hilbert space!)
Let $\alpha = 0$ be "trivial" type, $\phi_0 = \mathbb{1}$.
Acts as identity: $\mathbb{1} \times \phi_\alpha = \phi_\alpha$ (clearly), $N_{\alpha 0}^{\beta} = \cancel{1}\ \delta_{\alpha\beta}$
Also associativity: $\phi_\alpha \times (\phi_\beta \times \phi_\gamma) = (\phi_\alpha \times \phi_\beta) \times \phi_\gamma$
because order of grouping can't matter

So we obtain a "fusion ring" (commutative) defined by above rules
(comm alg over $\mathbb{Z}$.)

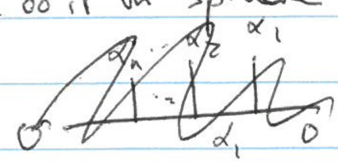
Can further assume that for any α, there is $\bar{\alpha}$
(anti qplde ~~type~~ type for α) s.t.

$$\phi_\alpha \times \phi_{\bar{\alpha}} = \underset{\substack{\uparrow \\ \text{coeff } 1!}}{\mathbb{1}} + \sum_{\gamma \neq 0} N_{\alpha\bar{\alpha}}^{\gamma} \phi_\gamma$$

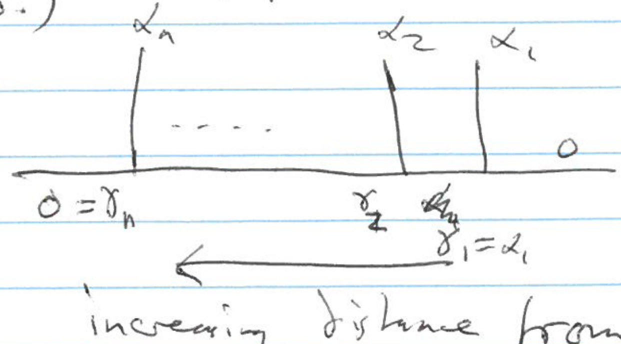
~~$N_{\alpha 0}^{\alpha} = 1$ always~~

If for some α, β, $\sum_\gamma N_{\alpha\beta}^{\gamma} > 1$,
then there is more than one way to combine,
i.e. more than one degenerate state,
in fact $N_{\alpha\beta}^{\gamma}$ of them.

"Twisting" is a change in b.c. on state at ∞.
In order to leave state at spatial infinity unchanged, need to fuse all q.p's to end in $\alpha = 0$.
(Then can do it on sphere also.)



"multiperipheral diagram"

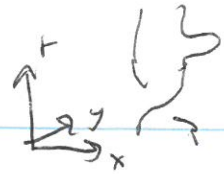


Then deg of such states is

$$\boxed{\deg = \sum_{\gamma_2 \ldots \gamma_{n-1}} N_{\gamma_{n-1}\alpha_n}^{0} N_{\gamma_{n-2}\alpha_{n-1}}^{\gamma_{n-1}} \cdots N_{\alpha_1\alpha_2}^{\gamma_2} \underbrace{N_{\alpha_1 0}^{\alpha_1}}_{=1}}$$

Exercise: check this!

Dragging quasiparticles



To define statistics, drag adiabatically (ie slowly) until at final time, qptles are at same position they started at, up to a permutation among qptles of same type.

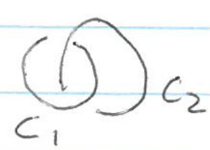
Produces a Berry phase (or unitary matrix) times original state (or in space of deg states).

Must do with all qptcles well-separated throughout (to avoid splitting)

Result <u>can</u> depend on ~~the~~ path taken
But change under small change in path (subject to preceding condition) can only be a phase factor even in deg case; phase does not depend on which other qptles are present. Dependence on precise path Has form

$$\exp i \sum_j \oint_{C_j} dx^\mu \, \lambda_{\mu,\alpha_j}(\underline{x}(s), t(s))$$

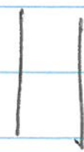
where path (link) has components C_j and type α_j moves around C_j. $\lambda_{\mu,\alpha}(\underline{x},t)$ is a real one-form ~~fn~~ on spacetime, ind of qptles present. It is a ~~local~~ coupling to background.



Change in path is local operation, cannot detect other qp's far away or change deg state, so this is only possible form

Remaining effect depends only on "isotopy class" of the exchange (or "topology").
Distinct isotopy classes cannot be deformed to one another

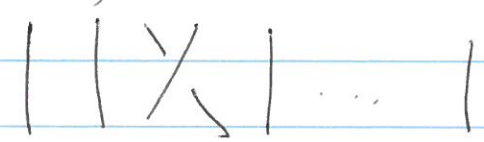
So  $\cong$  "isotopic" – can deform without lines crossing

but  $\ncong$ 

Suppose for simplicity all qp's present are same type.

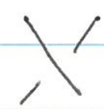
Project space to line, draw pictures

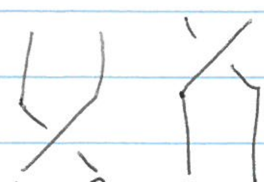
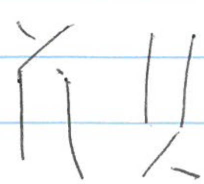
Basic exchange ("braid")

$\tau_{34} =$ 

1 2 3 4 5 - - n

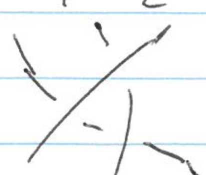
$\tau_{i,i+1}$ similarly.

$\tau^{-1} =$ 

Isotopy:  $\cong$ 

1 2 3 4

$\tau_{34}\tau_{12} = \tau_{12}\tau_{34}$
or any $\tau_{i,i+1}, \tau_{j,j+1}$ s.t. $i \neq j, j+1$, $i+1 \neq j, j+1$

 $\cong$ 

1 2 3   1 2 3

$\tau_{23}\tau_{12}\tau_{23} = \tau_{12}\tau_{23}\tau_{12}$

for any three [illegible]

These generators and relations define Artin's braid group B_n. He proved this captures all isotopy equiv of braids – i.e. exchanges.

$$\boxed{\text{Exchanges of } n \text{ identical type qp's form a group, } B_n}$$

Exercise: if have these relations and also $\tau_{i,i+1}^2 = id$, $i = 1, \ldots, n-1$, ~~then~~ we get the symmetric (permutation) group S_n. Try to prove this.

In QM (ie sys of qp's), these operations "represented" by unitary matrices. Sufficient to know matrices of $\tau_{i,i+1}$; the matrices must obey the braid relations above.

Simplest way is $\tau_{i,i+1} = e^{i\theta}$ (θ real), all i, so (rep^n) space is 1 dim^l. Called anyons or Abelian anyons (because $\tau_{i,i+1}$'s commute).

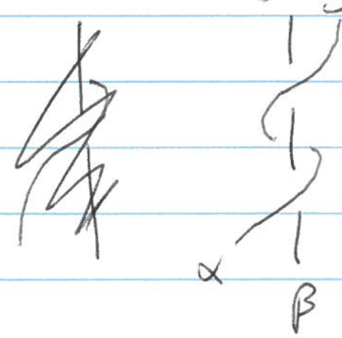
If $\tau_{i,i+1}$ are non-commuting matrices, speak of nonabelian stats or nonabelions.

~~Can these occur for qp's in real world?~~

Consistency as qp statistics investigated in 80s

— Doplicher, Roberts; Frohlich, ...

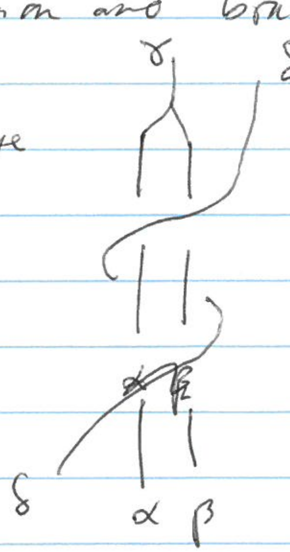
General picture also involves "mutual stats", effect of qp of one type going around qp of another type



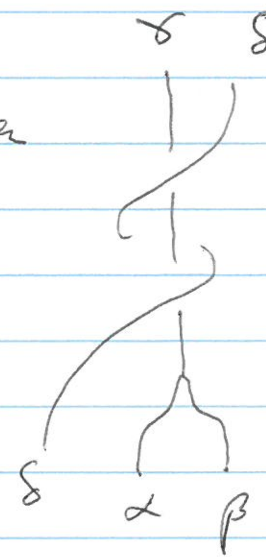
(cannot exchange distinct types meaningfully)

Also fusion and braiding must be consistent:

Braid then fuse



$\cong$ fuse then braid



— isotopic, so same matrix

Some additional structures also

Same structure was found in rational conformal field thy

(Moore, Seiberg 1989) and in quantum groups

Culminated in def of a modular tensor category

(Reshitikhin + Turaev 1990)

& connected with Chern-Simons gauge thy

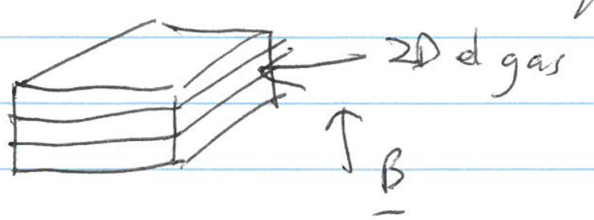
(Witten 1989)

Also connected with isotopy invariants of knots, links, 3-manifolds (Jones 1984)

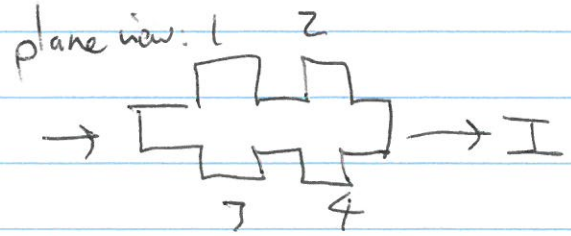
(Background to our construction: Moore, NR 1991)

QH basics

Semicond heterostructures / quantum wells

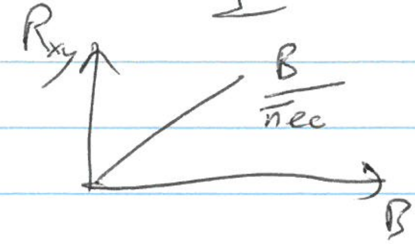


plane view:



resistance $R_{xx} = \dfrac{V_{12}}{I}$, $R_{xy} = \dfrac{V_{13}}{I}$

Density $\bar{n} = \dfrac{N}{A} \approx 10^{11}\,\text{cm}^{-2}$
area

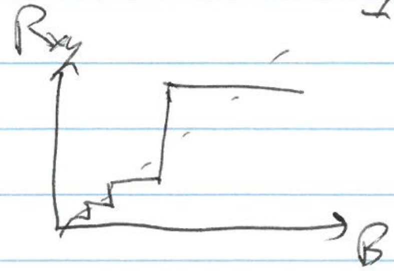


Classical Hall effect (non-rel el's)

Classically (clean system) $R_{xy} = \underbrace{\rho_{xy}}_{\text{resistivity}} = \dfrac{B}{\bar{n}ec} = \dfrac{B}{\bar{n}} \cdot \dfrac{e}{hc} \cdot \underbrace{\dfrac{h}{e^2}}_{\simeq 26\,\text{k}\Omega}$

$\Phi_0 = \dfrac{hc}{e}$

Quantized Hall effect:
Expt observation



$\sigma_{xy} = \dfrac{1}{\rho_{xy}} = \nu \dfrac{e^2}{h}$

$\rho_{xx} = 0$

and $R_{xx} = 0$ on plateau

$$\boxed{\nu = \frac{\bar{n}\,\Phi_0}{B} = \frac{N}{N_\phi}}$$

dimensionless density or filling factor

N_ϕ = # flux quanta $= \dfrac{A \cdot B}{\Phi_0}$

IQHE Klitzing 1980
FQHE Stormer et al 1982

with ν = integer or rational

Landau levels

Single ptcle $\quad H_1 = \frac{1}{2m_e}\left(-i\hbar\nabla - \frac{e}{c}\underline{A}\right)^2$

$\nabla \times \underline{A} = \underline{B}$

Evals $\quad E_n = (n+\tfrac{1}{2})\hbar\omega_c, \; n = 0,1,2,\ldots$

cyclotron freq $\quad \omega_c = \frac{eB}{m_e c} > 0$

"Symmetric gauge" $\underline{A} = \frac{1}{2}\underline{r}\times\underline{B}$, the

efns ~~are~~ for $n=0$ are

$$u_m(z) = \frac{z^m e^{-\frac{1}{4}|z|^2}}{\sqrt{2\pi 2^m m!}} \quad \left(\text{I set } l_B^2 = \frac{\hbar c}{eB} = 1\right)$$

$$z = x+iy, \quad m = 0,1,2,\ldots$$

$u_m(z)$ peaked at $|z| = \sqrt{2m}$ $\Rightarrow$ no. of states in LLL per unit area $= \frac{1}{2\pi}$

Exercise: check this from the info above

I.e. one state per LL per ~~ar~~ area covered by one flux quantum $\Phi_0 = \frac{hc}{e}$

(Same for higher LLs, $u_{m,n\neq 0}$ differ)

Many non-interacting els ~~states~~: ground state has lowest energy states filled

If occupy $n = 0, 1, \ldots, \nu - 1$ (ν levels filled)
the density is $\bar{n} = \frac{\nu}{2\pi}$ and uniform.

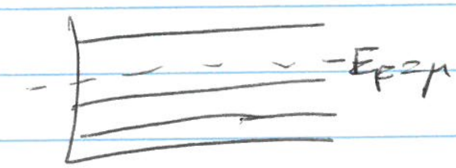
Then $\nu = 2\pi \bar{n} = \frac{\bar{n}\Phi_0}{B}$

hence term "filling factor"

For this trans. inv. Ham, can show $\sigma_{xy} = \frac{\nu e^2}{h}$, $\sigma_{xx} = 0 = \rho_{xx}$

$\nu = 0, 1, 2, \ldots$ special because creating/destroying el costs energy (change in $H - \mu N$) > 0

— energy gap



μ as a fn of $\bar{n}$ jumps at these values
so $\frac{d\mu}{d\bar{n}}$ has δ-fn spikes

$\Delta\kappa$

— compressibility $\kappa = \frac{d\bar{n}}{d\mu}$ is zero at these densities

— incompressible at $\nu = 0, 1, 2, \ldots$